MASARYK UNIVERSITY
Faculty of Science
Department of mathematics and statistics

Habilitation Thesis

Brno 2016 Petr Zemánek

MASARYK UNIVERSITY
Faculty of Science
Department of mathematics and statistics

Discrete Symplectic Systems and Square Summable Solutions

Habilitation Thesis

Petr Zemánek

2010 Mathematics Subject Classification:
Primary 39A10; 39A12; 39A70; 47A06.
Secondary 34B20; 34B05; 47A20; 47B25; 47B37; 47B39.

Keywords: *Discrete symplectic system; time-reversed system; linear Hamiltonian difference system; Weyl–Titchmarsh theory; eigenvalue; $M(\lambda)$-function; Atkinson-type condition; Weyl disk; Weyl circle; Weyl solution; square summable solution; limit point case; limit circle case; limit circle invariance; criteria; separated endpoints; jointly varying endpoints; periodic endpoints; definiteness condition; nonhomogeneous problem; linear relation; deficiency index; self-adjoint extension; uniqueness; Krein–von Neumann extension.*

A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.

David Hilbert, see [86, pg. 438]

Preface

This habilitation thesis is submitted as an integral part of the promotion to *associate professor* at Masaryk University (Faculty of Science) in the field Mathematics – Mathematical analysis. It reflects my research from the period of 2011 to 2016, which includes the postdoc position supported by the Program "Employment of Newly Graduated Doctors of Science for Scientific Excellence" (grant number CZ.1.07/2.3.00/30.0009) co-financed from European Social Fund and the state budget of the Czech Republic.

I would like to express gratitude to my collaborator, colleague, friend, and former advisor Roman Šimon Hilscher for his continual support, many hours of stimulating and inspiring discussions, and his comments to a pre-final version of the manuscript. I am also deeply indebted to Stephen Clark for the (hopefully successful and pleasant) collaboration and to the Department of Mathematics and Statistics (Missouri University of Science and Technology, Rolla, USA) for hosting my visits in 2012 and 2014, where some parts of the presented results were achieved. In addition, many thanks belong to the heads of our scientific team, Zuzana Došlá and Ondřej Došlý. Finally, I highly appreciate the financial support of the Dean of the Faculty of Science provided for the Spring semester 2016, which enabled me to focus on this text only.

Nevertheless, this work could not be written without the unflagging support and endless tolerance of my wife Péťa. I also apologize to our daughter Stella that we did not spend more time together during the first months of her life. I am extremely thankful to both of them with

$$\left(x^2 + \tfrac{9}{4}\,y^2 + z^2 - 1\right)^3 - x^2 z^3 - \tfrac{9}{200}\,y^2 z^3 = 0,$$
$$r(\theta) = 1 + \sqrt{-\ln\left(2\,\mathrm{e}^{-0.81^2} - \mathrm{e}^{-1.4^2 \sin^2[5\,(\theta-\pi/2)/2]}\right)}.$$

The text was typeset by using XƎLaTeX and pictures were generated by GeoGebra.

Brno, August 2016 Petr Zemánek

Table of contents

The approach turns out to be fruitful and successful, and leads to the effective construction as well as the theoretical understanding of an abundance of what we call symplectic difference scheme, or symplectic algorithms, or simply Hamiltonian algorithms, since they present the proper way, i.e., the Hamiltonian way for computing Hamiltonian dynamics.

Kang Feng, see [74, pg. 18]

Chapter 1

Introduction

In this habilitation thesis we present our recent contributions to the ongoing development of the theory of square summable solutions of discrete symplectic systems. It is based on results, which were achieved by the author and his scientific collaborators (S. Clark and R. Šimon Hilscher) during his postdoctoral research in the period 2011–2016. The major part of them was published in papers [A13, A15, A17, A18, A21] and in a more general setting also in papers [A16, A19, A20].

Systematic research in this area began in 2010, when M. Bohner and S. Sun in [26] and independently (and more extensively) S. Clark and the author in [A4] investigated square summable solutions of discrete symplectic systems with a special linear dependence on the spectral parameter, see the beginning of Chapter 2 for more details. In the present work we collect our results for discrete symplectic systems with general linear dependence on the spectral parameter as well as with a polynomial or analytic dependence. However, we emphasize that these results do not only improve the type of the dependence on the spectral parameter, but they significantly generalize and extend the results of [26] and [A4]. We also lay foundations of the "operator theory" for discrete symplectic systems, which is intimately connected with the topic of square summable solutions. The thesis consists of seven chapters and an appendix.

- In the next sections of this introductory chapter we summarize the used notation and some important results from linear algebra, define discrete symplectic systems, and show some of their special cases.
- In *Chapter* 2 we develop the limit point and limit circle classification for discrete symplectic systems, which depend linearly on the spectral parameter. In particular, we investigate the associated eigenvalue problem with separated boundary conditions, the Weyl disks and Weyl circles, their limiting behavior, and properties of square summable solutions including the precise analysis of the number of linearly independent square summable solutions as well as some criteria for the limit point and limit circle cases. This chapter is based on [A15].
- Since the theory of Chapter 2 is based on separated boundary conditions, we focus in

Chapter 3 on the spectral theory for discrete symplectic systems with general jointly varying endpoints. We characterize the eigenvalues, construct the $M(\lambda)$-function and Weyl disks, their matrix radii and centers, and discuss the number of linearly independent square summable solutions. These results include several particular cases, such as the periodic and antiperiodic endpoints. The method utilizes a new transformation to separated endpoints, which is simpler and more transparent than the one in the known literature. This chapter is based on [A13].

- In *Chapter 4* we extend the invariance of the limit circle case to two linear discrete systems depending linearly on spectral parameter. The main result is a discrete analogue of the corresponding continuous time statement, which was derived by Walker for a pair of non-hermitian linear Hamiltonian differential systems in [168]. This chapter is based on [A19].
- In *Chapter 5* we consider discrete symplectic systems with polynomial and analytic dependence on the spectral parameter. We derive fundamental properties of these systems (including the Lagrange identity) and discuss their connection with systems known in the literature. In analogy with the results of Chapter 2, we present a construction of the Weyl disks and determine the number of linearly independent square summable solutions. In addition, we prove the invariance of the limit circle case for a special quadratic dependence on the spectral parameter and its extension to the case of two (generally non-symplectic) discrete systems. We also provide several illustrative examples, one of which contradicts the invariance of the limit circle case for symplectic systems depending truly analytically (i.e., nonpolynomially) on the spectral parameter. This chapter is based on [A17].
- In *Chapter 6* we study the definiteness of the discrete symplectic system, pay an attention to a nonhomogeneous discrete symplectic system, and introduce the minimal and maximal linear relations associated with these systems. We also show some fundamental properties of the corresponding deficiency indices, including a relationship between the number of square summable solutions and the dimension of the defect subspace. Moreover, we give a sufficient condition for the existence of a densely defined operator associated with a discrete symplectic system. This chapter is based on [A18].
- In *Chapter 7* we characterize all self-adjoint extensions of the minimal linear relation. Especially for the scalar case on a finite discrete interval we present some equivalent forms, discuss their uniqueness, and describe the Krein–von Neumann extension. In addition, we establish a limit point criterion, which partially generalizes a classical limit point criterion for the second order Sturm–Liouville difference equations. This chapter is based on [A21].
- In order to make the thesis self-contained we conclude this work by a short overview of basic definitions and some important results from the theory of linear relations, which is utilized in Chapters 6 and 7.

We close each chapter by a section concerning bibliographical notes, in which we mention some open problems and possible directions for our future research. For readers' convenience, we provide also a list of symbols, which are used throughout the thesis. Finally, we include an overview of author's publications.

For completeness, we point out that the results of Chapters 2–5 were further extended to symplectic systems on time scales in [A16, A19, A20]. This generalization enables us to unify and compare the corresponding results for linear Hamiltonian differential systems and discrete symplectic systems. Moreover, since some of the studied problems were not considered for linear Hamiltonian differential systems (such as the jointly varying endpoints or the analytic dependence on the spectral parameter), it yields even new results for these systems.

1.1 Notation and auxiliary results

In this section we summarize the notation used through this thesis and recall several known facts from linear algebra (see also the list of symbols on page 141).

The sets of *natural numbers*, *integers*, *real* and *complex numbers* are, respectively, denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$. Moreover, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $\lambda \in \mathbb{C}$ the symbols $\bar{\lambda}$, $\mathrm{re}(\lambda)$, $\mathrm{im}(\lambda)$, and $\delta(\lambda)$ represent, respectively, the *complex conjugate* of λ, the *real* and *imaginary parts* of λ, and the *sign* of the imaginary part of λ i.e., $\delta(\lambda) := \mathrm{sgn}(\mathrm{im}(\lambda))$. We also use the symbols $\mathbb{C}_+$ and $\mathbb{C}_-$ for the *upper* and *lower complex half-planes*, i.e., we put $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \delta(\lambda) = 1\}$ and $\mathbb{C}_- := \{\lambda \in \mathbb{C} \mid \delta(\lambda) = -1\}$.

Typically, all vectors and matrices are written by small and capital letters, respectively. All matrices are considered over the field of complex numbers $\mathbb{C}$. For $r \in \mathbb{N}$ we denote the $r \times r$ *identity* and *zero matrices* by I_r and 0_r. If the dimension is clear from the context, we write only I and 0 (for simplicity, the zero vector will also be denoted by 0). For $r, s \in \mathbb{N}$ we mean by $\mathbb{C}^{r\times s}$ the *space of $r \times s$ complex matrices* $M = (m_{i,j})_{\substack{i=1,\dots,r\\ j=1,\dots,s}}$ and $\mathbb{C}^{r\times 1}$ is abbreviated as $\mathbb{C}^r$. If $M_1, \dots, M_m \in \mathbb{C}^{r\times r}$, then $\mathrm{diag}\{M_1, \dots, M_m\}$ represents the block diagonal matrix $M \in \mathbb{C}^{mr\times mr}$ with the matrices $M_1, \dots, M_m$ on the main diagonal. For a given matrix $M \in \mathbb{C}^{r\times s}$ we indicate by $M^\top$, $\overline{M}$, M^*, $\mathrm{rank}\, M$, $\mathrm{tr}\, M$, $\det M$, $M > 0$, $M \geq 0$, M^{adj}, $\mathrm{Ker}\, M$, $\mathrm{Ran}\, M$, $\dim \mathrm{Ran}\, M$, $\mathrm{sprad}\, M$, $\mathrm{im}(M) := (M - M^*)/(2i)$, and $\mathrm{re}(M) := (M + M^*)/2$, respectively, its *transpose*, *conjugate*, *conjugate transpose*, *rank*, *trace*, *determinant*, *positive definiteness*, *positive semidefiniteness*, *adjugate* (or *adjoint*) *matrix*, *kernel*, *range* (or image, i.e., the space spanned by the columns of M), the dimension of $\mathrm{Ran}\, M$, *spectral radius*, and *Hermitian components* (or real and imaginary parts, see [102, pg. 170] or [16, Fact 3.7.29]). In addition, by $M_{p,q}$ we mean the submatrix of $M \in \mathbb{C}^{r\times s}$ consisting of the first $p \leq r$ rows and of the first $q \leq s$ columns of the matrix M and we write only M_p in the case $p = q$, i.e., for the p-th leading principal submatrix of M. We recall that two Hermitian and positive semidefinite matrices $L, M \in \mathbb{C}^{r\times r}$ with $L \leq M$ satisfy

$$\mathrm{Ran}\, L \subseteq \mathrm{Ran}\, M \quad \text{and} \quad \mathrm{rank}\, L \leq \mathrm{rank}\, M, \tag{1.1}$$

where the equalities occur simultaneously, i.e., it holds $\mathrm{Ran}\, L = \mathrm{Ran}\, M$ if and only if $\mathrm{rank}\, L = \mathrm{rank}\, M$, see e.g. [16, Fact 8.10.2]. Furthermore, for any matrices $L \in \mathbb{C}^{r\times s}$, $M \in \mathbb{C}^{s\times p}$, $P \in \mathbb{C}^{r\times q}$, and $Q \in \mathbb{C}^{r\times r}$ we have

$$\mathrm{rank}\, L = \mathrm{rank}\, LL^* = \mathrm{rank}\, L^*L, \tag{1.2}$$

$$\mathrm{rank}\, L + \mathrm{rank}\, M - s \leq \mathrm{rank}\, LM \leq \min\{\mathrm{rank}\, L, \mathrm{rank}\, M\}, \tag{1.3}$$

$$\mathrm{rank}(L, P) + \dim[\mathrm{Ran}\, L \cap \mathrm{Ran}\, P] = \mathrm{rank}\, L + \mathrm{rank}\, P, \tag{1.4}$$

$$(I - Q)^{-1} = \sum_{k=0}^{\infty} Q^k, \quad \text{when } \mathrm{sprad}\, Q < 1, \tag{1.5}$$

see e.g. [16, Corollaries 2.5.1, 2.5.3, 2.5.10 and Facts 2.11.9, 4.10.5]. In the following statements we show two important properties of unitary matrices. The first proposition can be found in [102, Lemma 2.1.8] and the second statement is from [108, Theorem 5.3].

Proposition 1.1.1. *Let $r \in \mathbb{N}$ and $U_1, U_2, \dots \in \mathbb{C}^{r\times r}$ be a given sequence of unitary matrices. Then there exists a subsequence $U_{k_1}, U_{k_2}, \dots$ such that all of the entries of U_{k_j} converge (as a sequences of complex numbers) to the entries of a unitary matrix U as $j \to \infty$.*

Proposition 1.1.2. *Let $r \in \mathbb{N}$ and the matrices $L, M \in \mathbb{C}^{r\times r}$ be such that* $\operatorname{rank} L = \ell$ *and* $\operatorname{rank} M = m$. *Then* $\sup\{\operatorname{rank} LUM \mid U \in \mathbb{C}^{r\times r}$ *is unitary*$\} = \min\{\ell, m\}$.

We also point out, see e.g. [A4, Remark 2.6], that $\operatorname{im}(M) > 0$ or $\operatorname{im}(M) < 0$ implies the invertibility of the matrix M. Moreover, we write only M^{*-1} instead of $(M^*)^{-1}$ or $\left(M^{-1}\right)^*$ and similarly for parameter dependent matrices

$$M^*(\lambda) := [M(\lambda)]^*, \quad M^{-1}(\lambda) := [M(\lambda)]^{-1}, \quad \text{and} \quad M^{*-1}(\lambda) := [M^*(\lambda)]^{-1} = [M^{-1}(\lambda)]^*.$$

Finally, if we denote by the symbol $S^\perp$ the *orthogonal complement* of a subspace S of an inner product space, then the *codimension* of S is defined as $\operatorname{codim} S := \dim S^\perp$ and it holds $S_1^\perp \supseteq S_2^\perp$ for any subspaces $S_1 \subseteq S_2$. Moreover, for any $M \in \mathbb{C}^{r\times s}$ we have

$$\operatorname{Ran} M = (\operatorname{Ker} M^*)^\perp, \tag{1.6}$$

see [16, Theorem 2.4.3].

For any matrix $M = (m_{i,j})_{i,j=1,\dots,r} \in \mathbb{C}^{r\times r}$ we define its entrywise *Hölder norm* (or *ℓ^1-norm*) and *spectral norm*, respectively, as

$$\|M\|_1 := \sum_{i=1}^{r} \sum_{j=1}^{r} |m_{i,j}| \quad \text{and} \quad \|M\|_\sigma := \max\left\{\sqrt{\mu} \mid \mu \text{ is an eigenvalue of } M^*M\right\},$$

see [102, Section 5.6] or [16, Chapter 9]. These norms satisfy the estimates

$$\|M\|_\sigma \le \|M\|_1 \le r\sqrt{\operatorname{rank} M} \times \|M\|_\sigma, \tag{1.7}$$

see [16, Fact 9.8.12 (v)], and possess the *submultiplicative* and *self-adjoint* properties, i.e., $\|ML\|_a \le \|M\|_a \times \|L\|_a$ and $\|M^*\|_a = \|M\|_a$, where $a = 1$ or $a = \sigma$. The spectral norm is also *unitarily invariant*, i.e., $\|UMV\|_\sigma = \|M\|_\sigma$ for any unitary matrices $U, V \in \mathbb{C}^{r\times r}$, which implies that $\|L\|_\sigma \le \|M\|_\sigma$ for any Hermitian matrices $L, M \in \mathbb{C}^{r\times r}$ such that $L \le M$, see [16, Fact 9.9.5]. Moreover, the spectral norm is the matrix norm induced by the Euclidean vector norm on $\mathbb{C}^r$, i.e., by the norm $\|v\|_2 := (v^*v)^{1/2}$ for any $v \in \mathbb{C}^r$, see [16, Proposition 9.49]. In other words, the inequality

$$\|Mv\|_2 \le \|M\|_\sigma \|v\|_2 \tag{1.8}$$

holds true for any $M \in \mathbb{C}^{r\times r}$ and $v \in \mathbb{C}^r$.

A matrix $M \in \mathbb{C}^{r\times r}$ is said to be *nilpotent* provided there exists $m \in \mathbb{N}$ such that $M^m = 0$. The following proposition can be found in [16, Fact 3.17.9].

Proposition 1.1.3. *Let $L, M \in \mathbb{C}^{r\times r}$ be such that the matrix L is nilpotent and the matrices commute, i.e., $LM = ML$. Then* $\det(L + M) = \det M$.

Following [16, Chapter 4], for $\lambda \in \mathbb{C}$ we define the *polynomial matrix* $M(\lambda)$ as

$$M(\lambda) := \lambda^m M_m + \lambda^{m-1} M_{m-1} + \dots + \lambda M_1 + M_0,$$

where $M_m, \dots, M_0 \in \mathbb{C}^{r\times r}$. The matrix-valued function $M(\lambda)$ is called *singular* if $\det M(\lambda)$ is zero for all $\lambda \in \mathbb{C}$, otherwise $M(\lambda)$ is called *nonsingular*. Moreover, $M(\lambda)$ is called *unimodular* if $\det M(\lambda)$ is a nonzero constant. The latter condition is equivalent to the fact that $M(\lambda)$ is nonsingular and that $M^{-1}(\lambda)$ is also a polynomial matrix, see [16, Proposition 4.3.7].

For $M \in \mathbb{C}^{r\times r}$ we also define the *matrix exponential* $\exp(M)$ as

$$\exp(M) := \sum_{j=0}^{\infty} \frac{1}{j!} M^j.$$

Then for $L, M, P, Q \in \mathbb{C}^{r\times r}$ with L being nonsingular and $PQ = QP$ we have

$$\det[\exp(M)] = \mathrm{e}^{\operatorname{tr} M}, \quad L\exp(M)L^{-1} = \exp(LML^{-1}), \tag{1.9}$$

$$\exp(P)\exp(Q) = \exp(Q)\exp(P) = \exp(P+Q), \tag{1.10}$$

see [16, Corollary 11.2.4, Proposition 11.2.8(v), Corollary 11.1.6] .

If $\mathcal{I}$ is an interval in $\mathbb{R}$, then the associated *discrete interval* $\mathcal{I}_{\mathbb{Z}}$ is the set of integers in $\mathcal{I}$, i.e., $\mathcal{I}_{\mathbb{Z}} := \mathcal{I} \cap \mathbb{Z}$. In particular, $\mathbb{N} = [1,\infty)_{\mathbb{Z}}$. With $N \in \mathbb{N} \cup \{0,\infty\}$ we will be interested in the *discrete intervals*, which are bounded or unbounded above, i.e., $\mathcal{I}_{\mathbb{Z}} := [0, N+1)_{\mathbb{Z}}$. Then we define $\mathcal{I}_{\mathbb{Z}}^{+} := [0, N+1]_{\mathbb{Z}}$ with the understanding that $\mathcal{I}_{\mathbb{Z}}^{+} = \mathcal{I}_{\mathbb{Z}}$ when $N = \infty$. If N is finite we write rather $[0, N]_{\mathbb{Z}}$ instead of $[0, N+1)_{\mathbb{Z}}$.

