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Habilitation thesis reviewer's report

Masaryk University	
Faculty of Science	
Field of study	**Mathematics - Geometry**
Applicant	**Mgr. Josef Šilhan, Ph.D.**
Unit	**Department of Mathematics and Statistics, Faculty of Science, Masaryk University**
Habilitation thesis	**Invariant quantization and differential symmetries on AHS structures**
Reviewer	Michael G Eastwood FAA (PhD Princeton)
Unit	University of Adelaide

Reviewer's report (extent of text up to the reviewer)

See the following two pages.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer) ...

None

Conclusion

The habilitation thesis submitted by mgr. **Josef Šilhan,** Ph.D., entitled "***Invariant quantization and differential symmetries on AHS structures***" ***meets*** the requirements applicable to habilitation theses in the field of **Mathematics - Geometry**.

In Adelaide on 26th April 2016

Report on

Invariant quantization and differential symmetries on AHS structures

by Josef Šilhan

Habilitation Thesis at Masaryk University

Regarding the style of this thesis, the writing is fairly good and sufficiently clear. These is, however, no need for spelling mistakes (for example, 'researches,' 'developement,' 'articels,' and 'habilition' in the Preface). There are occasional gaps in the author's reasoning (for example, in his definitions of (commuting) symmetries of the Laplacian). Some statements are, strictly speaking, false (for example, his assertion that a full understanding of symmetries of the conformal Laplacian is available only on locally flat conformal structures (since it is easy to produce conformal structures with no symmetries at all and the empty set is fully understood)). His discussion of the notion of *invariance* in §1.3 is far too informal to be useful. The first sentence of §1.4, 'It turns out that there is indeed a preferred symmetry of the Laplacian for a given symbol' is false. I know what he means to say, but he is not saying it. The *prolongation connection* mentioned in §1.5 is not and cannot be obtained by prolongation! I am sure that the author knows these things but his presentation is occasionally imprecise. There is nothing wrong with the 'prolongation connection' itself (due to joint work of Šilhan with Hammerl, Somberg, and Souček) but it is not obtained by prolongation but is rather an excellent replacement, given that prolongation itself does not work.

In summary, I do not think that this thesis itself will be useful for researchers on the borderline of analysis and mathematical physics (as is the author's stated aim in the Preface). Instead, the author's published works and, in particular, the six articles reproduced in this Thesis, are very good indeed and well deserving of Habilitation status.

As already mentioned, the prolongation connection of [23] (further developed through key examples in [24]) is an excellent construction, which circumvents the lack of invariance inherent in naïve prolongation techniques (as in the joint work of Branson, Čap, Eastwood, and Gover).

My personal favourite article, however, is [30] 'Second order symmetries of the conformal Laplacian' written by Michael, Radoux, and Šilhan. This article closely analyses the second order symmetries of the conformal Laplacian in the curved setting and relates this analysis to the quantization operators developed in earlier works, reproduced in this Thesis. Rather amazingly, not only do these authors produce an example of a conformal metric in three dimensions (namely, a particular Stäckel metric) with non-trivial conformal Killing tensor K^{ab} such that there is no symmetry of the form

$$K^{ab}\nabla_a\nabla_b + \text{lower order terms},$$

but they do so by means of an obstruction! More specifically, they associate to K^{ab} a conformally invariant 'obstruction' 1-form $\mathbf{Obs}(K)$ with the following properties. Let

$$\Box \equiv \nabla^a\nabla_a - \tfrac{n-2}{4(n-1)}R$$

denote the conformal Laplacian or 'Yamabe operator' and let

$$\mathcal{D}_K \equiv K^{ab}\nabla_a\nabla_b f + \tfrac{n}{n+2}(\nabla_a K^{ab})\nabla_b + \tfrac{n(n-2)}{4(n+2)(n+1)}(\nabla_a\nabla_b K^{ab}) - \tfrac{n+2}{4(n+1)}R_{ab}K^{ab}$$

denote the operator constructed in [15], now regarded as an example of the 'quantization' procedure of Šilhan and co-workers. Then

- $\mathbf{Obs}(K) = 0 \Leftrightarrow \mathcal{D}_K$ is a symmetry of $\Box$,
- $\mathbf{Obs}(K) = -2df \Leftrightarrow \mathcal{D}_K + f$ is a symmetry of $\Box$,
- $\Box$ admits a symmetry with symbol $K^{ab} \Leftrightarrow d(\mathbf{Obs}(K)) = 0$.

This is the best possible solution to the search for second order symmetries of the conformal Laplacian. The particular three-dimensional Stäckel metric shows that $d(\mathbf{Obs}(K))$ is a genuine obstruction.

I have no hesitation is recommending that, on the basis of this thesis, Josef Šilhan be awarded Habilitation status in the field of Mathematics - Geometry at Masaryk University.