By $\mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{r\times s}$ we denote the *space of sequences*, defined on $\mathcal{I}_{\mathbb{Z}}$, of complex $r \times s$ matrices, where typically $r \in \{n, 2n\}$ and $1 \leq s \leq 2n$. Especially, we write only $\mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{r}$ in the case $s = 1$. If $M \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{r\times s}$, then $M(k) := M_k$ for $k \in \mathcal{I}_{\mathbb{Z}}$; if $M(\lambda) \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{r\times s}$, then $M(\lambda, k) := M_k(\lambda)$ for $k \in \mathcal{I}_{\mathbb{Z}}$. When $M \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{r\times s}$ and $L \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{s\times q}$, then $ML \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{r\times q}$, where $(ML)_k := M_k L_k$ for all $k \in \mathcal{I}_{\mathbb{Z}}$. The set $\mathbb{C}_0(\mathcal{I}_{\mathbb{Z}})^{r\times s}$ represents the subspace of $\mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{r\times s}$ consisting of all sequences *compactly supported* in the discrete interval $\mathcal{I}_{\mathbb{Z}}$.

The symbol Δ means the *forward difference operator* acting on $\mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{r\times s}$, i.e., we put $(\Delta z)_k := \Delta z_k = z_{k+1} - z_k$. Moreover, we let $z_k\big|_m^n := z_n - z_m$.

Finally, the next result follows directly from [9, Theorem IV.1.1] and it concerns a sufficient condition for the boundedness of any fundamental matrix of a recurrence relation.

Proposition 1.1.4. *Let $M \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{r\times r}$ be such that $\sum_{k=0}^{\infty} \|M_k - I\|_1 < \infty$. Then all solutions of the recurrence relation*

$$u_{k+1} = M_k u_k, \quad k \in [0,\infty)_{\mathbb{Z}}, \tag{1.11}$$

converge as $k \to \infty$, i.e., for any fundamental matrix $\mathcal{U} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{r\times r}$ of system (1.11) there exists $\kappa > 0$ such that $\|\mathcal{U}_k\|_1 < \kappa$ for all $k \in [0,\infty)_{\mathbb{Z}}$. In addition, if M_k is invertible for all $k \in [0,\infty)_{\mathbb{Z}}$, then $\lim_{k\to\infty} u_k \neq 0$ for any nontrivial solution $u \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{r}$ of system (1.11).

1.2 Discrete symplectic systems

Let us define the real $2n \times 2n$ skew-symmetric matrix

$$\mathcal{J} := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1.12}$$

Then $\det \mathcal{J} = 1$ and $\mathcal{J}$ can be seen as a matrix analogue of the complex unit i, because $\mathcal{J}^2 = -I$. Moreover, $\mathcal{J}^{\top}\mathcal{J} = I$ and $\mathcal{J}^{-1} = -\mathcal{J} = \mathcal{J}^{\top}$. A matrix $M \in \mathbb{C}^{2n\times 2n}$ is called *Hamiltonian* if the matrix $\mathcal{J}M$ is Hermitian, i.e.,

$$M^{*}\mathcal{J} + \mathcal{J}M = 0,$$

and it is said to be *symplectic*[1,2] whenever

$$M^* \mathcal{J} M = \mathcal{J}. \tag{1.13}$$

The simplest examples of symplectic matrices are I_{2n} and $\mathcal{J}$. From (1.13) one easily observes that every symplectic matrix is invertible and satisfies $|\det M| = 1$ (in the case of a real symplectic matrix we have even $\det M = 1$, see e.g. [117, pg. 3] or [115, Appendix 3, Theorem 5]). In addition, M is symplectic if and only if $M^{-1} = -\mathcal{J} M^* \mathcal{J}$. Therefore the set of $2n \times 2n$ symplectic matrices over $\mathbb{C}$ forms a group with respect to the standard matrix multiplication. We also note that condition (1.13) is equivalent to

$$M \mathcal{J} M^* = \mathcal{J},$$

i.e., M is symplectic if and only if M^* is symplectic.

A *discrete symplectic system* is the first order system of recurrence relations

$$z_{k+1} = \mathcal{S}_k z_k, \tag{1.14}$$

where k belongs to some discrete interval $\mathcal{I}_{\mathbb{Z}}$ and $\mathcal{S} \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{2n \times 2n}$ with $\mathcal{S}_k$ being symplectic matrices for all $k \in \mathcal{I}_{\mathbb{Z}}$. These systems naturally arise in the discrete calculus of variations and optimal theory as Jacobi systems obtained from the weak Pontryagin maximum principle applied to the second variation of a functional, see e.g. [89–92, 94, 151]. Moreover, these systems can be found also in numerical integration schemes for Hamiltonian systems or in the theory of continued fractions, see e.g. [31, 47, 73–77, 140] and [3, Chapter 2], respectively. The origin of a systematic treatment of discrete symplectic systems goes back to [3], see also [4, 20]. However, some aspects of this theory can be observed at least 30 years earlier in [9, Section 3]. In the last two decades, the theory of discrete symplectic systems has been developed in various directions such as the Reid roundabout theorem, see e.g. [18, 24, 53, 87, 92–94, 137, 138], trigonometric and hyperbolic systems, see e.g. [5, 21, 22, 58, 59, A3], Sturmian, spectral, and oscillation theory, see e.g. [25, 50, 54–57, 62–65, 67, 68, 149].

Since the matrices $\mathcal{S}_k$ are symplectic on $\mathcal{I}_{\mathbb{Z}}$, it follows from their invertibility that any initial value problem associated with system (1.14) and with an initial condition given at an arbitrary point $k_0 \in \mathcal{I}_{\mathbb{Z}}$ possesses a unique solution $z \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$. Moreover, any fundamental matrix of system (1.14) is symplectic on $\mathcal{I}_{\mathbb{Z}}$ if and only if it is symplectic at some index $k \in \mathcal{I}_{\mathbb{Z}}$. The same property has also any fundamental matrix of the *linear Hamiltonian differential system*

$$z'(t) = H(t) z(t), \tag{1.15}$$

where t belongs to some interval $\mathcal{I}$ and $H(t)$ is a piecewise continuous Hamiltonian matrix on $\mathcal{I}$, see e.g. [115, Appendix 3, Theorem 3]. Therefore system (1.14) can be regarded as

[1] The term *symplectic* in this context was suggested by the German mathematician Hermann Klaus Hugo Weyl (1885–1955) in his book [173, pg. 165]: *The name "complex group" formerly advocated by me in allusion to line complexes, as these are defined by the vanishing of antisymmetric bilinear forms, has become more and more embarrassing through collision with the word "complex" in the connotation of complex number. I therefore propose to replace it by the corresponding Greek adjective "symplectic". Dickson calls the group the "Abelian linear group" in homage to Abel who first studied it.*

[2] A matrix $M \in \mathbb{C}^{2n \times 2n}$ satisfying equality (1.13) is also referred as *conjugate* symplectic. Moreover, if condition (1.13) is replaced by $M^\top \mathcal{J} M = \mathcal{J}$ and applied to matrices $M \in \mathbb{C}^{2n \times 2n}$ or $M \in \mathbb{R}^{2n \times 2n}$, then it is called *complex* or *real* symplectic, respectively, see e.g. [117]. Nevertheless, we suppress the adjective "conjugate", because we will consider only complex matrices and identity (1.13) throughout this thesis.

the proper discrete counterpart of system (1.15), see also [52] and Remark 1.2.1(iv) below. We note that the Hamiltonian property of the matrix $H(t)$ implies the block structure $H(t) = \left(\begin{smallmatrix} A(t) & B(t) \\ C(t) & -A^*(t) \end{smallmatrix}\right)$ for some piecewise continuous $n \times n$ matrix-valued functions $A(t)$, $B(t)$, and $C(t)$ such that $B(t) = B^*(t)$ and $C(t) = C^*(t)$ on $\mathcal{I}$. Moreover, system (1.15) can be equivalently written as

$$-\mathcal{J} z'(t) = \widetilde{H}(t)\, z(t), \tag{1.16}$$

where $\widetilde{H}(t)$ is Hermitian on $\mathcal{I}$. For completeness we remark that some authors deal rather with a matrix $\widetilde{\mathcal{J}}$ instead of $\mathcal{J}$ given in (1.12), where $\widetilde{\mathcal{J}} = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right) = -\mathcal{J}$ or more generally $\widetilde{\mathcal{J}}$ is any nonsingular $2n \times 2n$ matrix satisfying $\widetilde{\mathcal{J}}^* = -\widetilde{\mathcal{J}}$, see e.g. [101] or [9, Chapter 9]. Moreover, it is also possible to replace $\mathcal{J}$ by a $2n \times 2n$ matrix-valued function $\widetilde{\mathcal{J}}(t)$ such that $\det \widetilde{\mathcal{J}}(t) \neq 0$ and $\widetilde{\mathcal{J}}^*(t) = -\widetilde{\mathcal{J}}(t)$ for all $t \in \mathcal{I}$, see e.g. [116].

Remark 1.2.1. In order to emphasize the importance and generality of system (1.14) we show now that it includes several equations or systems, which have been intensively studied in the literature. Therefore we fix the numbers $n \in \mathbb{N}$, $N \in \mathbb{N}_0$ and, for simplicity, divide the vector z_k into two blocks of the same size and the coefficient matrix $\mathcal{S}_k$ of system (1.14) into four blocks of the same size as

$$z_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix} \quad \text{and} \quad \mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}.$$

Then the symplecticity of the matrices $\mathcal{S}_k$ and $\mathcal{S}_k^*$ is equivalent with the conditions

$$\mathcal{A}_k^* \mathcal{D}_k - \mathcal{C}_k^* \mathcal{B}_k = I = \mathcal{A}_k \mathcal{D}_k^* - \mathcal{B}_k \mathcal{C}_k^* \tag{1.17}$$

$$\text{and the matrices} \quad \mathcal{A}_k^* \mathcal{C}_k, \quad \mathcal{B}_k^* \mathcal{D}_k, \quad \mathcal{A}_k \mathcal{B}_k^*, \quad \mathcal{C}_k \mathcal{D}_k^* \quad \text{are Hermitian.} \tag{1.18}$$

Let us also note that although in parts (i)–(iii) below we consider only finite discrete intervals, the discussed equivalences remain valid (with appropriate modifications) for any type of an unbounded discrete interval.

(i) Let the number $m \in \mathbb{N}$ be fixed and $P^{[0]} \in \mathbb{C}([0,N]_{\mathbb{Z}})^{n\times n}$, $P^{[1]} \in \mathbb{C}([0,N+1]_{\mathbb{Z}})^{n\times n}, \dots,$ $P^{[m]} \in \mathbb{C}([0,N+m]_{\mathbb{Z}})^{n\times n}$ be sequences of complex-valued $n \times n$ Hermitian matrices with $\det P_k^{[m]} \neq 0$ for all $k \in [0, N+m]_{\mathbb{Z}}$. Then the *$n$-vector-valued Sturm–Liouville difference equation of order $2m$*, i.e.,

$$\sum_{s=0}^{m} (-1)^s \Delta^s \left(P_k^{[s]} \Delta^s y_{k+1-s}\right) = 0, \tag{1.19}$$

is equivalent to a discrete symplectic system of a special form. More specifically, if $y \in \mathbb{C}([1-m, N+m+1]_{\mathbb{Z}})^n$ solves equation (1.19) on $[0,N]_{\mathbb{Z}}$, then $z \in \mathbb{C}([0,N+1]_{\mathbb{Z}})^{2mn}$ with the components

$$x_k := \begin{pmatrix} y_k \\ \vdots \\ \Delta^{r-1} y_{k+1-r} \\ \vdots \\ \Delta^{m-1} y_{k+1-m} \end{pmatrix}, \quad u_k := \begin{pmatrix} \sum_{s=1}^{m} (-\Delta)^{s-1} \left(P_k^{[s]} \Delta^s y_{k+1-s}\right) \\ \vdots \\ \sum_{s=r}^{m} (-\Delta)^{s-r} \left(P_k^{[s]} \Delta^s y_{k+1-s}\right) \\ \vdots \\ P_k^{[m]} \Delta^m y_{k+1-m} \end{pmatrix} \tag{1.20}$$

solves symplectic system (1.14) on $[0,N]_{\mathbb{Z}}$, where $\mathcal{S} \in \mathbb{C}([0,N]_{\mathbb{Z}})^{2mn\times 2mn}$ with the $mn \times mn$ blocks

$$\mathcal{A}_k := \begin{pmatrix} I & I & \cdots & I \\ 0 & I & \cdots & I \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I \end{pmatrix}, \; \mathcal{D}_k := \begin{pmatrix} I & 0 & \cdots & \cdots & 0 & P_k^{[0]}\left(P_k^{[m]}\right)^{-1} \\ -I & I & 0 & \cdots & 0 & P_k^{[1]}\left(P_k^{[m]}\right)^{-1} \\ 0 & -I & I & \cdots & 0 & P_k^{[2]}\left(P_k^{[m]}\right)^{-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & -I & I + P_k^{[m-1]}\left(P_k^{[m]}\right)^{-1} \end{pmatrix}, \tag{1.21}$$

$$\mathcal{B}_k := \begin{pmatrix} 0 & \cdots & 0 & \left(P_k^{[m]}\right)^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \left(P_k^{[m]}\right)^{-1} \end{pmatrix}, \quad \mathcal{C}_k := \begin{pmatrix} P_k^{[0]} & P_k^{[0]} & \cdots & P_k^{[0]} \\ 0 & P_k^{[1]} & \cdots & P_k^{[1]} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_k^{[m-1]} \end{pmatrix}. \tag{1.22}$$

On the other hand, let $z \in \mathbb{C}([0,N+1]_{\mathbb{Z}})^{2mn}$ solve system (1.14) on $[0,N]_{\mathbb{Z}}$ with $\mathcal{S}_k$ having the same block structure as in (1.21)–(1.22) and denote the first n components of x_k as y_k for all $k \in [0,N+1]_{\mathbb{Z}}$. Then according to the transformation in (1.20) we can extend the definition of y_k to the interval $[1-m,N+m+1]_{\mathbb{Z}}$. In particular, the relation between the components of x_k and y_k applied at $k=0$ yields $y_{-1},\dots,y_{1-m}$, while the relation between the components of u_k and y_k applied at $k=N+1$, together with the invertibility of $P_k^{[m]}$ on $[N+1,N+m]_{\mathbb{Z}}$, yields $y_{N+2},\dots,y_{N+m+1}$. Then we have $y \in \mathbb{C}([1-m,N+m+1]_{\mathbb{Z}})^n$, which satisfies equation (1.19) on $[0,N]_{\mathbb{Z}}$.

(ii) From the previous part it follows that any symplectic system (1.14) with $\mathcal{A}_k \equiv I$ and $\det \mathcal{B}_k \neq 0$ on the discrete interval $[0,N]_{\mathbb{Z}}$ can be reduced to the second order Sturm–Liouville difference equation

$$-\Delta\left(P_k^{[1]}\Delta y_k\right) + P_k^{[0]} y_{k+1} = 0. \tag{1.23}$$

Indeed, if $N \geq 1$ and we put $y_k := x_k$ for all $k \in [0,N+1]_{\mathbb{Z}}$, then equality (1.23) is satisfied on $[0,N-1]_{\mathbb{Z}}$ with $P_k^{[1]} := \mathcal{B}_k^{-1}$ and $P_k^{[0]} := \mathcal{C}_k$. Moreover, equation (1.23) is a special case of the *Jacobi equation*

$$-\Delta\left(P_k\Delta y_k + R_k^* y_{k+1}\right) + Q_k y_{k+1} + R_k \Delta y_k = 0, \tag{1.24}$$

where $k \in [0,N]_{\mathbb{Z}}$, $P,R \in \mathbb{C}([0,N+1]_{\mathbb{Z}})^{n\times n}$ with matrices P_k being Hermitian on $[0,N+1]_{\mathbb{Z}}$ and $P_k + R_k^*$ invertible for all $k \in [0,N+1]_{\mathbb{Z}}$, and $Q \in \mathbb{C}([0,N]_{\mathbb{Z}})^{n\times n}$ with Q_k being Hermitian on $[0,N]_{\mathbb{Z}}$. But also equation (1.24) can be written as a discrete symplectic system and, under an additional assumption, vice versa. More precisely, if $y \in \mathbb{C}([0,N+2]_{\mathbb{Z}})^n$ solves equation (1.24), then the pair $x_k := y_k$, $k \in [0,N+2]_{\mathbb{Z}}$, and $u_k := P_k\Delta y_k + R_k^* y_{k+1}$, $k \in [0,N+1]_{\mathbb{Z}}$, solves system (1.14) on $[0,N]_{\mathbb{Z}}$, where

$$\mathcal{A}_k := (P_k + R_k^*)^{-1} P_k, \quad \mathcal{D}_k := (P_k + R_k^* + R_k + Q_k)(P_k + R_k^*)^{-1},$$
$$\mathcal{B}_k := (P_k + R_k^*)^{-1}, \quad \mathcal{C}_k := Q_k (P_k + R_k^*)^{-1} P_k - R_k (P_k + R_k^*)^{-1} R_k^*.$$

On the other hand, if $N \geq 1$ and $z \in \mathbb{C}([0,N+1]_{\mathbb{Z}})^{2n}$ solves system (1.14) with $\det \mathcal{B}_k \neq 0$ on $[0,N]_{\mathbb{Z}}$, then $y_k := x_k$, $k \in [0,N+1]_{\mathbb{Z}}$, satisfies equation (1.24) for all $k \in [0,N-1]_{\mathbb{Z}}$ with the coefficient matrices (observe that $P_k + R_k^* = \mathcal{B}_k^{-1}$)

$$P_k := \mathcal{B}_k^{-1}\mathcal{A}_k, \quad R_k := (I - \mathcal{A}_k^*)\mathcal{B}_k^{*-1}, \quad Q_k := (\mathcal{D}_k - I)\mathcal{B}_k^{-1} + (\mathcal{A}_k^* - I)\mathcal{B}_k^{*-1}.$$

(iii) Upon expanding the difference operators in (1.23) or in (1.24), we obtain special cases of the *symmetric three-term recurrence relation*, i.e.,

$$S_{k+1}\, y_{k+2} - T_{k+1}\, y_{k+1} + S_k^*\, y_k = 0, \tag{1.25}$$

where we have $k \in [0,N]_{\mathbb{Z}}$, $S \in \mathbb{C}([0,N+1]_{\mathbb{Z}})^{n\times n}$ with $\det S_k \neq 0$ on $[0,N+1]_{\mathbb{Z}}$, and $T \in \mathbb{C}([1,N+1]_{\mathbb{Z}})^{n\times n}$ with $T_k^* = T_k$ on $[1,N+1]_{\mathbb{Z}}$. In particular, equation (1.24) leads to (1.25) with $S_k = P_k + R_k^*$ and $T_k = P_k + P_{k-1} + R_{k-1}^* + R_{k-1} + Q_{k-1}$. Equation (1.25) is also equivalent to a special discrete symplectic system. Indeed, if $y \in \mathbb{C}([0,N+2]_{\mathbb{Z}})^n$ solves equation (1.25) and we put $x_k := y_k$ for $k \in [0,N+2]_{\mathbb{Z}}$ and $u_k := S_k x_{k+1}$ for $k \in [0,N+1]_{\mathbb{Z}}$, then z_k solves system (1.14) on $[0,N]_{\mathbb{Z}}$ with the $n \times n$ blocks

$$\mathcal{A}_k := 0, \quad \mathcal{B}_k := S_k^{-1}, \quad \mathcal{C}_k := -S_k^*, \quad \mathcal{D}_k := T_{k+1}\, S_k^{-1}. \tag{1.26}$$

On the other hand, if $N \geq 1$ and $z \in \mathbb{C}([0,N+1]_{\mathbb{Z}})^{2n}$ solves system (1.14) with $\det \mathcal{B}_k \neq 0$ on $[0,N]_{\mathbb{Z}}$, then $y_k := x_k$, $k \in [0,N+1]_{\mathbb{Z}}$, satisfies equation (1.25) for all $k \in [0,N-1]_{\mathbb{Z}}$ with the coefficient matrices

$$S_k := \mathcal{B}_k^{-1}, \quad T_k := \mathcal{B}_k^{-1} \mathcal{A}_k + \mathcal{D}_{k-1}\, \mathcal{B}_{k-1}^{-1}.$$

This follows immediately from the previous part and the relation between equations (1.24) and (1.25).

(iv) Another extremely important example of system (1.14) is provided by the *linear Hamiltonian difference system*

$$\Delta x_k = A_k\, x_{k+1} + B_k\, u_k, \quad \Delta u_k = C_k\, x_{k+1} - A_k^*\, u_k, \tag{1.27}$$

where k belongs to a discrete interval $\mathcal{I}_{\mathbb{Z}}$, $z \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$, and $A, B, C \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{n\times n}$ with the matrices B_k and C_k being Hermitian on $\mathcal{I}_{\mathbb{Z}}$ and the matrix $I - A_k$ invertible for all $k \in \mathcal{I}_{\mathbb{Z}}$. If we denote by the superscript $^{[s]}$ the partial shift in the first component of z_k, i.e., $z_k^{[s]} := \left(\begin{smallmatrix} x_{k+1} \\ u_k \end{smallmatrix}\right)$, then system (1.27) can be written as

$$\Delta z_k = H_k\, z_k^{[s]} \quad \text{or equivalently} \quad -\mathcal{J}\, \Delta z_k = \widetilde{H}_k\, z_k^{[s]},$$

where $H \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{2n\times 2n}$ with the matrices $H_k = \left(\begin{smallmatrix} A_k & B_k \\ C_k & -A_k^* \end{smallmatrix}\right)$ being Hamiltonian for all $k \in \mathcal{I}_{\mathbb{Z}}$, while $\widetilde{H} \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{2n\times 2n}$ with $\widetilde{H}_k = \left(\begin{smallmatrix} -C_k & A_k^* \\ A_k & B_k \end{smallmatrix}\right)$ being Hermitian on $\mathcal{I}_{\mathbb{Z}}$, compare with systems (1.15) and (1.16). This system was introduced in [69,70] as a discrete analogue of (1.15), however the invertibility of $I - A_k$ guarantees that system (1.27) can be written as symplectic system (1.14) with the coefficient matrix

$$\mathcal{S}_k := \begin{pmatrix} (I - A_k)^{-1} & (I - A_k)^{-1} B_k \\ C_k\, (I - A_k)^{-1} & I - A_k^* + C_k\, (I - A_k)^{-1} B_k \end{pmatrix}. \tag{1.28}$$

On the other hand, system (1.14) can be written as (1.27) only if $\mathcal{A}_k$ is invertible for all $k \in \mathcal{I}_{\mathbb{Z}}$, in which case

$$A_k := I - \mathcal{A}_k^{-1}, \quad B_k := \mathcal{A}_k^{-1}\, \mathcal{B}_k, \quad C_k := \mathcal{C}_k\, \mathcal{A}_k^{-1}. \tag{1.29}$$

1.3 Bibliographical notes

The equivalence described in Remark 1.2.1(i) is motivated by [4, Section 3.5], [20, Remark 2], and [111, Lemma 3]. We note that, analogously to [139, Chapter 3], it seems to be possible to write equation (1.19) as system (1.14) also without the regularity assumption for the coefficient matrix $P^{[m]}$. A solution of this problem will appear soon. The relation between system (1.14) and equations (1.24) and (1.25) from Remark 1.2.1(ii)–(iii) was discussed in [150]. In [4, Section 3.6] similar transformation of equation (1.24) into system (1.14) was derived as a consequence of a relation between equation (1.24) and system (1.27), which requires the additional assumption $\det P_k \neq 0$. The connection between systems (1.14) and (1.27) was shown in [2, Theorem 3], see also [4, Section 3.4].

A modern mathematical proof is not very different from a modern machine, or a modern test setup: the simple fundamental principles are hidden and almost invisible under a mass of technical details.

HERMANN WEYL, SEE [174, PG. 453]

Chapter 2

WEYL–TITCHMARSH THEORY FOR GENERAL LINEAR DEPENDENCE ON SPECTRAL PARAMETER

In this chapter we present the theory of square summable solutions of discrete symplectic systems in the form

$$z_{k+1}(\lambda) = \mathbb{S}_k(\lambda) z_k(\lambda) \quad \text{with} \ \ \mathbb{S}_k(\lambda) := \mathcal{S}_k + \lambda \mathcal{V}_k, \tag{$\mathcal{S}_\lambda$}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter and $\mathcal{S}_k$ and $\mathcal{V}_k$ are complex $2n\times 2n$ matrices satisfying

$$\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{S}_k^* \mathcal{J} \mathcal{V}_k \ \text{ is Hermitian,} \quad \mathcal{V}_k^* \mathcal{J} \mathcal{V}_k = 0, \quad \text{and} \quad \Psi_k := \mathcal{J} \mathcal{V}_k \mathcal{J} \mathcal{S}_k^* \mathcal{J} \geq 0. \tag{2.1}$$

Here $\mathcal{J}$ stands for the matrix defined in (1.12) but it is also possible to use its generalization as discussed at the end of the paragraph preceding Remark 1.2.1, see also Section 3.3. The indices k belong to a bounded or unbounded discrete interval as will be specified later. The dependence on λ in ($\mathcal{S}_\lambda$) is linear, but other than that quite general. The properties in (2.1) imply that the matrix $\mathcal{S}_k$ is symplectic and that the coefficient matrix $\mathbb{S}_k(\lambda)$ of ($\mathcal{S}_\lambda$) satisfies the symplectic-type identity

$$\mathbb{S}_k^*(\bar{\lambda}) \mathcal{J} \mathbb{S}_k(\lambda) = \mathcal{J} \quad \text{for all} \ \ \lambda \in \mathbb{C}. \tag{2.2}$$

The Hermitian matrix Ψ_k will play a role of a weight for the associated semi-inner product, see (2.26) and (2.54). Throughout this thesis we also use the standard convention that by ($\mathcal{S}_\nu$) we refer to the system as in ($\mathcal{S}_\lambda$) with λ replaced by ν.

Identity (2.2) shows that the matrix $\mathbb{S}_k(\lambda)$ satisfies properties, which are similar to those of symplectic matrices, see Section 1.2. This fact also motivates the above terminology "symplectic system", although system ($\mathcal{S}_\lambda$) corresponds to the discrete symplectic system as introduced in Section 1.2 only when $\lambda \in \mathbb{R}$, such as for $\lambda = 0$. On the other hand, system ($\mathcal{S}_\lambda$) can be viewed as a perturbation of the original symplectic system $z_{k+1} = \mathcal{S}_k z_k$, i.e., system ($\mathcal{S}_0$), for which the fundamental properties of symplectic systems are satisfied with appropriate (but natural) modifications. Such properties of system ($\mathcal{S}_\lambda$) are derived in Section 2.1. We note that system ($\mathcal{S}_\lambda$) with $\lambda \in \mathbb{R}$ was already investigated in [9, Sections 3.1 and 3.3], where the weight matrix Ψ_k is also obtained for this special case. We remark that Ψ_k and $\mathcal{J}$ correspond to C_n and $-J$ in [9, Formula (3.3.10)]. Particularly, in [9, Section 3.3] it was observed that

$$\mathbb{S}_k(\lambda) = \mathcal{S}_k + \lambda \mathcal{V}_k = (I + \lambda \mathcal{J} \Psi_k) \mathcal{S}_k. \tag{2.3}$$

Hence, system ($\mathcal{S}_\lambda$) is reduced to the equivalent form

$$z_{k+1}(\lambda) = (I + \lambda \mathcal{J} \Psi_k) \mathcal{S}_k z_k(\lambda) \quad \text{with} \quad \Psi_k^* = \Psi_k \quad \text{and} \quad \Psi_k \mathcal{J} \Psi_k = 0, \tag{2.4}$$

which is more convenient in some applications, e.g., when calculating the determinant $|\det \mathbb{S}_k(\lambda)| = \det(I + \lambda \mathcal{J} \Psi_k) = 1$, see Lemma 2.1.3. Note that in the scalar case $n = 1$ the properties of Ψ_k in (2.4) imply that Ψ_k is a real 2×2 matrix. Therefore, the present chapter extends this special form of Ψ_k from the scalar (and hence real) case to any even-dimensional complex case, cf. again [9, Section 3.3].

The origin of the theory of square integrable or summable solutions goes back to the paper of H. Weyl [172], where the second-order Sturm–Liouville differential equation was considered and the famous Weyl alternative was proven by using a geometrical approach[3]. Weyl's results were re-proved (by using more analytical methods) and further extended by Titchmarsh in the series of papers summarized in [165, 166]. In honor of the pioneers, this theory is usually referred as the Weyl–Titchmarsh theory. Of course, it has been developed in many directions during the last hundred years and we do not attempt to delineate all details of its long history and a considerable literature, see an outstanding overview given in [71]. Rather than that we now discuss only some crucial moments, which are closely related to the topic of this thesis. As a natural generalization of the theory for Sturm–Liouville differential equations, Atkinson initiated the study of the Weyl–Titchmarsh theory for the linear Hamiltonian differential system

$$z'(t) = [H(t) + \lambda W(t)] z(t) \quad \text{or equivalently} \quad -\mathcal{J} z'(t) = \left[\widetilde{H}(t) + \lambda \widetilde{W}(t)\right] z(t) \tag{2.5}$$

where $H(\cdot)$ and $W(\cdot)$ are suitable Hamiltonian matrix-valued functions with $-\mathcal{J} W(t)$ being positive semidefinite, see [9] and also e.g. [30, 34, 35, 39, 97–101, 108–110, 123, 141]. As far as we know, the first related results devoted to the second-order difference equations were independently given in [85, 125], see also the references mentioned in connection with equations (2.8)–(2.10) below. However the discrete Weyl–Titchmarsh theory appears to be substantially underdeveloped in contrast to the continuous time case. Surprisingly, its extension to discrete systems had not attracted almost any attention (except [9, Chapter 3]) until 2004, when the Weyl–Titchmarsh theory for the linear Hamiltonian difference system

$$\Delta \begin{pmatrix} x_k \\ u_k \end{pmatrix} = (H_k + \lambda W_k) \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix}, \quad H_k := \begin{pmatrix} A_k & B_k \\ C_k & -A_k^* \end{pmatrix}, \quad W_k := \begin{pmatrix} E_k & F_k \\ G_k & -E_k^* \end{pmatrix} \tag{2.6}$$

was established in [36] as an answer to a remark of Professor Allan Krall, see [36, pg. 152]. In (2.6) the coefficient matrices B_k, C_k and F_k, G_k are Hermitian, i.e., H_k and W_k are Hamiltonian. Furthermore, W_k is such that $-\mathcal{J} W_k \geq 0$, and the matrix $\widetilde{A}_k(\lambda) := (I - A_k - \lambda E_k)^{-1}$

[3] Yes, it is the same Hermann Weyl as in Section 1.2, see the first footnote on page 6. Hence the topic of this thesis can be seen as a connection of two (originally unrelated) concepts, which were significantly influenced by H. Weyl.

exists for all k and $\lambda \in \mathbb{C}$, which guarantees the existence of a solution of any initial value problem associated with (2.6) in the backward time. A special case of system (2.6) with $E_k \equiv 0$ (which implies that $\tilde{A}_k(\lambda) \equiv \tilde{A}_k$ is constant in λ) was independently and more intensively studied in [142], see also e.g. [10, 122, 133, 158].

As already mentioned at the beginning of Chapter 1, the study of the Weyl–Titchmarsh theory for discrete symplectic systems was initiated in [26] and [A4], where system ($\mathcal{S}_\lambda$) was considered in the special case when its first equation does not depend on λ. In this case the matrices $\mathcal{S}_k$, $\mathcal{V}_k$, and Ψ_k have necessarily the form

$$\mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \quad \mathcal{V}_k = \begin{pmatrix} 0 & 0 \\ -\mathcal{W}_k \mathcal{A}_k & -\mathcal{W}_k \mathcal{B}_k \end{pmatrix}, \quad \Psi_k = \begin{pmatrix} \mathcal{W}_k & 0 \\ 0 & 0 \end{pmatrix} \tag{2.7}$$

with $\mathcal{W}_k \in \mathbb{C}^{n\times n}$ being Hermitian and positive semidefinite for all k, see also [23, Remark 3(iii)]. The theory of discrete symplectic systems ($\mathcal{S}_\lambda$) with (2.7) has been developed in several directions. For example, the results in [25, 54, 56, 65, 66] cover the oscillation theorems, Sturmian theory, properties of finite eigenvalues, and the Rayleigh principle. Let us note that the form of $\mathcal{V}_k$ in (2.7) follows from the perturbation of the second equation in system ($\mathcal{S}_0$) by the term $\lambda \mathcal{W}_k x_{k+1}$. This approach is naturally motivated by the connection between discrete symplectic systems and any even order vector-valued Sturm–Liouville difference equation, Jacobi equation, and symmetric three-term recurrence relation discussed in Remark 1.2.1(i)–(iii). More specifically, system ($\mathcal{S}_\lambda$) with the coefficients of the form (2.7) includes equations (1.19), (1.24), and (1.25) with the term $\lambda \mathcal{W}_k y_{k+1}$ on the right-hand side, i.e., the equations

$$\sum_{s=0}^{m} (-1)^s \Delta^s \left(P_k^{[s]} \Delta^s y_{k+1-s}\right) = \lambda \mathcal{W}_k y_{k+1}, \tag{2.8}$$

$$-\Delta\left(P_k \Delta y_k + R_k^* y_{k+1}\right) + Q_k y_{k+1} + R_k \Delta y_k = \lambda \mathcal{W}_k y_{k+1}, \tag{2.9}$$

$$S_{k+1} y_{k+2} - T_{k+1} y_{k+1} + S_k^* y_k = \lambda \mathcal{W}_k y_{k+1}, \tag{2.10}$$

which were studied e.g. in [9, 11, 12, 15, 32, 33, 96, 103, 121, 147, 156, 157, 164, 170] and [105, Chapter 7], see also [39–41]. In particular, for equation (2.8) we get system ($\mathcal{S}_\lambda$) with the matrix $\mathcal{S}_k$ as in Remark 1.2.1(i) and

$$\mathcal{V}_k = -\begin{pmatrix} 0 & 0 \\ \mathcal{V}_k^{[1]} & \mathcal{V}_k^{[2]} \end{pmatrix}, \quad \mathcal{V}_k^{[1]} = \begin{pmatrix} \mathcal{W}_k & \cdots & \mathcal{W}_k \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad \mathcal{V}_k^{[2]} = \begin{pmatrix} 0 & \cdots & 0 & \mathcal{W}_k \left(P_k^{[m]}\right)^{-1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2.11}$$

which yields $\Psi_k = \operatorname{diag}\{\mathcal{W}_k, 0 \dots, 0\} \in \mathbb{C}^{mn\times mn}$. Similarly, for equation (2.8) with $m = 1$ and equations (2.9), (2.10) we obtain system ($\mathcal{S}_\lambda$) with $\Psi_k = \operatorname{diag}\{\mathcal{W}_k, 0\}$. The aim of this chapter is to present a generalization and extension of the results in [9, 26] and [A4] to the discrete symplectic systems of the form ($\mathcal{S}_\lambda$). Moreover, it turns out that theses results are more general than those in [26] and [A4] theoretically and also practically. We show (see Example 2.5.2) that the present theory applies to certain system ($\mathcal{S}_\lambda$) with (2.7), to which the results in [26] and [A4] cannot be used.

It is easy to see that there is (almost) no restriction on matrices $W(t)$ and W_k in systems (2.5) and (2.6), respectively, while the form of $\mathcal{V}_k$ in (2.7) is very special. This inconsistency represents one of our motivations for a thorough study of the discrete symplectic systems with general linear dependence on λ and their Weyl–Titchmarsh theory as presented in this chapter. However even for system ($\mathcal{S}_\lambda$) there remains one remarkable difference, which follows immediately from the third condition in (2.1): the matrix $\mathcal{V}_k$ has to be singular for all k. Moreover, it is worth noticing that there is an interesting overlap between system (2.6) with $E_k \equiv 0$ and system ($\mathcal{S}_\lambda$), see also Remark 1.2.1(iv). More precisely, system ($\mathcal{S}_\lambda$) can be written as a linear Hamiltonian difference system only if the $n \times n$ left-upper block of $\mathbb{S}_k(\lambda)$ is invertible for all $\lambda \in \mathbb{C}$. However, in this instance the dependence on λ may be nonlinear and the matrix E_k may be nonzero. On the other hand, system (2.6) can be written as system ($\mathcal{S}_\lambda$) only if $G_k(I - A_k)^{-1} F_k \equiv 0$, see also [142, Formula (2.3)]. Without this additional assumption we obtain a discrete symplectic system with a special quadratic dependence on λ. This observation motivates our study of discrete symplectic systems with polynomial and analytic dependence on λ and their Weyl–Titchmarsh theory in Chapter 5. If, in addition, also $F_k \equiv 0$ in (2.6) we get system ($\mathcal{S}_\lambda$) with the special linear dependence described in (2.7). For completeness, we note that the study of discrete *symplectic* systems has also an advantage over the approach based on the linear difference *Hamiltonian* systems in an easier unification with the continuous time theory through the calculus on time scales, see [A7, A16]. Some problems with a unification of the Weyl–Titchmarsh theory for continuous and discrete Hamiltonian systems are discussed in [6].

This chapter is organized as follows. In Section 2.1 we present the fundamental properties of system ($\mathcal{S}_\lambda$) and in Section 2.2 we study the spectral theory on a bounded interval. In the subsequent sections we focus on the Weyl–Titchmarsh theory for system ($\mathcal{S}_\lambda$). In Section 2.3 we introduce the corresponding Weyl disks and Weyl circles both in the regular and singular cases. In Section 2.4 we consider the space ℓ^2_Ψ of square summable sequences with respect to the weight Ψ_k and investigate the limit point and limit circle cases. Finally, in Section 2.5 we provide several examples illustrating our theory.

2.1 Preliminaries

First, we derive some important "symplectic" properties of the matrix $\mathbb{S}_k(\lambda)$ defined in ($\mathcal{S}_\lambda$). Observe that (2.2) and (2.3) imply that $\mathbb{S}_k(\lambda)$ and $I + \lambda \mathcal{J} \Psi_k$ are invertible with

$$\mathbb{S}_k^{-1}(\lambda) = -\mathcal{J}\, \mathbb{S}_k^*(\bar{\lambda})\, \mathcal{J}, \quad (I + \lambda \mathcal{J} \Psi_k)^{-1} = I - \lambda \mathcal{J} \Psi_k \quad \text{for all } \lambda \in \mathbb{C}. \tag{2.12}$$

From the invertibility of $\mathbb{S}_k(\lambda)$ we obtain the (global) existence and uniqueness of solutions of any initial value problem associated with system ($\mathcal{S}_\lambda$). Formula (2.12) also yields the following straightforward facts about the coefficients $\mathcal{S}_k$ and $\mathcal{V}_k$ from (2.1).

Lemma 2.1.1. *Let $n \in \mathbb{N}$ be given. For any $k \in [0, \infty)_\mathbb{Z}$ the following conditions are equivalent.*

- (i) *The matrices $\mathcal{S}_k$ and $\mathcal{V}_k$ satisfy the first three conditions in (2.1), i.e., $\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}$, $\mathcal{S}_k^* \mathcal{J} \mathcal{V}_k$ is Hermitian, and $\mathcal{V}_k^* \mathcal{J} \mathcal{V}_k = 0$.*
- (ii) *The matrix $\mathbb{S}_k(\lambda)$ in ($\mathcal{S}_\lambda$) satisfies (2.2), i.e., $\mathbb{S}_k^*(\bar{\lambda})\, \mathcal{J}\, \mathbb{S}_k(\lambda) = \mathcal{J}$ for all $\lambda \in \mathbb{C}$.*
- (iii) *The matrices $\mathcal{S}_k$ and $\mathcal{V}_k$ satisfy $\mathcal{S}_k \mathcal{J} \mathcal{S}_k^* = \mathcal{J}$ and $\mathcal{V}_k \mathcal{J} \mathcal{V}_k^* = 0$, and $\mathcal{V}_k \mathcal{J} \mathcal{S}_k^*$ is Hermitian.*
- (iv) *The matrix $\mathbb{S}_k(\lambda)$ in ($\mathcal{S}_\lambda$) satisfies $\mathbb{S}_k(\lambda)\, \mathcal{J}\, \mathbb{S}_k^*(\bar{\lambda}) = \mathcal{J}$ for all $\lambda \in \mathbb{C}$.*

Condition (iii) in Lemma 2.1.1 implies that the matrix Ψ_k is indeed Hermitian, as required in the main assumption (2.1). This shows that if $\mathcal{S}_k$ and $\mathcal{V}_k$ are any given matrices satisfying the first three properties in (2.1), then the matrix $\Psi_k := \mathcal{J} \mathcal{V}_k \mathcal{J} \mathcal{S}_k^* \mathcal{J}$ is Hermitian and, moreover, $\Psi_k \mathcal{J} \Psi_k = 0$. The latter equality in fact characterizes the matrices $\mathcal{V}_k$ for which Ψ_k has this property, i.e., the matrices $\mathcal{V}_k$ and Ψ_k determine each other. More precisely, if Ψ_k is any $2n \times 2n$ Hermitian matrix such that $\Psi_k \mathcal{J} \Psi_k = 0$, then following (2.4) we define $\mathcal{V}_k := \mathcal{J} \Psi_k \mathcal{S}_k$. It is easy to see that the second and third properties in (2.1) are in this case satisfied.

For convenience, we summarize the notation employed throughout this chapter.

Notation 2.1.2. The numbers $n \in \mathbb{N}$ and $N \in [0,\infty)_{\mathbb{Z}}$ are fixed and $\mathcal{S}, \mathcal{V}, \Psi \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ are such that (2.1) is satisfied for all $k \in [0,\infty)_{\mathbb{Z}}$.

Now we focus on the determinant of the matrix $\mathbb{S}_k(\lambda)$. Let us recall that the absolute value of the determinant of any symplectic matrix is equal to 1 as a simple consequence of the formula (1.13).

Lemma 2.1.3. *For every $\lambda \in \mathbb{C}$ and $k \in [0,\infty)_{\mathbb{Z}}$ we have $|\det \mathbb{S}_k(\lambda)| = |\det \mathcal{S}_k| = 1$.*

Proof. First, from the expression given in (2.3) we obtain $\det \mathbb{S}_k(\lambda) = \det(I + \lambda \mathcal{J} \Psi_k) \times \det \mathcal{S}_k$. Since $(\lambda \mathcal{J} \Psi_k)^2 = 0$, the matrix $\lambda \mathcal{J} \Psi_k$ is nilpotent of degree 2. Thus, by Proposition 1.1.3 with $L := \lambda \mathcal{J} \Psi_k$ and $M := I$, we get $\det(\lambda \mathcal{J} \Psi_k + I) = \det(L+M) = \det M = 1$, which implies $\det \mathbb{S}_k(\lambda) = \det \mathcal{S}_k$. Hence the statement follows from the first condition in (2.1), because

$$|\det \mathbb{S}_k(\lambda)| = |\det \mathcal{S}_k| \overset{(2.1)}{=} \det \mathcal{J} = 1. \qquad \blacksquare$$

Remark 2.1.4. In some special cases the result of Lemma 2.1.3 can be also verified directly. For example, when the dependence on λ is special as displayed in (2.7), we have by [66, pg. 1232] that

$$\mathbb{S}_k(\lambda) = \begin{pmatrix} I & 0 \\ -\lambda \mathcal{W}_k & I \end{pmatrix} \mathcal{S}_k, \quad \text{which implies} \quad \det \mathbb{S}_k(\lambda) = \det \mathcal{S}_k \;\; \text{for all } \lambda \in \mathbb{C}. \tag{2.13}$$

The following statements are direct consequences of formula (2.2) and they provide basic properties of solutions of system $(\mathcal{S}_\lambda)$ on $[0,\infty)_{\mathbb{Z}}$. Nevertheless, it is easy to see that the results remain valid (with appropriate modifications) for solutions of system $(\mathcal{S}_\lambda)$ on any discrete interval $\mathcal{I}_{\mathbb{Z}} \subseteq [0,\infty)_{\mathbb{Z}}$.

Lemma 2.1.5 (Wronskian-type identity). *Let $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If the sequences $Z(\lambda), Z(\bar{\lambda}) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times m}$ solve systems $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_{\bar{\lambda}})$ on $[0,\infty)_{\mathbb{Z}}$, respectively, then*

$$Z_k^*(\bar{\lambda}) \mathcal{J} Z_k(\lambda) = Z_0^*(\bar{\lambda}) \mathcal{J} Z_0(\lambda) \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}. \tag{2.14}$$

Proof. Identity (2.14) follows directly from $(\mathcal{S}_\lambda)$, $(\mathcal{S}_{\bar{\lambda}})$, and (2.2), because

$$Z_{k+1}^*(\bar{\lambda}) \mathcal{J} Z_{k+1}(\lambda) = Z_k^*(\bar{\lambda}) \mathbb{S}_k^*(\bar{\lambda}) \mathcal{J} \mathbb{S}_k(\lambda) Z_k(\lambda) = Z_k^*(\bar{\lambda}) \mathcal{J} Z_k(\lambda)$$

for any $k \in [0,\infty)_{\mathbb{Z}}$. $\blacksquare$

Lemma 2.1.6. *Let $\lambda \in \mathbb{C}$ and $\Phi(\lambda)$ be a fundamental matrix of system $(\mathcal{S}_\lambda)$ on $[0,\infty)_{\mathbb{Z}}$ such that*

$$\Phi_0^*(\bar{\lambda}) \mathcal{J} \Phi_0(\lambda) = \mathcal{J}. \tag{2.15}$$

Then for any $k \in [0,\infty)_{\mathbb{Z}}$ we have

$$\Phi_k^*(\bar{\lambda}) \mathcal{J} \Phi_k(\lambda) = \mathcal{J}, \quad \Phi_k^{-1}(\lambda) = -\mathcal{J} \Phi_k^*(\bar{\lambda}) \mathcal{J}, \quad \text{and} \quad \Phi_k(\lambda) \mathcal{J} \Phi_k^*(\bar{\lambda}) = \mathcal{J}. \tag{2.16}$$

Proof. The first identity in (2.16) follows from Lemma 2.1.5 and identity (2.15). The other two identities in (2.16) follow from the first one, since $\Phi_k(\lambda) \Phi_k^{-1}(\lambda) = I = \Phi_k^{-1}(\lambda) \Phi_k(\lambda)$. $\blacksquare$

One easily observes that identity (2.15) is satisfied especially when $\Phi_0(\lambda) \equiv \Phi_0$ does not depend on λ and Φ_0 is symplectic.

The matrix Ψ_k plays a key role in the Lagrange identity for system ($\mathcal{S}_\lambda$), which is one of the main tools in the whole Weyl–Titchmarsh theory, see also [9, Formula (3.7.6)].

Theorem 2.1.7 (Lagrange identity). *Let $\lambda, \nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If the sequences $Z(\lambda), Z(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times m}$ solve systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) on $[0,\infty)_{\mathbb{Z}}$, respectively, then for any $k \in [0,\infty)_{\mathbb{Z}}$ we have*

$$\Delta\Big[Z_k^*(\lambda)\,\mathcal{J}\,Z_k(\nu)\Big] = (\bar{\lambda}-\nu)\,Z_{k+1}^*(\lambda)\,\Psi_k\,Z_{k+1}(\nu), \tag{2.17}$$

$$Z_{k+1}^*(\lambda)\,\mathcal{J}\,Z_{k+1}(\nu) = Z_0^*(\lambda)\,\mathcal{J}\,Z_0(\nu) + (\bar{\lambda}-\nu)\sum_{j=0}^{k} Z_{j+1}^*(\lambda)\,\Psi_j\,Z_{j+1}(\nu). \tag{2.18}$$

Proof. Let $Z_k(\lambda)$ and $Z_k(\nu)$ satisfy ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) on $[0,\infty)_{\mathbb{Z}}$, respectively. Then

$$\begin{aligned}\Delta\Big[Z_k^*(\lambda)\,\mathcal{J}\,Z_k(\nu)\Big] &= Z_{k+1}^*(\lambda)\Big[\mathcal{J} - \mathbb{S}_k^{*-1}(\lambda)\,\mathcal{J}\,\mathbb{S}_k^{-1}(\nu)\Big]Z_{k+1}(\nu)\\ &\overset{(2.12)}{=} Z_{k+1}^*(\lambda)\Big[\mathcal{J} + \mathcal{J}\,\mathbb{S}_k(\bar{\lambda})\,\mathcal{J}\,\mathbb{S}_k^*(\bar{\nu})\,\mathcal{J}\Big]Z_{k+1}(\nu) \overset{(2.1)}{=} (\bar{\lambda}-\nu)\,Z_{k+1}^*(\lambda)\,\Psi_k\,Z_{k+1}(\nu),\end{aligned}$$

which shows identity (2.17). Equality (2.18) then follows from (2.17) by summation. ∎

2.2 Spectral theory on bounded interval

In this section we study the spectral properties of the corresponding regular eigenvalue problem with separated boundary conditions. Matrices describing these boundary conditions belong to the set

$$\Gamma := \Big\{\alpha \in \mathbb{C}^{n\times 2n} \mid \alpha\,\alpha^* = I,\ \alpha\,\mathcal{J}\,\alpha^* = 0\Big\}. \tag{2.19}$$

It is known e.g. in [A4, Remark 2.7] that for any $\alpha \in \Gamma$ the $2n\times 2n$ matrix $(\alpha^*,\ -\mathcal{J}\alpha^*)$ is unitary and symplectic and it satisfies

$$\alpha^*\alpha - \mathcal{J}\alpha^*\alpha\mathcal{J} = I, \quad \text{i.e.,} \quad \begin{pmatrix}\alpha^* & -\mathcal{J}\alpha^*\end{pmatrix}^{-1} = \begin{pmatrix}\alpha\\ \alpha\mathcal{J}\end{pmatrix}, \quad \text{and} \quad \operatorname{Ker}\alpha = \operatorname{Ran}\mathcal{J}\alpha^*. \tag{2.20}$$

For $\alpha \in \Gamma$ we denote by $\Phi(\lambda,\alpha) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ the fundamental matrix of system ($\mathcal{S}_\lambda$) determined by the initial condition $\Phi_0(\lambda,\alpha) = \big(\alpha^*,\ -\mathcal{J}\alpha^*\big)$, i.e.,

$$\Phi_{k+1}(\lambda) = (\mathcal{S}_k + \lambda\mathcal{V}_k)\,\Phi_k(\lambda), \quad k \in [0,\infty)_{\mathbb{Z}}, \quad \Phi_0(\lambda) = \begin{pmatrix}\alpha^* & -\mathcal{J}\alpha^*\end{pmatrix}, \quad \lambda \in \mathbb{C}. \tag{2.21}$$

Then the initial value $\Phi_0(\lambda,\alpha)$ is unitary, symplectic, does not depend on λ, and its inverse is $\Phi_0^{-1}(\lambda,\alpha) = \Phi_0^*(\lambda,\alpha)$. However we usually suppress the dependence on α, i.e., we write only $\Phi(\lambda)$ instead of $\Phi(\lambda,\alpha)$. In addition, we need to emphasize the two “halves” of the fundamental matrix $\Phi(\lambda)$, hence we put

$$\Phi_k(\lambda) = \begin{pmatrix}Z_k(\lambda) & \widetilde{Z}_k(\lambda)\end{pmatrix}, \tag{2.22}$$

where $Z(\lambda) = Z(\lambda,\alpha) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ and $\widetilde{Z}(\lambda) = \widetilde{Z}(\lambda,\alpha) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ are the $2n\times n$ solutions of system ($\mathcal{S}_\lambda$) satisfying the initial conditions $Z_0(\lambda) = \alpha^*$ and $\widetilde{Z}_0(\lambda) = -\mathcal{J}\alpha^*$.

Definition 2.2.1. With the fundamental matrix $\Phi(\lambda)$ and its blocks specified in (2.22), we define for $M \in \mathbb{C}^{n\times n}$ the *Weyl solution*[4] $\mathcal{X}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ of system $(\mathcal{S}_\lambda)$ as

$$\mathcal{X}_k(\lambda) = \mathcal{X}_k(\lambda,\alpha,M) := \Phi_k(\lambda)\,(I,\ M^*)^* = Z_k(\lambda) + \widetilde{Z}_k(\lambda)M, \quad k \in [0,\infty)_{\mathbb{Z}}. \tag{2.23}$$

For the following part (including the beginning of Section 2.3) we restrict our attention to the finite discrete interval $[0,N]_{\mathbb{Z}}$ with the fixed $N \in [0,\infty)_{\mathbb{Z}}$ as stated in Notation 2.1.2. Then for $\alpha,\beta \in \Gamma$ we consider the following (regular) eigenvalue problem

$$(\mathcal{S}_\lambda), \quad k \in [0,N]_{\mathbb{Z}}, \quad \lambda \in \mathbb{C}, \quad \alpha z_0(\lambda) = 0, \quad \beta z_{N+1}(\lambda) = 0. \tag{2.24}$$

Note that when $\alpha = (I,\ 0) = \beta$, the boundary conditions in (2.24) reduce to the Dirichlet boundary conditions $x_0 = 0 = x_{N+1}$. On the other hand, the periodic or antiperiodic boundary conditions $z_0(\lambda) = \pm z_{N+1}(\lambda)$ cannot be obtained through any choice of the matrices $\alpha,\beta \in \Gamma$. These particular cases are special examples of jointly varying end-points, which are investigated in Chapter 3. We recall that a number $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of problem (2.24) if, for this particular value λ, there exists a nontrivial solution $z(\lambda) \in \mathbb{C}([0,N+1]_{\mathbb{Z}})^{2n}$ of problem (2.24). In this case, the function $z(\lambda)$ is said to be the *eigenfunction* corresponding to the eigenvalue λ and the dimension of all these eigenfunctions corresponding to λ is called the *geometric multiplicity* of λ.

Moreover, we introduce the following definiteness assumption, called *Atkinson's condition*, compare with [9, Formula (3.7.10)]. Throughout this chapter we will distinguish several forms of this definiteness assumption depending on how many solutions of $(\mathcal{S}_\lambda)$ is involved. This distinction also serves as an indicator of the minimal assumptions needed in each result, compare with Hypotheses 2.3.4 and 2.3.7 below.

Hypothesis 2.2.2 (Weak Atkinson condition – finite). For any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ every column $z(\lambda)$ of the solution $\widetilde{Z}(\lambda)$ satisfies

$$\sum_{k=0}^{N} z_{k+1}^*(\lambda)\,\Psi_k\, z_{k+1}(\lambda) > 0. \tag{2.25}$$

Identity (2.18) and Hypothesis 2.2.2 imply the following characterization of the eigenvalues and eigenfunctions of problem (2.24).

Theorem 2.2.3. *Let $\alpha,\beta \in \Gamma$ be given. Then the following statements hold.*

(i) *A number $\lambda \in \mathbb{C}$ is an eigenvalue of (2.24) if and only if $\det \beta \widetilde{Z}_{N+1}(\lambda) = 0$. In this case, the eigenfunctions corresponding to the eigenvalue λ have the form $z(\lambda) = \widetilde{Z}(\lambda)\,d$ on $[0,N+1]_{\mathbb{Z}}$ with nonzero $d \in \operatorname{Ker} \beta \widetilde{Z}_{N+1}(\lambda)$. Moreover, the geometric multiplicity of λ is equal to its algebraic multiplicity, i.e., to $\dim \operatorname{Ker} \beta \widetilde{Z}_{N+1}(\lambda)$.*

(ii) *A number $\lambda \in \mathbb{C}$ is an eigenvalue of problem (2.24) if and only if $\det(-\widetilde{Z}_{N+1}(\lambda),\ \mathcal{J}\beta^*) = 0$. In this case, the algebraic and geometric multiplicities of the eigenvalue λ are equal to the value of $\dim \operatorname{Ker}(-\widetilde{Z}_{N+1}(\lambda),\ \mathcal{J}\beta^*)$.*

(iii) *Under Hypothesis 2.2.2, the eigenvalues of (2.24) are real and eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the semi-inner product*

$$\langle z,\, \tilde{z}\rangle_{\Psi,N} := \sum_{k=0}^{N} z_{k+1}^*\,\Psi_k\, \tilde{z}_{k+1}. \tag{2.26}$$

[4] The symbol $\mathcal{X}$ stands for the Greek letter Chi (/'ki:/).

Proof. The proof follows standard arguments from linear algebra about eigenvalue problems for Hermitian matrices or self-adjoint differential or difference equations. Alternatively, see the proofs of [A4, Lemmas 3.1, 2.9 and Theorem 2.11]. ■

Next we proceed by defining the Weyl–Titchmarsh $M(\lambda)$-function for problem (2.24).

Definition 2.2.4. Let $\alpha, \beta \in \Gamma$. Whenever the matrix $\beta \widetilde{Z}_k(\lambda)$ is invertible for some value $\lambda \in \mathbb{C}$ and $k \in [0, N+1]_{\mathbb{Z}}$, we define the *Weyl–Titchmarsh $M(\lambda)$-function* as the $n \times n$ matrix

$$M_k(\lambda) = M_k(\lambda, \alpha, \beta) := -[\beta \widetilde{Z}_k(\lambda)]^{-1} \beta Z_k(\lambda). \tag{2.27}$$

It follows from Theorem 2.2.3 that the $M(\lambda)$-function is well defined at $k = N+1$ for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, when Hypothesis 2.2.2 holds. Now we show the "symmetry" property of the $M(\lambda)$-function.

Lemma 2.2.5. *Let $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C}$ be given. If $k \in [0, N+1]_{\mathbb{Z}}$ is such that $M_k(\lambda)$ and $M_k(\bar{\lambda})$ exist, then*

$$M_k^*(\lambda) = M_k(\bar{\lambda}). \tag{2.28}$$

Moreover, $M_k(\cdot)$ is an analytic function in its argument λ.

Proof. Let $k \in [0, N+1]_{\mathbb{Z}}$. By the definition of $M_k(\lambda)$, the partition of $\Phi_k(\lambda)$ in (2.22), and the third formula in (2.16) we have

$$\begin{aligned} M_k^*(\lambda) - M_k(\bar{\lambda}) &= \left[\beta \widetilde{Z}_k(\bar{\lambda})\right]^{-1} \beta\, \Phi_k(\bar{\lambda})\, \mathcal{J}\, \Phi_k^*(\lambda)\, \beta^* \left[\beta \widetilde{Z}_k(\lambda)\right]^{*-1} \\ &= \left[\beta \widetilde{Z}_k(\bar{\lambda})\right]^{-1} \beta \mathcal{J} \beta^* \left[\beta \widetilde{Z}_k(\lambda)\right]^{*-1} \overset{(2.19)}{=} 0, \end{aligned}$$

which proves identity (2.28). The analytic property of $M_k(\cdot)$ follows from the fact that $Z_k(\lambda)$ and $\widetilde{Z}_k(\lambda)$, and hence $\beta Z_k(\lambda)$ and $\beta \widetilde{Z}_k(\lambda)$, are polynomials in λ. ■

Remark 2.2.6.

(i) The Weyl solution $\mathcal{X}(\lambda)$ from Definition 2.2.1 trivially satisfies the initial boundary condition $\alpha\, \mathcal{X}_0(\lambda) = I$. In addition, if $\beta \widetilde{Z}_k(\lambda)$ is invertible for some $k \in [0, N+1]_{\mathbb{Z}}$, then $\beta\, \mathcal{X}_k(\lambda) = \beta \widetilde{Z}_k(\lambda) [M - M_k(\lambda)]$. This shows that for $M = M_k(\lambda)$ we have $\beta\, \mathcal{X}_k(\lambda) = 0$. In particular, when $k = N+1$ and $M = M_{N+1}(\lambda)$, the Weyl solution $\mathcal{X}(\lambda)$ satisfies the second boundary condition in (2.24).

(ii) We also point out that the matrix $P := -\beta \mathcal{J}\, \mathcal{X}_k(\lambda) \in \mathbb{C}^{n \times n}$, where the Weyl solution $\mathcal{X}(\lambda)$ is defined by (2.23) with $M = M_k(\lambda)$, is invertible for any $\beta \in \Gamma$ and any given $k \in [0, N+1]_{\mathbb{Z}}$, which will be a very useful fact in the proof of the next theorem. Indeed, the calculation

$$n = \operatorname{rank} \begin{pmatrix} \beta \\ \beta \mathcal{J} \end{pmatrix} \mathcal{X}_k(\lambda) = \operatorname{rank} \begin{pmatrix} 0 \\ \beta \mathcal{J}\, \mathcal{X}_k(\lambda) \end{pmatrix} = \operatorname{rank} P$$

shows that P is invertible.

In the following theorem we specify the dependence of the Weyl-Titchmarsh $M(\lambda)$-function on the matrix α determining the initial boundary condition of the fundamental matrix $\Phi(\lambda) = \Phi(\lambda, \alpha)$ in (2.22). We consider the matrix $M_k(\lambda, \alpha, \beta)$ defined in (2.27) and the matrix $M_k(\lambda, \gamma, \beta)$ given also by (2.27) but with α replaced by $\gamma \in \Gamma$, i.e., $M_k(\lambda, \gamma, \beta)$ is defined through the $2n \times n$ columns of the fundamental matrix $\Phi(\lambda, \gamma)$ which satisfies $\Phi_0(\lambda, \gamma) = (\gamma^*, -\mathcal{J} \gamma^*)$. The proofs of the next theorem and its corollary follow the similar arguments as the corresponding proofs in [A4, Lemma 3.10 and Corollary 3.11]. Note that the assumptions of Theorem 2.2.7 and Corollary 2.2.8 below are in particular satisfied when $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $k = N+1$, and Hypothesis 2.2.2 holds.

Theorem 2.2.7. *Let $\beta \in \Gamma$ and $\lambda \in \mathbb{C}$. Assume that for $\alpha, \gamma \in \Gamma$ and $k \in [0, N+1]_{\mathbb{Z}}$ the matrices $M_k(\lambda,\alpha,\beta)$ and $M_k(\lambda,\gamma,\beta)$ exist. Then we have*

$$M_k(\lambda,\alpha,\beta) = [\alpha\mathcal{J}\gamma^* + \alpha\gamma^* M_k(\lambda,\gamma,\beta)]\,[\alpha\gamma^* - \alpha\mathcal{J}\gamma^* M_k(\lambda,\gamma,\beta)]^{-1}. \tag{2.29}$$

Proof. Let $\mathcal{X}(\alpha) := \mathcal{X}(\lambda,\alpha,M_k(\lambda,\alpha,\beta))$ and $\mathcal{X}(\gamma) := \mathcal{X}(\lambda,\gamma,M_k(\lambda,\gamma,\beta))$ be the Weyl solutions as in (2.23) corresponding to $M = M_k(\lambda,\alpha,\beta)$ and $M = M_k(\lambda,\gamma,\beta)$, respectively. Since $\beta\,\mathcal{X}_k(\alpha) = 0 = \beta\,\mathcal{X}_k(\gamma)$ by Remark 2.2.6(i), it follows from the third equality in (2.20) that there exist matrices $P(\alpha), P(\gamma) \in \mathbb{C}^{n\times n}$ such that $\mathcal{X}_k(\alpha) = \mathcal{J}\beta^* P(\alpha)$ and $\mathcal{X}_k(\gamma) = \mathcal{J}\beta^* P(\gamma)$. Moreover, the matrices $P(\alpha)$ and $P(\gamma)$ are invertible by Remark 2.2.6(ii), and hence

$$\mathcal{X}_k(\alpha)\,P^{-1}(\alpha) = \mathcal{J}\beta^* = \mathcal{X}_k(\gamma)\,P^{-1}(\gamma), \quad \text{i.e.,} \quad \mathcal{X}_k(\alpha) = \mathcal{X}_k(\gamma)\,P \quad \text{with } P := P^{-1}(\gamma)\,P(\alpha).$$

By the uniqueness of solutions of system ($\mathcal{S}_\lambda$), it follows $\mathcal{X}(\alpha) = \mathcal{X}(\gamma)\,P$ on $[0,N+1]_{\mathbb{Z}}$, i.e.,

$$\mathcal{X}_j(\alpha) = \mathcal{X}_j(\gamma)\,P \quad \text{for all } j \in [0,N+1]_{\mathbb{Z}}. \tag{2.30}$$

The choice $j = 0$ then yields

$$\begin{pmatrix} I \\ M_k(\lambda,\alpha,\beta) \end{pmatrix} = \Phi_0^{-1}(\lambda,\alpha)\,\Phi_0(\lambda,\gamma) \begin{pmatrix} I \\ M_k(\lambda,\gamma,\beta) \end{pmatrix} P = \begin{pmatrix} \alpha\gamma^* - \alpha\mathcal{J}\gamma^* M_k(\lambda,\gamma,\beta) \\ \alpha\mathcal{J}\gamma^* + \alpha\gamma^* M_k(\lambda,\gamma,\beta) \end{pmatrix} P. \tag{2.31}$$

The first row of the latter identity implies $P = [\alpha\gamma^* - \alpha\mathcal{J}\gamma^* M_k(\lambda,\gamma,\beta)]^{-1}$, and then the second row of (2.31) yields identity (2.29). ∎

As a consequence of (2.30) and Theorem 2.2.7 we get a formula relating the Weyl solutions corresponding to the matrices $M = M_k(\lambda,\alpha,\beta)$ and $M = M_k(\lambda,\gamma,\beta)$ and the initial conditions with $\alpha, \gamma \in \Gamma$.

Corollary 2.2.8. *Let $\beta \in \Gamma$ and $\lambda \in \mathbb{C}$. Assume that for $\alpha, \gamma \in \Gamma$ and $k \in [0,N+1]_{\mathbb{Z}}$ the matrices $M_k(\lambda,\alpha,\beta)$ and $M_k(\lambda,\gamma,\beta)$ exist. Then for all $j \in [0,N+1]_{\mathbb{Z}}$ we have*

$$\mathcal{X}_j(\lambda,\alpha,M_k(\lambda,\alpha,\beta)) = \mathcal{X}_j(\lambda,\gamma,M_k(\lambda,\gamma,\beta))\,[\alpha\gamma^* - \alpha\mathcal{J}\gamma^* M_k(\lambda,\gamma,\beta)]^{-1}.$$

2.3 Weyl disk and Weyl circle

In this section we study the properties of the Weyl disks and the Weyl circles, which are defined through the following $\mathcal{E}(M)$-function. For a given $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$ we define the matrix-valued function $\mathcal{E}_k(M) = \mathcal{E}_k(M,\lambda,\alpha) : [0,N+1]_{\mathbb{Z}} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ as

$$\mathcal{E}_k(M) := i\delta(\lambda)\,\mathcal{X}_k^*(\lambda,\alpha,M)\,\mathcal{J}\,\mathcal{X}_k(\lambda,\alpha,M). \tag{2.32}$$

In the abbreviated form we write $\mathcal{E}(M) = i\delta(\lambda)\,\mathcal{X}^*(\lambda)\,\mathcal{J}\,\mathcal{X}(\lambda)$. The matrix $\mathcal{E}_k(M)$ is Hermitian for any $k \in [0,N+1]_{\mathbb{Z}}$ and $M \in \mathbb{C}^{n\times n}$, which can be seen from the equality $(i\mathcal{J})^* = i\mathcal{J}$. Moreover, the Lagrange identity (Theorem 2.1.7) yields that

$$\mathcal{E}_k(M) = -2\delta(\lambda)\,\operatorname{im}(M) + 2\,|\operatorname{im}(\lambda)| \sum_{j=0}^{k-1} \mathcal{X}_{j+1}^*(\lambda)\,\Psi_j\,\mathcal{X}_{j+1}(\lambda), \quad k \in [0,N+1]_{\mathbb{Z}}, \tag{2.33}$$

where for $k = 0$ the sum is zero by definition.

Definition 2.3.1. Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. For all $k \in [0, N+1]_{\mathbb{Z}}$ we define the *Weyl disk* $D_k(\lambda) = D_k(\lambda, \alpha)$ and the *Weyl circle* $C_k(\lambda) = C_k(\lambda, \alpha)$, respectively, by

$$D_k(\lambda) := \left\{M \in \mathbb{C}^{n\times n} \mid \mathcal{E}_k(M) \le 0\right\}, \qquad C_k(\lambda) := \left\{M \in \mathbb{C}^{n\times n} \mid \mathcal{E}_k(M) = 0\right\}.$$

A natural question now arises concerning the elements of $D_k(\lambda)$ and $C_k(\lambda)$. For example, from (2.33) we obtain $\mathcal{E}_0(M) = -2\delta(\lambda)\,\mathrm{im}(M)$, which implies that the Weyl circle $C_0(\lambda)$ coincides with the set of all $n \times n$ Hermitian matrices, while the interior of the Weyl disk $D_0(\lambda)$ is a proper subset of the set of all invertible $n \times n$ matrices. In the following two theorems we present characterizations of matrices M lying on the Weyl circle and in the interior of the Weyl disk.

Theorem 2.3.2. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $k \in [0, N+1]_{\mathbb{Z}}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M belongs to the Weyl circle $C_k(\lambda)$ if and only if there exists $\beta \in \Gamma$ such that $\beta\,\mathcal{X}_k(\lambda) = 0$. In this case $M = M_k(\lambda)$, whenever the matrix $M_k(\lambda)$ defined in (2.27) exists.*

Proof. Assume that $M \in C_k(\lambda)$, i.e., $\mathcal{E}_k(M) = 0$. Then for the matrix $\gamma := \mathcal{X}_k^*(\lambda)\mathcal{J}$ we get $i\delta(\lambda)\gamma\,\mathcal{X}_k(\lambda) = \mathcal{E}_k(M) = 0$, which implies $\gamma\,\mathcal{X}_k(\lambda) = 0$ and also $\gamma\mathcal{J}\gamma^* = 0$. Moreover, since $\operatorname{rank}\gamma = n$, we have $\gamma\gamma^* > 0$. The matrix $\beta := (\gamma\gamma^*)^{-1/2}\gamma$ satisfies $\beta\,\mathcal{X}_k(\lambda) = 0$, $\beta\mathcal{J}\beta^* = 0$, and $\beta\beta^* = I$. Thus $\beta \in \Gamma$ as stated in the theorem.

Conversely, assume that for a given matrix $M \in \mathbb{C}^{n\times n}$ there exists $\beta \in \Gamma$ such that $\beta\,\mathcal{X}_k(\lambda) = 0$. Then $\mathcal{X}_k(\lambda) = \mathcal{J}\beta^*P$ for $P := -\beta\mathcal{J}\,\mathcal{X}_k(\lambda)$, see Remark 2.2.6(ii). It follows that $\mathcal{E}_k(M) = i\delta(\lambda)P^*\beta\mathcal{J}\beta^*P = 0$, so that $M \in C_k(\lambda)$. Finally, if $M_k(\lambda)$ exists, then $\beta\widetilde{Z}_k(\lambda)$ is invertible and $\beta Z_k(\lambda) + \beta\widetilde{Z}_k(\lambda)M = \beta\,\mathcal{X}_k(\lambda) = 0$, i.e., $M = M_k(\lambda)$. ∎

Theorem 2.3.3. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $k \in [0, N+1]_{\mathbb{Z}}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M satisfies $\mathcal{E}_k(M) < 0$ if and only if there exists $\beta \in \mathbb{C}^{n\times 2n}$ such that $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ and $\beta\,\mathcal{X}_k(\lambda) = 0$. In this case we have with such a matrix β that $M = M_k(\lambda)$, whenever the matrix $M_k(\lambda)$ exists, and β may be chosen so that $\beta\beta^* = I$.*

Proof. For $M \in \mathbb{C}^{n\times n}$ we consider the Weyl solution $\mathcal{X}(\lambda)$ given by (2.23) with $n \times n$ blocks $\varphi(\lambda)$ and $\psi(\lambda)$, i.e., $\mathcal{X}_j(\lambda) = (\varphi_j^*(\lambda),\ \psi_j^*(\lambda))^*$ for all $j \in [0, N+1]_{\mathbb{Z}}$. Assume first $\mathcal{E}_k(M) < 0$. Then the matrices $\varphi_k(\lambda)$ and $\psi_k(\lambda)$ are invertible, since for a vector $d \in \mathbb{C}^n$ such that $\varphi_k(\lambda)d = 0$ or $\psi_k(\lambda)d = 0$ we have $d^*\mathcal{E}_k(M)d = i\delta(\lambda)d^*[\varphi_k^*(\lambda)\psi_k(\lambda) - \psi_k^*(\lambda)\varphi_k(\lambda)]d = 0$, so that $\mathcal{E}_k(M) < 0$ implies $d = 0$. We put $\gamma := (I, -\varphi_k(\lambda)\psi_k^{-1}(\lambda))$ and then we have $\gamma\,\mathcal{X}_k(\lambda) = 0$ and $\mathcal{E}_k(M) = -i\delta(\lambda)\psi_k^*(\lambda)\gamma\mathcal{J}\gamma^*\psi_k(\lambda)$. Since $\mathcal{E}_k(M) < 0$ and $\psi_k(\lambda)$ is invertible, it follows that $i\delta(\lambda)\gamma\mathcal{J}\gamma^* > 0$. Finally, the matrix $\beta := (\gamma\gamma^*)^{-1/2}\gamma$ satisfies $\beta\,\mathcal{X}_k(\lambda) = 0$, $\beta\mathcal{J}\beta^* > 0$, and $\beta\beta^* = I$ as required in the theorem.

Conversely, assume that for a given matrix $M \in \mathbb{C}^{n\times n}$ there exists $\beta = (\beta_1, \beta_2) \in \mathbb{C}^{n\times 2n}$ such that $\beta\,\mathcal{X}_k(\lambda) = 0$ and $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$. Since $2i\,\mathrm{im}(\beta_1\beta_2^*) = \beta\mathcal{J}\beta^*$, we can see that the condition $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ is equivalent to $\mathrm{im}(\beta_1\beta_2^*) > 0$ for $\mathrm{im}(\lambda) < 0$ and to $\mathrm{im}(\beta_1\beta_2^*) < 0$ for $\mathrm{im}(\lambda) > 0$. In both cases, the positive or negative definiteness of $\mathrm{im}(\beta_1\beta_2^*)$ implies the invertibility of $\beta_1\beta_2^*$, and consequently the invertibility of β_1 and β_2 alone. Hence, from $\beta\,\mathcal{X}_k(\lambda) = 0$ we obtain the equality $\varphi_k(\lambda) = -\beta_1^{-1}\beta_2\psi_k(\lambda)$, and then

$$\mathcal{E}_k(M) = i\delta(\lambda)[\varphi_k^*(\lambda)\psi_k(\lambda) - \psi_k^*(\lambda)\varphi_k(\lambda)] = -i\delta(\lambda)\psi_k^*(\lambda)\beta_1^{-1}(\beta\mathcal{J}\beta^*)\beta_1^{*-1}\psi_k(\lambda). \tag{2.34}$$

If $\psi_k(\lambda)d = 0$ for some $d \in \mathbb{C}^n$, then $\varphi_k(\lambda)d = -\beta_1^{-1}\beta_2\psi_k(\lambda)d = 0$. Since $\operatorname{rank}\mathcal{X}_k(\lambda) = n$, it follows that $d = 0$, i.e., $\psi_k(\lambda)$ is invertible. Therefore, identity (2.34) and the assumption $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ imply $\mathcal{E}_k(M) < 0$. Finally, the identity $M = M_k(\lambda)$ follows with the same argument as in the final part of the proof of Theorem 2.3.2. ∎

The following result shows that the matrix $\delta(\lambda)\,\mathrm{im}(M)$ is positive semidefinite when M belongs to the Weyl disk $D_k(\lambda)$. Under an additional Atkinson-type assumption we also obtain that $\delta(\lambda)\,\mathrm{im}(M)$ is positive definite.

Hypothesis 2.3.4. For a given matrix $M \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$, each column $z(\lambda)$ of the Weyl solution $\mathcal{X}(\lambda)$ satisfies (2.25).

Theorem 2.3.5. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and $k \in [0, N+1]_{\mathbb{Z}}$. For every matrix $M \in D_k(\lambda)$ we have*

$$\delta(\lambda)\,\mathrm{im}(M) \geq |\,\mathrm{im}(\lambda)|\sum_{j=0}^{k-1} \mathcal{X}^*_{j+1}(\lambda)\,\Psi_j\,\mathcal{X}_{j+1}(\lambda) \geq 0. \tag{2.35}$$

Moreover, if $k = N+1$ and Hypothesis 2.3.4 holds, then $\delta(\lambda)\,\mathrm{im}(M) > 0$ and thus M is invertible.

Proof. For a matrix $M \in D_k(\lambda)$, inequality (2.35) follows from (2.33) via $\mathcal{E}_k(M) \leq 0$. Moreover, under Hypothesis 2.3.4 we have $\delta(\lambda)\,\mathrm{im}(M) > 0$, which yields that M is invertible. ■

Since the number $N \in [0,\infty)_{\mathbb{Z}}$ was chosen arbitrarily, see Notation 2.1.2, in the remaining part of this section we focus on the Weyl disks when k belongs to the unbounded interval $[0,\infty)_{\mathbb{Z}}$. The first result shows that the Weyl disks are nested with increasing k.

Theorem 2.3.6. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then we have $D_k(\lambda) \subseteq D_j(\lambda)$ for every $k, j \in [0,\infty)_{\mathbb{Z}}$ such that $k \geq j$.*

Proof. Let $M \in D_k(\lambda)$, i.e., $\mathcal{E}_k(M) \leq 0$. From identity (2.33) used at indices k and j and from the fact $\Psi_\ell \geq 0$ for all $\ell \in [j, k-1]_{\mathbb{Z}}$ we get $\mathcal{E}_j(M) \leq \mathcal{E}_k(M) \leq 0$. Therefore, $M \in D_j(\lambda)$. ■

Our next goal is to identify the center and the matrix radii of the Weyl disks $D_k(\lambda)$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$, see Theorem 2.3.8. First we analyze the structure of the $\mathcal{E}_k(M)$ function. From the definition of $\mathcal{E}_k(M)$ in (2.32) and from (2.23) one easily derives

$$\mathcal{E}_k(M) = \begin{pmatrix} I & M^* \end{pmatrix} \mathcal{K}_k(\lambda) \begin{pmatrix} I \\ M \end{pmatrix}, \quad \mathcal{K}_k(\lambda) := i\delta(\lambda)\,\Phi^*_k(\lambda)\,\mathcal{J}\,\Phi_k(\lambda) = \begin{pmatrix} \mathcal{F}_k(\lambda) & \mathcal{G}^*_k(\lambda) \\ \mathcal{G}_k(\lambda) & \mathcal{H}_k(\lambda) \end{pmatrix}, \tag{2.36}$$

where $\mathcal{F}_k(\lambda)$, $\mathcal{G}_k(\lambda)$, $\mathcal{H}_k(\lambda)$ are the $n \times n$ matrices

$$\mathcal{F}_k(\lambda) := i\delta(\lambda)\,Z^*_k(\lambda)\,\mathcal{J}\,Z_k(\lambda), \;\; \mathcal{G}_k(\lambda) := i\delta(\lambda)\,\widetilde{Z}^*_k(\lambda)\,\mathcal{J}\,Z_k(\lambda), \;\; \mathcal{H}_k(\lambda) := i\delta(\lambda)\,\widetilde{Z}^*_k(\lambda)\,\mathcal{J}\,\widetilde{Z}_k(\lambda).$$

Since $\mathcal{K}_k(\lambda)$ is Hermitian, it follows that $\mathcal{F}_k(\lambda)$ and $\mathcal{H}_k(\lambda)$ are also Hermitian. The Lagrange identity (Theorem 2.1.7) with $\nu = \lambda$ then implies

$$\mathcal{K}_k(\lambda) = i\delta(\lambda)\,\mathcal{J} + 2\,|\,\mathrm{im}(\lambda)|\sum_{j=0}^{k-1} \Phi^*_{j+1}(\lambda)\,\Psi_j\,\Phi_{j+1}(\lambda),$$

from which we get the formula

$$\mathcal{H}_k(\lambda) = 2\,|\,\mathrm{im}(\lambda)|\sum_{j=0}^{k-1} \widetilde{Z}^*_{j+1}(\lambda)\,\Psi_j\,\widetilde{Z}_{j+1}(\lambda). \tag{2.37}$$

Therefore, the following Atkinson-type condition is used in order to guarantee the invertibility (in fact, the positive definiteness) of $\mathcal{H}_k(\lambda)$ for large k; cf. Hypothesis 5.3.2. Note also that if $\mathcal{H}_m(\lambda)$ is invertible for some $m \in [0,\infty)_{\mathbb{Z}}$, then it is invertible for all $k \in [m,\infty)_{\mathbb{Z}}$, because the sequence of matrices $\mathcal{H}_k(\lambda)$ is nondecreasing in k as a consequence of the fourth condition in (2.1) and identity (2.37).

Hypothesis 2.3.7 (Weak Atkinson condition – infinite). There exists $N_0 \in [0,\infty)_{\mathbb{Z}}$ such that each column $z(\lambda)$ of $\widetilde{Z}(\lambda)$ satisfies inequality (2.25) with $N = N_0$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Under Hypothesis 2.3.7, the matrices $\mathcal{H}_k(\lambda)$ are positive definite (and hence invertible) for all $k \in [N_0+1,\infty)_{\mathbb{Z}}$. For these values of k it is then possible to represent $\mathcal{E}_k(M)$ as

$$\mathcal{E}_k(M) = \mathcal{F}_k(\lambda) - \mathcal{G}_k^*(\lambda)\mathcal{H}_k^{-1}(\lambda)\mathcal{G}_k(\lambda) + [\mathcal{G}_k^*(\lambda)\mathcal{H}_k^{-1}(\lambda) + M^*]\,\mathcal{H}_k(\lambda)\,[\mathcal{H}_k^{-1}(\lambda)\mathcal{G}_k(\lambda) + M],$$

see also [A4, Identity (4.11)]. By using the third identity in (2.16), it follows that the matrices $\mathcal{K}_k(\lambda)$ defined in (2.36) satisfy the symplectic-type relation

$$\begin{pmatrix} \mathcal{F}_k(\lambda)\mathcal{G}_k(\bar\lambda) - \mathcal{G}_k^*(\lambda)\mathcal{F}_k(\bar\lambda) & \mathcal{F}_k(\lambda)\mathcal{H}_k(\bar\lambda) - \mathcal{G}_k^*(\lambda)\mathcal{G}_k^*(\bar\lambda) \\ \mathcal{G}_k(\lambda)\mathcal{G}_k(\bar\lambda) - \mathcal{H}_k(\lambda)\mathcal{F}_k(\bar\lambda) & \mathcal{G}_k(\lambda)\mathcal{H}_k(\bar\lambda) - \mathcal{H}_k(\lambda)\mathcal{G}_k^*(\bar\lambda) \end{pmatrix} = \mathcal{K}_k^*(\lambda)\,\mathcal{J}\,\mathcal{K}_k(\bar\lambda) = -\mathcal{J}.$$

This implies that

$$\mathcal{G}_k^*(\lambda)\mathcal{H}_k^{-1}(\lambda)\mathcal{G}_k(\lambda) - \mathcal{F}_k(\lambda) = \mathcal{H}_k^{-1}(\bar\lambda) > 0 \quad \text{for all } k \in [N_0+1,\infty)_{\mathbb{Z}}. \tag{2.38}$$

In the following theorem we justify the terminology "disk" and "circle" for $D_k(\lambda)$ and $C_k(\lambda)$, respectively. In the scalar case $n = 1$, the sets $D_k(\lambda)$ and $C_k(\lambda)$ indeed represent a disk and a circle in the complex plane similarly as in the original paper of H. Weyl, see [172]. For this purpose, we introduce the set $\mathbb{U}$ of unitary matrices in $\mathbb{C}^{n\times n}$ and the set $\mathbb{V}$ of contractive matrices in $\mathbb{C}^{n\times n}$, i.e.,

$$\mathbb{U} := \Big\{U \in \mathbb{C}^{n\times n} \mid U^*U = I\Big\} \quad \text{and} \quad \mathbb{V} := \Big\{V \in \mathbb{C}^{n\times n} \mid V^*V \le I\Big\}. \tag{2.39}$$

Theorem 2.3.8. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.3.7 holds. Then for every $k \in [N_0+1,\infty)_{\mathbb{Z}}$ the Weyl disk and Weyl circle admit the representations*

$$D_k(\lambda) = \Big\{P_k(\lambda) + R_k(\lambda)\,V\,R_k(\bar\lambda) \mid V \in \mathbb{V}\Big\}, \tag{2.40}$$

$$C_k(\lambda) = \Big\{P_k(\lambda) + R_k(\lambda)\,U\,R_k(\bar\lambda) \mid U \in \mathbb{U}\Big\}, \tag{2.41}$$

where the matrices $P_k(\lambda)$ and $R_k(\lambda)$, $R_k(\bar\lambda)$ are defined by

$$P_k(\lambda) := -\mathcal{H}_k^{-1}(\lambda)\mathcal{G}_k(\lambda) \quad \textit{and} \quad R_k(\lambda) := \mathcal{H}_k^{-1/2}(\lambda),\ \ R_k(\bar\lambda) := \mathcal{H}_k^{-1/2}(\bar\lambda). \tag{2.42}$$

Consequently, the sets $D_k(\lambda)$ are closed and convex for every $k \in [N_0+1,\infty)_{\mathbb{Z}}$.

Proof. Let $k \in [N_0+1,\infty)_{\mathbb{Z}}$ be fixed. Identity (2.37) and Hypothesis 2.3.7 imply that the matrices $\mathcal{H} := \mathcal{H}_k(\lambda)$ and $\widetilde{\mathcal{H}} := \mathcal{H}_k(\bar\lambda)$ are positive definite, so that $P := P_k(\lambda)$, $R := R_k(\lambda)$, and $\widetilde{R} := R_k(\bar\lambda)$ are well defined. For any matrix $M \in D_k(\lambda)$ we then have

$$\begin{aligned} 0 \ge \mathcal{E}_k(M) &= \mathcal{F} - \mathcal{G}^*\mathcal{H}^{-1}\mathcal{G} + (\mathcal{G}^*\mathcal{H}^{-1} + M^*)\,\mathcal{H}\,(\mathcal{H}^{-1}\mathcal{G} + M) \\ &= -\widetilde{\mathcal{H}}^{-1} + (\mathcal{G}^*\mathcal{H}^{*-1} + M^*)\,\mathcal{H}\,(\mathcal{H}^{-1}\mathcal{G} + M) = -\widetilde{R}^2 + (M^* - P^*)\,R^{-2}\,(M-P), \end{aligned} \tag{2.43}$$

where the equality from (2.38) was used. Identity (2.43) can be also written as

$$\widetilde{R}^{-1}(M^* - P^*)\,R^{-2}\,(M-P)\,\widetilde{R}^{-1} \le I, \quad \text{i.e.,} \quad V^*V \le I \quad \text{with } V := R^{-1}(M-P)\,\widetilde{R}^{-1}. \tag{2.44}$$

Therefore, the above defined matrix V belongs to the set $\mathbb{V}$ and $M = P + RV\widetilde{R}$. This calculation can be reversed, i.e., every matrix $V \in \mathbb{V}$ gives a unique matrix $M := P + RV\widetilde{R}$, which then belongs to $D_k(\lambda)$. This leads to a bijection (even a homeomorphism) between the matrices $M \in D_k(\lambda)$ and the matrices $V \in \mathbb{V}$. Therefore, the Weyl disk $D_k(\lambda)$ has the representation $D_k(\lambda) = P + R\mathbb{V}\tilde{R}$, as it is stated in (2.40). In a similar way, the elements of the Weyl circle $C_k(\lambda)$ are in one-to-one correspondence with the matrices V given in (2.44), which in this case satisfy the relation $V^*V = I$. This means that the Weyl circle $C_k(\lambda)$ admits the representation $C_k(\lambda) = P + R\mathbb{U}\tilde{R}$ given in (2.41). Finally, the Weyl disk $D_k(\lambda)$ is closed and convex, because the set $\mathbb{V}$ has the same properties. ∎

The matrix $P_k(\lambda)$ in Theorem 2.3.8 is called the *center* of the Weyl disk or the Weyl circle, and the matrices $R_k(\lambda)$ and $R_k(\bar{\lambda})$ are called the *matrix radii* of the Weyl disk or the Weyl circle. Given (2.37), these matrices are well defined whenever Hypothesis 2.3.7 is satisfied. Moreover, the matrices $\mathcal{H}_k(\lambda)$ are nondecreasing for $k \to \infty$, so that the matrix radii $R_k(\lambda)$ and $R_k(\bar{\lambda})$ are nonincreasing as $k \to \infty$. And since $R_k(\lambda)$ and $R_k(\bar{\lambda})$ are Hermitian and positive definite for $k \in [N_0+1,\infty)_{\mathbb{Z}}$, their limits

$$R_+(\lambda) := \lim_{k\to\infty} R_k(\lambda), \qquad R_+(\bar{\lambda}) := \lim_{k\to\infty} R_k(\bar{\lambda}) \tag{2.45}$$

exist and satisfy $R_+(\lambda) \geq 0$ and $R_+(\bar{\lambda}) \geq 0$. Now we show that the limit of the center $P_k(\lambda)$ also exists. The proof of this fact relies on properties of the spectral matrix norm $\|\cdot\|_\sigma$ recalled in Section 1.1.

Theorem 2.3.9. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.3.7 holds. Then the center $P_k(\lambda)$ of the Weyl disk $D_k(\lambda)$ converges for $k \to \infty$ to a limiting matrix $P_+(\lambda) \in \mathbb{C}^{n\times n}$, i.e.,*

$$P_+(\lambda) := \lim_{k\to\infty} P_k(\lambda). \tag{2.46}$$

Proof. The proof utilizes three main tools: the representation of the Weyl disks in Theorem 2.3.8, the convergence of the matrix radii $R_k(\lambda)$ and $R_k(\bar{\lambda})$ to $R_+(\lambda)$ and $R_+(\bar{\lambda})$, and the Cauchy criterion for sequences. Let $k \geq j \in [N_0+1,\infty)_{\mathbb{Z}}$. Then $D_k(\lambda) \subseteq D_j(\lambda)$ by Theorem 2.3.6. Hence for a matrix $M \in D_k(\lambda)$ there exist by Theorem 2.3.8 (unique) matrices $V_k, V_j \in \mathbb{V}$ such that

$$M = P_k(\lambda) + R_k(\lambda)\, V_k\, R_k(\bar{\lambda}) \quad \text{and} \quad M = P_j(\lambda) + R_j(\lambda)\, V_j\, R_j(\bar{\lambda}). \tag{2.47}$$

By comparing both equalities in (2.47) we can express the matrix V_j in terms of V_k as

$$V_j = R_j^{-1}(\lambda)\,[P_k(\lambda) - P_j(\lambda) + R_k(\lambda)\, V_k\, R_k(\bar{\lambda})]\,R_j^{-1}(\bar{\lambda}). \tag{2.48}$$

The right-hand side of equation (2.48) defines a continuous mapping $T : \mathbb{V} \to \mathbb{V}$, which assigns to each matrix $V = V_k$ the matrix $T(V) = V_j$. Since the set $\mathbb{V}$ is convex and compact, the Brouwer fixed point theorem implies that the mapping T has a fixed point, i.e., there exists a matrix $V \in \mathbb{V}$ such that $T(V) = V$. Going back to equation (2.48), we get from $T(V) = V$ the expression

$$P_k(\lambda) - P_j(\lambda) = [R_j(\lambda) - R_k(\lambda)]\, V\, R_j(\bar{\lambda}) + R_k(\lambda)\, V\, [R_j(\bar{\lambda}) - R_k(\bar{\lambda})].$$

The matrices $V \in \mathbb{V}$ satisfy $\|V\|_\sigma \leq 1$, so that from the above equality we obtain

$$\|P_k(\lambda) - P_j(\lambda)\|_\sigma \leq \|R_j(\lambda) - R_k(\lambda)\|_\sigma \times \|R_j(\bar{\lambda})\|_\sigma + \|R_k(\lambda)\|_\sigma \times \|R_j(\bar{\lambda}) - R_k(\bar{\lambda})\|_\sigma. \tag{2.49}$$

Since the sequences of the matrix radii $R(\lambda), R(\bar{\lambda}) \in \mathbb{C}([N_0+1,\infty)_{\mathbb{Z}})^{n\times n}$ converge, they are bounded in the spectral norm, i.e., there exists $K > 0$ such that $\|R_\ell(\lambda)\|_\sigma \leq K$ and $\|R_\ell(\bar{\lambda})\|_\sigma \leq K$ for all $\ell \in [N_0+1,\infty)_{\mathbb{Z}}$. Choose now an arbitrary $\varepsilon > 0$. The convergence of $R_\ell(\lambda)$ and $R_\ell(\bar\lambda)$ for $\ell \to \infty$ yields the existence of an index $m \in [N_0+1,\infty)_{\mathbb{Z}}$ such that $\|R_j(\nu) - R_k(\nu)\|_\sigma < \varepsilon/(2K)$ for $\nu \in \{\lambda, \bar\lambda\}$ and for every $k \geq j \geq m$. From inequality (2.49) we then get $\|P_k(\lambda) - P_j(\lambda)\|_\sigma < \varepsilon$ for all $k \geq j \geq m$. This shows that the sequence $P(\lambda) \in \mathbb{C}([N_0+1,\infty)_{\mathbb{Z}})^{n\times n}$ is a Cauchy sequence. Hence the completeness of $\mathbb{C}^{n\times n}$ in the spectral norm implies the result. ∎

From Theorems 2.3.6 and 2.3.8 it follows that the Weyl disks $D_k(\lambda)$ are closed, convex, and nested with increasing $k \in [N_0+1,\infty)_{\mathbb{Z}}$, where the number N_0 is from Hypothesis 2.3.7. Therefore, the limit of $D_k(\lambda)$ as $k \to \infty$ exists and it is closed, convex, and nonempty.

Definition 2.3.10. Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Under Hypothesis 2.3.7, we define the *limiting Weyl disk* as the set

$$D_+(\lambda) := \lim_{k\to\infty} D_k(\lambda) = \bigcap\nolimits_{k\in[N_0+1,\infty)_{\mathbb{Z}}} D_k(\lambda)$$

The matrix $P_+(\lambda)$ defined in (2.46) and the matrices $R_+(\lambda)$ and $R_+(\bar\lambda)$ from (2.45) are called the *center* and the *matrix radii* of the limiting Weyl disk $D_+(\lambda)$.

Based on Theorem 2.3.8, the limiting Weyl disk $D_+(\lambda)$ has the following representation.

Corollary 2.3.11. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, and suppose that Hypothesis 2.3.7 holds. Then*

$$D_+(\lambda) = \left\{P_+(\lambda) + R_+(\lambda)\, V R_+(\bar\lambda) \mid V \in \mathbb{V}\right\}.$$

The next corollary shows that the Weyl solutions $\mathcal{X}(\lambda)$ corresponding to matrices $M \in D_+(\lambda)$ have finite "norms" with respect to the weight matrices Ψ_k. And conversely, the matrices M for which the corresponding Weyl solution $\mathcal{X}(\lambda)$ satisfies the estimate below belongs necessarily to the limiting Weyl disk $D_+(\lambda)$. This result illustrates the significance of the limiting Weyl disk, because it yields a lower bound for the number of linearly independent square summable solutions in Section 2.4.

Corollary 2.3.12. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, $M \in \mathbb{C}^{n\times n}$, and suppose that Hypothesis 2.3.7 holds. Then the matrix M belongs to the limiting Weyl disk $D_+(\lambda)$ if and only if*

$$\sum_{k=0}^{\infty} \mathcal{X}^*_{k+1}(\lambda)\, \Psi_k\, \mathcal{X}_{k+1}(\lambda) \leq \frac{\operatorname{im}(M)}{\operatorname{im}(\lambda)}.$$

Proof. It follows directly from Theorem 2.3.5, when it is applied at each $k \in [N_0+1,\infty)_{\mathbb{Z}}$. ∎

Under an additional assumption on the Weyl solution $\mathcal{X}(\lambda)$, compare with Hypothesis 2.3.4, we get from Theorem 2.3.5 also an information about the positive definiteness of the matrix $\delta(\lambda)\operatorname{im}(M)$.

Hypothesis 2.3.13. There exists $N_1 \in [0,\infty)_{\mathbb{Z}}$ such that for a given matrix $M \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$, each column $z(\lambda)$ of the Weyl solution $\mathcal{X}(\lambda)$ satisfies (2.25) with $N = N_1$.

Corollary 2.3.14. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, $M \in D_+(\lambda)$, and suppose that Hypotheses 2.3.7 and 2.3.13 hold. Then $\delta(\lambda)\operatorname{im}(M) > 0$ and hence M is invertible.*

Proof. The result follows from Corollary 2.3.12 and the second part of Theorem 2.3.5, when it is applied at each $k \in [N_0+1,\infty)_{\mathbb{Z}} \cap [N_1+1,\infty)_{\mathbb{Z}}$. ∎

The last result of this section describes the matrices M which lie in the "interior" of $D_+(\lambda)$. This statement requires a strengthened version of Hypothesis 2.3.13. This stronger assumption guarantees that the Weyl disks are strictly nested, i.e.,

$$D_+(\lambda) \subsetneq D_m(\lambda) \subsetneq D_k(\lambda) \quad \text{for all } m > k \text{ large enough.}$$

Hypothesis 2.3.15. There exists $N_2 \in [0,\infty)_{\mathbb{Z}}$ such that for all $m > k \in [N_2,\infty)_{\mathbb{Z}}$ and for a given matrix $M \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$, each column $z(\lambda)$ of the Weyl solution $\mathcal{X}(\lambda)$ satisfies the inequality

$$\sum_{j=k}^{m-1} z^*_{j+1}(\lambda)\,\Psi_j\, z_{j+1}(\lambda) > 0.$$

Theorem 2.3.16. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $M \in \mathbb{C}^{n\times n}$, and suppose that Hypotheses 2.3.7 and 2.3.15 hold. Then $M \in D_+(\lambda)$ if and only if $\mathcal{E}_k(M) < 0$ for all $k \in [N_0+1,\infty)_{\mathbb{Z}} \cap [N_2+1,\infty)_{\mathbb{Z}}$.*

Proof. Let us start with the condition $M \in D_+(\lambda)$, which is equivalent to $M \in D_k(\lambda)$ for all $k \in [N_0+1,\infty)_{\mathbb{Z}} \cap [N_2+1,\infty)_{\mathbb{Z}}$. With such an index k and with $m > k$ we get from the representation of $\mathcal{E}_k(M)$ and $\mathcal{E}_m(M)$ in (2.33) that

$$\mathcal{E}_k(M) = \mathcal{E}_m(M) - 2|\operatorname{im}(\lambda)| \sum_{j=k}^{m-1} \mathcal{X}^*_{j+1}(\lambda)\,\Psi_j\,\mathcal{X}_{j+1}(\lambda).$$

The second term on the right-hand side of the latter equality is positive due to Hypothesis 2.3.15, while the first term satisfies $\mathcal{E}_m(M) \le 0$. Thus, $M \in D_+(\lambda)$ is truly equivalent to $\mathcal{E}_k(M) < 0$ for all $k \in [N_0+1,\infty)_{\mathbb{Z}} \cap [N_2+1,\infty)_{\mathbb{Z}}$. ∎

Remark 2.3.17.

(i) As in [A4, Formula (4.57)], we can define a $M_+(\lambda)$-function corresponding to the matrices from the limiting Weyl disk $D_+(\lambda)$. In particular, for $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and under Hypothesis 2.3.7 we define $M_+(\lambda) \in D_+(\lambda)$ as the limit of a subsequence of the matrices $M_k(\lambda,\alpha,\beta_k) \in D_k(\lambda)$, i.e.,

$$M_+(\lambda) := \lim_{j\to\infty} M_{k_j}(\lambda,\alpha,\beta_{k_j}), \tag{2.50}$$

where $i\delta(\lambda)\beta_{k_j}\mathcal{J}\beta^*_{k_j} \ge 0$ and $\beta_{k_j}\beta^*_{k_j} = I$, see also Remark 2.4.4 below. The function $M_+(\lambda)$ defined in (2.50) is called a *half-line Weyl–Titchmarsh $M(\lambda)$-function* and it satisfies

$$M^*_+(\lambda) = M_+(\bar\lambda) \qquad \text{for all } \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

Moreover, it is analytic on $\mathbb{C}_+$ and $\mathbb{C}_-$ (and consequently it is a Herglotz function) as a limit of uniformly bounded analytic functions $M_k(\lambda)$ with λ restricted to compact subsets of the upper and/or lower half-planes of $\mathbb{C}$, see [142, Lemma 2.14].

(ii) In [26, Definition 3.4], the *matrices on the limiting Weyl circle* were defined as the elements $M \in D_+(\lambda)$ for which there exists a sequence $\{k_j\}_{j=1}^\infty$ such that $k_j \to \infty$ as $j \to \infty$ and $\lim_{j\to\infty} \mathcal{E}_{k_j}(M) = 0$. By using Corollary 2.3.12 and Theorem 2.3.2, this condition is equivalent to

$$\sum_{k=0}^{\infty} \mathcal{X}^*_{k+1}(\lambda)\,\Psi_k\,\mathcal{X}_{k+1}(\lambda) = \frac{\operatorname{im}(M)}{\operatorname{im}(\lambda)}. \tag{2.51}$$

Moreover, with the aid of the Lagrange identity (Theorem 2.1.7) we obtain for every $k \in [0,\infty)_{\mathbb{Z}}$ that

$$\mathcal{X}^*_{k+1}(\lambda)\mathcal{J}\mathcal{X}_{k+1}(\lambda) = 2i\operatorname{im}(\lambda)\left[\frac{\operatorname{im}(M)}{\operatorname{im}(\lambda)} - \sum_{j=0}^{k} \mathcal{X}^*_{j+1}(\lambda)\,\Psi_j\,\mathcal{X}_{j+1}(\lambda)\right]. \tag{2.52}$$

Hence one easily concludes that the sum on the right-hand side of (2.52) converges for $k \to \infty$ and (2.51) is satisfied if and only if $\lim_{k\to\infty} \mathcal{X}^*_{k+1}(\lambda)\mathcal{J}\mathcal{X}_{k+1}(\lambda) = 0$. The latter statement is a generalization of [26, Theorems 3.8(ii) and 3.9] and [142, Theorem 6.3] to systems ($\mathcal{S}_\lambda$) with a general linear dependence on λ.

2.4 Square summable solutions

In this section we extend the classification of system ($\mathcal{S}_\lambda$) being in the limit point and limit circle case from [A4, Section 4] to a general linear dependence on λ. In the literature there are different (but equivalent) approaches to this issue and we essentially follow [36] and [A4]. Hence we consider the linear space of weighted square summable sequences (with the weight Ψ) with entries in $\mathbb{C}^{2n}$, i.e.,

$$\ell^2_\Psi = \ell^2_\Psi[0,\infty)_\mathbb{Z} := \Big\{z \in \mathbb{C}([0,\infty)_\mathbb{Z})^{2n} \mid \|z\|_\Psi < \infty\Big\}, \tag{2.53}$$

where

$$\|z\|_\Psi := \sqrt{\langle z, z\rangle_\Psi} \quad \text{and} \quad \langle z, \tilde{z}\rangle_\Psi := \sum_{k=0}^{\infty} z^*_{k+1}\Psi_k\tilde{z}_{k+1}. \tag{2.54}$$

However we point out that ℓ^2_Ψ is never a Hilbert space, because $\|\cdot\|_\Psi$ is only a semi-norm and $\langle\cdot,\cdot\rangle_\Psi$ is only a semi-inner product as a consequence of the fourth condition in (2.1), compare with (6.60). We also denote by $\mathcal{N}(\lambda)$ the linear space of all square summable solutions of system ($\mathcal{S}_\lambda$), i.e.,

$$\mathcal{N}(\lambda) := \Big\{z \in \ell^2_\Psi \mid z \text{ solves system } (\mathcal{S}_\lambda)\Big\}.$$

Just for the sake of curiosity, we note that the space of all non-square summable solutions does not form a linear space. In the next result we show the ℓ^2_Ψ-properties of the Weyl solution $\mathcal{X}(\lambda)$ with respect to the choice of M.

Theorem 2.4.1. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. Then the columns of $\mathcal{X}(\lambda)$ form a system of linearly independent solutions of system ($\mathcal{S}_\lambda$). If in addition Hypothesis 2.3.7 holds and $M \in D_+(\lambda)$, then the columns of the Weyl solution $\mathcal{X}(\lambda) = \mathcal{X}(\lambda,\alpha,M)$ are square summable, i.e., they belong to $\mathcal{N}(\lambda)$ and so $\dim\mathcal{N}(\lambda) \geq n$.*

Proof. Let $j \in \{1,\dots,n\}$ and denote by $z^{[j]} := \mathcal{X}(\lambda)e_j$ the columns of the Weyl solution $\mathcal{X}(\lambda)$, where e_j stands for the j-th unit vector of the standard basis in $\mathbb{C}^n$. If $c_1 z^{[1]}_k + \dots + c_n z^{[n]}_k = 0$ for some $k \in [0,\infty)_\mathbb{Z}$ and $c_1,\dots,c_n \in \mathbb{C}$, then $\mathcal{X}_k(\lambda)c = 0$ with $c := (c_1,\dots,c_n)^\top$. Since $\Phi_k(\lambda)$ is invertible on $[0,\infty)_\mathbb{Z}$, it follows from (2.23) that $(I, M^*)^*c = 0$, i.e., $c = 0$ and the solutions $z^{[1]},\dots,z^{[n]}$ are linearly independent. Under Hypothesis 2.3.7 the limiting Weyl disk $D_+(\lambda)$ exists and then for $M \in D_+(\lambda)$ we have by Corollary 2.3.12 that

$$\left\|z^{[j]}\right\|^2_\Psi = \sum_{k=0}^{\infty} z^{[j]*}_{k+1}\Psi_k z^{[j]}_{k+1} \leq e^*_j\,\frac{\operatorname{im}(M)}{\operatorname{im}(\lambda)}\,e_j < \infty \quad \text{for all } j \in \{1,\dots,n\}.$$

Therefore, in this case $z^{[j]} \in \mathcal{N}(\lambda)$ for all $j \in \{1,\dots,n\}$. ∎

As a consequence of Theorem 2.4.1 we get under Hypothesis 2.3.7 the estimate

$$n \leq \dim\mathcal{N}(\lambda) \leq 2n \quad \text{for each } \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{2.55}$$

This fact motivates the following notions, which correspond to the Weyl dichotomy in the scalar case (i.e., for $n = 1$ we have either $\dim\mathcal{N}(\lambda) = 1$ or $\dim\mathcal{N}(\lambda) = 2$).

Definition 2.4.2. Let $\lambda \in \mathbb{C}$. System ($\mathcal{S}_\lambda$) is said to be in the *limit point case* if $\dim \mathcal{N}(\lambda) = n$ and to be in the *limit circle case* if $\dim \mathcal{N}(\lambda) = 2n$.

Any other case, i.e., $n + 1 \le \dim \mathcal{N}(\lambda) \le 2n - 1$, is simply called *intermediate*. The special cases introduced in Definition 2.4.2 are singled out because of their *sui generis* characteristics. In particular, in the limit point case there exists a unique $M \in \mathbb{C}^{n\times n}$ such that the corresponding Weyl solution $\mathcal{X}(\lambda)$ is square summable, while in the limit circle case the Weyl solution $\mathcal{X}(\lambda)$ is square summable for any $M \in \mathbb{C}^{n\times n}$ and this behavior is invariant with respect to λ, see Theorem 2.4.17.

In the following theorem we show that the Weyl disks $D_k(\lambda)$ collapse to a singleton as $k \to \infty$ if system ($\mathcal{S}_\lambda$) is in the limit point case. In particular, the center $P_+(\lambda)$ given by (2.46) is the only matrix belonging to $D_+(\lambda)$. This justifies the above terminology of being in the *limit point case* for system ($\mathcal{S}_\lambda$) with $\dim \mathcal{N}(\lambda) = n$. For this particular situation we show in the proof below that the columns of $\widetilde{Z}(\lambda)$ do not belong to $\mathcal{N}(\lambda)$. This result is a generalization of [A4, Lemma 4.11].

Theorem 2.4.3. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.3.7 holds. System ($\mathcal{S}_\lambda$) is in the limit point case if and only if the limiting matrix radius $R_+(\lambda) = 0$. In this case the limiting Weyl disk satisfies $D_+(\lambda) = \{P_+(\lambda)\}$ and $D_+(\bar\lambda) = \{P_+(\bar\lambda)\}$.*

Proof. Assume that system ($\mathcal{S}_\lambda$) is in the limit point case, i.e., $\dim \mathcal{N}(\lambda) = n$. Since the columns of the fundamental matrix $\Phi(\lambda)$ introduced in (2.22) span all solutions of system ($\mathcal{S}_\lambda$), the definition of $\mathcal{X}(\lambda)$ as $Z(\lambda) + \widetilde{Z}(\lambda)M$ with $M \in D_+(\lambda)$ implies that the columns of $\widetilde{Z}(\lambda)$ and $\mathcal{X}(\lambda)$ also form a basis of all solutions of system ($\mathcal{S}_\lambda$). Hence, from $\dim \mathcal{N}(\lambda) = n$ and Theorem 2.4.1 we conclude that the columns of $\widetilde{Z}(\lambda)$ do not belong to $\mathcal{N}(\lambda)$. It then follows from formula (2.37) that the matrix $\mathcal{H}_k(\lambda)$ is nondecreasing for $k \in [0,\infty)_{\mathbb{Z}}$ without any upper bound, i.e., its eigenvalues (being real) tend to ∞. Therefore, the function $R_k(\lambda)$ has its limit at ∞ equal to zero, i.e., $R_+(\lambda) = 0$. This argument can be reversed, i.e., if $R_+(\lambda) = 0$, then the eigenvalues of $\mathcal{H}_k(\lambda)$ tend to ∞ and formula (2.37) yields that the columns of $\widetilde{Z}(\lambda)$ do not belong to ℓ^2_Ψ. Since the columns of $Z(\lambda)$ and $\widetilde{Z}(\lambda)$ form a basis of all solutions of ($\mathcal{S}_\lambda$), it then follows that $\dim \mathcal{N}(\lambda) \le n$. But since at the same time $\dim \mathcal{N}(\lambda) \ge n$ by Theorem 2.4.1, we obtain $\dim \mathcal{N}(\lambda) = n$ and system ($\mathcal{S}_\lambda$) is in the limit point case. Finally, if $R_+(\lambda) = 0$ (or, equivalently, system ($\mathcal{S}_\lambda$) is in the limit point case), then the equality $D_+(\lambda) = \{P_+(\lambda)\}$ follows from Corollary 2.3.11. At the same time we get from Corollary 2.3.11 that $D_+(\bar\lambda) = \{P_+(\bar\lambda)\}$. ∎

Remark 2.4.4.

(i) In the continuous time setting, i.e., for linear Hamiltonian differential systems or Sturm–Liouville differential equations, it can happen that the limiting Weyl disk $D_+(\lambda)$ is a singleton consisting, of course, of the limiting center $P_+(\lambda)$, but the corresponding limiting matrix radius $R_+(\lambda)$ is not the zero matrix, i.e, it satisfies $\operatorname{rank} R_+(\lambda) \ge 1$. Such an example is constructed in [120] for the fourth order Sturm–Liouville differential equation. Although the same behavior can be expected also in the discrete case, a specific example is still missing, see also Remark 2.4.21.

(ii) In addition, Theorem 2.4.3 gives a simpler characterization of the half-line Weyl–Titchmarsh $M(\lambda)$-function $M_+(\lambda)$ from (2.50). In particular, in the limit point case the limit in (2.50) can be taken over all $k \in [N_0 + 1, \infty)_{\mathbb{Z}}$ without going to subsequences and also with $\beta_k \equiv \beta \in \Gamma$. That is, we have in this case

$$M_+(\lambda) = \lim_{k\to\infty} M_k(\lambda, \alpha, \beta).$$

In the next part of this section we extend the result in Theorem 2.4.3 in a way which includes the precise effect of the matrix radii $R_+(\lambda)$ and $R_+(\bar{\lambda})$ on the number of linearly independent square summable solutions of system ($\mathcal{S}_\lambda$). Specifically, we study the relationship between the number

$$r(\lambda) := \operatorname{rank} R_+(\lambda), \quad \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{2.56}$$

and the dimension of $\mathcal{N}(\lambda)$. The statements in Theorem 2.4.5–Corollary 2.4.9 extend the results in [142, Section 4] from special linear Hamiltonian difference systems to discrete symplectic systems. In addition, these results were established as new even for system ($\mathcal{S}_\lambda$) with the special linear dependence on λ in (2.7). At the same time, they can be regarded as discrete time analogues of the corresponding results for linear Hamiltonian differential systems in [108, Section 5] and [141].

From the definition of $R_+(\lambda)$ in (2.45) and from (2.42) one can see that the value of $r(\lambda)$ depends also on $\alpha \in \Gamma$, i.e., we should write $r(\lambda, \alpha)$ instead of $r(\lambda)$ in (2.56). But, obviously, the number of linearly independent square summable solutions of system ($\mathcal{S}_\lambda$) does not depend on the choice $\alpha \in \Gamma$ and we will see in Theorem 2.4.8 below that also $r(\lambda, \alpha)$ is independent of α, so that the notation in (2.56) is justified.

Theorem 2.4.5. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.3.7 holds. Then system ($\mathcal{S}_\lambda$) has at least $m := n + \min\{r(\lambda), r(\bar{\lambda})\}$ linearly independent square summable solutions, i.e., we have $\dim \mathcal{N}(\lambda) \geq m$.*

Proof. Consider the Weyl solution $\widetilde{\mathcal{X}}(\lambda)$ defined through the matrix $M = P_+(\lambda)$, i.e.,

$$\widetilde{\mathcal{X}}_k(\lambda) := \Phi_k(\lambda) \begin{pmatrix} I \\ P_+(\lambda) \end{pmatrix} = \left(\tilde{z}_k^{[1]}, \ \ldots, \ \tilde{z}_k^{[n]}\right) \quad \text{for all } k \in [0, \infty)_{\mathbb{Z}},$$

where $\tilde{z}^{[j]} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ for $j \in \{1, \ldots, n\}$. Then, by Theorem 2.4.1, the sequences $\tilde{z}^{[1]}, \ldots, \tilde{z}^{[n]}$ represent n linearly independent square summable solutions of system ($\mathcal{S}_\lambda$). We also consider the Weyl solution $\widehat{\mathcal{X}}(\lambda)$ defined by the matrix $M = P_+(\lambda) + R_+(\lambda)\, U R_+(\bar{\lambda})$, i.e.,

$$\widehat{\mathcal{X}}_k(\lambda) := \Phi_k(\lambda) \begin{pmatrix} I \\ M \end{pmatrix} = \left(\hat{z}_k^{[1]}, \ \ldots, \ \hat{z}_k^{[n]}\right) \quad \text{for all } k \in [0, \infty)_{\mathbb{Z}},$$

where $U \in \mathbb{U}$ is such that $\operatorname{rank} R_+(\lambda)\, U R_+(\bar{\lambda}) = \min\{r(\lambda), r(\bar{\lambda})\}$, see Proposition 1.1.2. It follows from Theorem 2.4.1 that $\hat{z}^{[1]}, \ldots, \hat{z}^{[n]}$ are also square summable solutions of ($\mathcal{S}_\lambda$), because by Corollary 2.3.11 the above matrix M belongs to the limiting disk $D_+(\lambda)$. If we put $z^{[j]} := \tilde{z}^{[j]}$ and $z^{[n+j]} := \hat{z}^{[j]}$ for $j \in \{1, \ldots, n\}$, then

$$\left(z_k^{[1]}, \ \ldots, \ z_k^{[2n]}\right) = \Phi_k(\lambda) \begin{pmatrix} I & I \\ P_+(\lambda) & M \end{pmatrix} = \Phi_k(\lambda) \begin{pmatrix} I & 0 \\ P_+(\lambda) & R_+(\lambda)\, U R_+(\bar{\lambda}) \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

for all $k \in [0, \infty)_{\mathbb{Z}}$. Since the rank of the middle matrix on the right-hand side above is equal to m and the other two matrices are invertible, we obtain that $\operatorname{rank}\left(z_k^{[1]}, \ldots, z_k^{[2n]}\right) = m$ as well, from which the statement follows. ∎

Before we present a precise relationship between $r(\lambda)$ and $\dim \mathcal{N}(\lambda)$ in Theorem 2.4.8 below, we proceed with some preliminary results. In the following theorem we establish a connection between the value of $r(\lambda)$ and the asymptotic behavior of the eigenvalues of the matrix $\mathcal{H}_k(\lambda)$ as $k \to \infty$. For a given $\lambda \in \mathbb{C}\setminus\mathbb{R}$ we denote by $\mu_k^{[1]} \leq \cdots \leq \mu_k^{[n]}$ the eigenvalues of the positive semidefinite matrix $\mathcal{H}_k(\lambda)$ arranged in the nondecreasing order (suppressing the argument λ).

Theorem 2.4.6. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.3.7 holds. Then for $\ell \in \{1,\dots,n\}$ we have $r(\lambda) = \ell$ if and only if*

$$0 < \lim_{k\to\infty} \mu_k^{[j]} =: \rho^{[j]} < \infty, \quad 1 \le j \le \ell, \quad \text{and} \quad \lim_{k\to\infty} \mu_k^{[j]} = \infty, \quad \ell+1 \le j \le n.$$

Moreover, the numbers $\left(\rho^{[1]}\right)^{-1/2}, \dots, \left(\rho^{[\ell]}\right)^{-1/2}$ are the positive eigenvalues of $R_+(\lambda)$.

Proof. Let $k \in [N_0+1,\infty)_{\mathbb{Z}}$, where N_0 is from Hypothesis 2.3.7. Since $\mathcal{H}_k(\lambda)$ is Hermitian, there exists a unitary matrix U_k such that

$$U_k^* \mathcal{H}_k(\lambda)\, U_k = \operatorname{diag}\left\{\mu_k^{[1]}, \dots, \mu_k^{[n]}\right\}. \tag{2.57}$$

From the definition of $R_k(\lambda)$ in (2.42) we have

$$U_k^* R_k(\lambda)\, U_k = \operatorname{diag}\left\{\left(\mu_k^{[1]}\right)^{-1/2}, \dots, \left(\mu_k^{[n]}\right)^{-1/2}\right\}, \tag{2.58}$$

so that $\left(\mu_k^{[1]}\right)^{-1/2}, \dots, \left(\mu_k^{[n]}\right)^{-1/2}$ are all the eigenvalues of $R_k(\lambda)$. Since the set $\mathbb{U}$ of unitary matrices is compact, there exists a subsequence $\{U_{k_j}\}_{j=1}^{\infty}$ which converges as $j \to \infty$ to a unitary matrix U_+, see also Proposition 1.1.1. Hence, from (2.58) and (2.45) we get

$$U_+^* R_+(\lambda)\, U_+ = \operatorname{diag}\left\{\lim_{j\to\infty} \left(\mu_{k_j}^{[1]}\right)^{-1/2}, \dots, \lim_{j\to\infty} \left(\mu_{k_j}^{[n]}\right)^{-1/2}\right\}.$$

This implies that $r(\lambda) = \ell$ if and only if the limits

$$\lim_{j\to\infty} \mu_{k_j}^{[1]} = \rho^{[1]}, \; \dots, \; \lim_{j\to\infty} \mu_{k_j}^{[\ell]} = \rho^{[\ell]} \quad \text{and} \quad \lim_{j\to\infty} \mu_{k_j}^{[\ell+1]} = \cdots = \lim_{j\to\infty} \mu_{k_j}^{[n]} = \infty,$$

where $\rho^{[1]}, \dots, \rho^{[\ell]}$ are finite and positive. Therefore $\left(\rho^{[1]}\right)^{-1/2}, \dots, \left(\rho^{[\ell]}\right)^{-1/2}$ are the positive eigenvalues of $R_+(\lambda)$, while the remaining $n-\ell$ eigenvalues of $R_+(\lambda)$ are zero. ∎

The following lemma will be utilized in the first part of the proof of the subsequent Theorem 2.4.8, which is the main result of this section.

Lemma 2.4.7. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $q \in \{0,1,\dots,n\}$, and suppose that Hypothesis 2.3.7 holds. System ($\mathcal{S}_\lambda$) has exactly $n+q$ linearly independent square summable solutions if and only if there exists an $n \times q$ matrix Q with $\operatorname{rank} Q = q$ such that the columns of $\widetilde{Z}(\lambda) Q$ belong to ℓ^2_Ψ, and $\widetilde{Z}(\lambda)\eta \in \ell^2_\Psi$ implies $\eta \in \operatorname{Ran} Q$.*

Proof. Let us assume that system ($\mathcal{S}_\lambda$) has $n+q$ linearly independent square summable solutions $z^{[1]}, \dots, z^{[n+q]}$. By Theorem 2.4.1, these solutions can be ordered so that the first n solutions $z^{[1]}, \dots, z^{[n]}$ correspond to the columns of the Weyl solution, which is defined through the center $P_+(\lambda)$, i.e.,

$$\widetilde{\mathcal{X}}_k(\lambda) := \Phi_k(\lambda) \begin{pmatrix} I \\ P_+(\lambda) \end{pmatrix} = \left(z_k^{[1]}, \; \dots, \; z_k^{[n]}\right) \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}. \tag{2.59}$$

Then there exists a constant $2n \times q$ matrix K such that $\operatorname{rank} K = q$ and

$$\left(z_k^{[n+1]}, \; \dots, \; z_k^{[n+q]}\right) = \Phi_k(\lambda) K \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}.$$

If we write $K = (K_1^*,\ K_2^*)^*$ with $n \times q$ blocks K_1, K_2, then we obtain for all $k \in [0, \infty)_{\mathbb{Z}}$ that

$$\left(z_k^{[1]}, \ldots, z_k^{[n+q]}\right) = \Phi_k(\lambda) \begin{pmatrix} I & K_1 \\ P_+(\lambda) & K_2 \end{pmatrix} \quad \text{and} \quad \operatorname{rank} \begin{pmatrix} I & K_1 \\ P_+(\lambda) & K_2 \end{pmatrix} = n + q, \tag{2.60}$$

because $\Phi_k(\lambda)$ is invertible on $[0, \infty)_{\mathbb{Z}}$. If we put $Q := K_2 - P_+(\lambda) K_1 \in \mathbb{C}^{n \times q}$, then

$$\begin{pmatrix} I_n & K_1 \\ P_+(\lambda) & K_2 \end{pmatrix} \begin{pmatrix} I_n & -K_1 \\ 0 & I_q \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ P_+(\lambda) & Q \end{pmatrix}. \tag{2.61}$$

It now follows from (2.60) that $\operatorname{rank} Q = q$. In addition, from the first equality in (2.60), (2.61), and (2.22) we get

$$\left(z_k^{[1]}, \ldots, z_k^{[n+q]}\right) \begin{pmatrix} I_n & -K_1 \\ 0 & I_q \end{pmatrix} = \Phi_k(\lambda) \begin{pmatrix} I_n & 0 \\ P_+(\lambda) & Q \end{pmatrix} = \left(\widetilde{\mathcal{X}}_k(\lambda) \quad \widetilde{Z}_k(\lambda) Q\right),$$

which implies that the columns of $\widetilde{Z}(\lambda) Q$ belong to ℓ^2_Ψ. Hence $(\widetilde{\mathcal{X}}_k(\lambda),\ \widetilde{Z}_k(\lambda) Q)$ consists of $n+q$ linearly independent square summable solutions. Finally, if we have $\widetilde{Z}(\lambda)\eta \in \ell^2_\Psi$ for some $\eta \in \mathbb{C}^n$, then by the previous part there exists $\xi = (\xi_1^*, \xi_2^*)^* \in \mathbb{C}^{n+q}$ such that

$$\widetilde{Z}_k(\lambda)\eta = \left(\widetilde{\mathcal{X}}_k(\lambda) \quad \widetilde{Z}_k(\lambda) Q\right) \xi \overset{(2.59)}{=} Z_k(\lambda)\xi_1 + \widetilde{Z}_k(\lambda)[P_+(\lambda)\xi_1 + Q\xi_2].$$

Since $\Phi_k(\lambda) = (Z_k(\lambda),\ \widetilde{Z}_k(\lambda))$, the above equality can be written as

$$\Phi_k(\lambda) \begin{pmatrix} \xi_1 \\ P_+(\lambda)\xi_1 + Q\xi_2 - \eta \end{pmatrix} = 0,$$

from which we get $\xi_1 = 0$ and $P_+(\lambda)\xi_1 + Q\xi_2 - \eta = 0$, i.e., $\eta = Q\xi_2 \in \operatorname{Ran} Q$ as required.

Conversely, assume that there exists a matrix $Q \in \mathbb{C}^{n\times q}$ with $\operatorname{rank} Q = q$ such that the columns of $\widetilde{Z}(\lambda) Q$ belong to ℓ^2_Ψ and that $\eta \in \operatorname{Ran} Q$ whenever $\widetilde{Z}(\lambda)\eta \in \ell^2_\Psi$. Let $\widetilde{\mathcal{X}}(\lambda)$ be as in (2.59). Then the equality

$$T_k(\lambda) := \left(\widetilde{\mathcal{X}}_k(\lambda) \quad \widetilde{Z}_k(\lambda) Q\right) = \Phi_k(\lambda) \begin{pmatrix} I_n & 0 \\ P_+(\lambda) & Q \end{pmatrix}$$

implies that $\operatorname{rank} T_k(\lambda) = n + q$ for all $k \in [0, \infty)_{\mathbb{Z}}$. This shows that system $(\mathcal{S}_\lambda)$ has at least $n+q$ linearly independent square summable solutions, namely these are the columns of $T_k(\lambda)$. Let $z \in \mathcal{N}(\lambda)$ be arbitrary. We show that z_k is a linear combination of the columns of $T_k(\lambda)$, i.e., we prove that $z_k \in \operatorname{Ran} T_k(\lambda)$ for all $k \in [0, \infty)_{\mathbb{Z}}$. Since the matrix

$$\left(\widetilde{\mathcal{X}}_k(\lambda) \quad \widetilde{Z}_k(\lambda)\right) = \Phi_k(\lambda) \begin{pmatrix} I & 0 \\ P_+(\lambda) & I \end{pmatrix}$$

is also a fundamental matrix of system $(\mathcal{S}_\lambda)$, there exists $\zeta = (\zeta_1^*, \zeta_2^*)^* \in \mathbb{C}^{2n}$ such that

$$z_k = \left(\widetilde{\mathcal{X}}_k(\lambda) \quad \widetilde{Z}_k(\lambda)\right) \zeta = \widetilde{\mathcal{X}}_k(\lambda)\zeta_1 + \widetilde{Z}_k(\lambda)\zeta_2 \quad \text{for all } k \in [0, \infty)_{\mathbb{Z}}.$$

This implies that $\widetilde{Z}(\lambda)\zeta_2 = z - \widetilde{\mathcal{X}}(\lambda)\zeta_1 \in \mathcal{N}(\lambda)$. Thus, by the current assumption, the vector $\zeta_2 \in \operatorname{Ran} Q$, i.e., $\zeta_2 = Qv$ for some vector $v \in \mathbb{C}^q$. It then follows that

$$z_k = \widetilde{\mathcal{X}}_k(\lambda)\zeta_1 + \widetilde{Z}_k(\lambda) Q v = \left(\widetilde{\mathcal{X}}_k(\lambda) \quad \widetilde{Z}_k(\lambda) Q\right) \begin{pmatrix} \zeta_1 \\ v \end{pmatrix} \in \operatorname{Ran} T_k(\lambda) \quad \text{for all } k \in [0, \infty)_{\mathbb{Z}}.$$

Therefore, system $(\mathcal{S}_\lambda)$ has exactly $n+q$ linearly independent square summable solutions. ■

Now we can give an exact relation between the number of linearly independent square summable solutions of system ($\mathcal{S}_\lambda$) and the rank of the limiting matrix radius $R_+(\lambda)$ of the limiting Weyl disk. The result below extends and makes more precise the statements in Theorems 2.4.1 and 2.4.5.

Theorem 2.4.8. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\backslash\mathbb{R}$ be given, suppose that Hypothesis 2.3.7 holds, and define $r(\lambda)$ by (2.56). Then system ($\mathcal{S}_\lambda$) has exactly $n+r(\lambda)$ linearly independent square summable solutions, i.e., $\dim \mathcal{N}(\lambda) = n+r(\lambda)$. Furthermore, the number $r(\lambda)$ is independent of the coefficient matrix α determining the initial boundary condition in (2.24).*

Proof. Since $\dim \mathcal{N}(\lambda)$, i.e., the number of square summable solutions of system ($\mathcal{S}_\lambda$), does not depend on the choice of α, the number $r(\lambda)$ also does not depend on α. Similarly as in [141, 142], the proof is divided into two parts. In the first part we derive the estimate $\dim \mathcal{N}(\lambda) \leq n + r(\lambda)$, while the opposite inequality will be given in the second part of the proof. We abbreviate $r := r(\lambda)$.

Assume that there exists a number q with $r < q \leq n$ such that system ($\mathcal{S}_\lambda$) has exactly $n+q$ square summable solutions. By Lemma 2.4.7, there exists a constant $n \times q$ matrix Q with $\operatorname{rank} Q = q$ such that the columns of $\widetilde{Z}(\lambda) Q$ belong to ℓ^2_Ψ and $\eta \in \operatorname{Ran} Q$ whenever $\widetilde{Z}(\lambda)\eta \in \ell^2_\Psi$. Using (2.57), for every $k \in [N_0+1, \infty)_\mathbb{Z}$ there is a unitary matrix U_k such that

$$\mathcal{H}_k(\lambda) = U_k \operatorname{diag}\left\{\mu_k^{[1]}, \dots, \mu_k^{[n]}\right\} U_k^*. \tag{2.62}$$

By Proposition 1.1.1, there exists a subsequence $\{k_j\}_{j=1}^\infty$ such that $k_j \to \infty$ as $j \to \infty$ and $U_{k_j} \to U_+$, where U_+ is unitary, see also the proof of Theorem 2.4.6. If we put $K_k := U_k^* Q = \left(K_k^{[1]*},\, K_k^{[2]*}\right)^* \in \mathbb{C}^{n\times q}$ with $K_k^{[1]} \in \mathbb{C}^{r\times q}$ and $K_k^{[2]} \in \mathbb{C}^{(n-r)\times q}$, then

$$K = \left(K^{[1]*},\, K^{[2]*}\right)^* := \lim_{j\to\infty} K_{k_j} = U_+^* Q \quad \text{with} \quad K^{[1]} := \lim_{j\to\infty} K_{k_j}^{[1]} \quad \text{and} \quad K^{[2]} := \lim_{j\to\infty} K_{k_j}^{[2]}.$$

It can be easily seen that $\operatorname{rank} K = q$, because U_+ is unitary and $\operatorname{rank} Q = q$. Moreover, since $q > r$, it follows that $\operatorname{rank} K^{[1]} \leq r$ and $\operatorname{rank} K^{[2]} \geq 1$. Hence there exists $\xi \in \mathbb{C}^q$ such that $K^{[2]}\xi \neq 0$, and then for $z_k := \widetilde{Z}_k(\lambda) Q \xi$ on $[0,\infty)_\mathbb{Z}$ we have $z \in \ell^2_\Psi$ by Lemma 2.4.7. On the other hand, from (2.37), (2.62), and the above definition of K_k we get

$$\begin{aligned}
\|z\|^2_\Psi &= \lim_{k\to\infty} \xi^* Q^* \left(\sum_{j=0}^{k-1} \widetilde{Z}^*_{j+1}(\lambda)\, \Psi_j\, \widetilde{Z}_{j+1}(\lambda)\right) Q \xi \overset{(2.37)}{=} \frac{1}{2|\operatorname{im}(\lambda)|} \lim_{k\to\infty} \xi^* Q^* \mathcal{H}_k(\lambda) Q \xi \\
&\overset{(2.62)}{=} \frac{1}{2|\operatorname{im}(\lambda)|} \lim_{k\to\infty} \xi^* Q^* U_k \operatorname{diag}\left\{\mu_k^{[1]}, \dots, \mu_k^{[n]}\right\} U_k^* Q \xi \\
&= \frac{1}{2|\operatorname{im}(\lambda)|} \lim_{k\to\infty} \xi^* \left(K_k^{[1]*} \operatorname{diag}\left\{\mu_k^{[1]}, \dots, \mu_k^{[r]}\right\} K_k^{[1]} + K_k^{[2]*} \operatorname{diag}\left\{\mu_k^{[r+1]}, \dots, \mu_k^{[n]}\right\} K_k^{[2]}\right) \xi.
\end{aligned}$$

By Theorem 2.4.6 (with $\ell = r$) we know that the eigenvalues $\mu_k^{[1]}, \dots, \mu_k^{[r]}$ have finite limits as $k \to \infty$, denoted by $\rho^{[1]}, \dots, \rho^{[r]}$, while the eigenvalues $\mu_k^{[r+1]}, \dots, \mu_k^{[n]}$ tend to ∞. This implies that

$$\lim_{j\to\infty} \xi^* K_{k_j}^{[1]*} \operatorname{diag}\left\{\mu_{k_j}^{[1]}, \dots, \mu_{k_j}^{[r]}\right\} K_{k_j}^{[1]} \xi = \xi^* K^{[1]*} \operatorname{diag}\left\{\rho^{[1]}, \dots, \rho^{[r]}\right\} K^{[1]} \xi < \infty,$$
$$\lim_{j\to\infty} \xi^* K_{k_j}^{[2]*} \operatorname{diag}\left\{\mu_{k_j}^{[r+1]}, \dots, \mu_{k_j}^{[n]}\right\} K_{k_j}^{[2]} \xi = \infty,$$

because $K_{k_j}^{[2]}\xi \to K^{[2]}\xi \neq 0$ as $j \to \infty$. This shows that $\|z\|_\Psi^2 = \infty$, which contradicts $z \in \ell_\Psi^2$. Thus, system ($\mathcal{S}_\lambda$) has at most $n+r$ linearly independent square summable solutions.

Conversely, we will show that $\dim \mathcal{N}(\lambda) \geq n+r$ by constructing $n+r$ linearly independent solutions of system ($\mathcal{S}_\lambda$) from ℓ_Ψ^2. By Theorem 2.4.1, we know that the columns of the Weyl solution defined in (2.59) form n linearly independent square summable solutions of ($\mathcal{S}_\lambda$). For $k \in [N_0+1,\infty)_\mathbb{Z}$, let $U_k \in \mathbb{C}^{n\times n}$ be a unitary matrix such that (2.57) and (2.62) hold. We put $U_k = \big(U_k^{[1]},\ U_k^{[2]}\big)$ with full rank blocks $U_k^{[1]} \in \mathbb{C}^{n\times r}$ and $U_k^{[2]} \in \mathbb{C}^{n\times(n-r)}$. It follows that the dimension of kernel of $U_k^{[2]*}$ is equal to r. Hence, if $\xi_k^{[1]},\dots,\xi_k^{[r]}$ is an orthonormal basis for $\operatorname{Ker} U_k^{[2]*}$, then the matrix $Q_k := \big(\xi_k^{[1]},\dots,\xi_k^{[r]}\big) \in \mathbb{C}^{n\times r}$ satisfies

$$Q_k^* Q_k = I_r \quad \text{and} \quad U_k^{[2]*} Q_k = 0_{(n-r)\times r} \quad \text{for all } k \in [N_0+1,\infty)_\mathbb{Z}, \tag{2.63}$$

where $0_{(n-r)\times r}$ means the $(n-r)\times r$ zero matrix. By the aid of Proposition 1.1.1 again, there exist a subsequence such that $U_{k_j} \to U_+$ and $Q_{k_j} \to Q_+ \in \mathbb{C}^{n\times r}$ for $j \to \infty$, where the matrix $U_+ = \big(U_+^{[1]},\ U_+^{[2]}\big)$ is unitary, $U_+^{[1]} \in \mathbb{C}^{n\times r}$ and $U_+^{[2]} \in \mathbb{C}^{n\times(n-r)}$ and, by (2.63), the matrix Q_+ satisfies $Q_+^* Q_+ = I_r$, $\operatorname{rank} Q_+ = r$, and $U_+^{[2]*} Q_+ = 0_{(n-r)\times r}$. If we denote by e_m for $1 \leq m \leq r$ the m-th unit vector in $\mathbb{C}^r$ (similarly as in the proof of Theorem 2.4.1), then

$$\left\|\widetilde{Z}(\lambda) Q_+ e_m\right\|_\Psi^2 \overset{(2.37)}{=} \frac{1}{2|\operatorname{im}(\lambda)|} \lim_{k\to\infty} e_m^* Q_+^* \mathcal{H}_k(\lambda) Q_+ e_m. \tag{2.64}$$

Fix now $k \in [N_0+1,\infty)_\mathbb{Z}$. Then for every $k_j \geq k$ we have by the monotonicity of $\mathcal{H}(\lambda)$ that

$$\begin{aligned} e_m^* Q_{k_j}^* \mathcal{H}_k(\lambda) Q_{k_j} e_m \leq e_m^* Q_{k_j}^* \mathcal{H}_{k_j}(\lambda) Q_{k_j} e_m &\overset{(2.62)}{=} e_m^* Q_{k_j}^* U_{k_j} \operatorname{diag}\left\{\mu_{k_j}^{[1]},\dots,\mu_{k_j}^{[n]}\right\} U_{k_j}^* Q_{k_j} e_m \\ &\overset{(2.63)}{=} e_m^* Q_{k_j}^* U_{k_j}^{[1]} \operatorname{diag}\left\{\mu_{k_j}^{[1]},\dots,\mu_{k_j}^{[r]}\right\} U_{k_j}^{[1]*} Q_{k_j} e_m. \end{aligned} \tag{2.65}$$

Upon taking $j \to \infty$ in inequality (2.65) we get

$$e_m^* Q_+^* \mathcal{H}_k(\lambda) Q_+ e_m \leq e_m^* Q_+^* U_+^{[1]} \operatorname{diag}\left\{\rho^{[1]},\dots,\rho^{[r]}\right\} U_+^{[1]*} Q_+ e_m =: T < \infty, \tag{2.66}$$

where $\rho^{[1]},\dots,\rho^{[r]}$ are the finite limits of the eigenvalues $\mu_{k_j}^{[1]},\dots,\mu_{k_j}^{[r]}$ as $j \to \infty$, see Theorem 2.4.6. Since the estimate in (2.66) holds for every $k \in [N_0+1,\infty)_\mathbb{Z}$, it follows from equality (2.64) that

$$2|\operatorname{im}(\lambda)| \times \left\|\widetilde{Z}(\lambda) Q_+ e_m\right\|_\Psi^2 = \lim_{k\to\infty} e_m^* Q_+^* \mathcal{H}_k(\lambda) Q_+ e_m \overset{(2.66)}{\leq} T < \infty.$$

This shows that the columns of $\widetilde{Z}(\lambda) Q_+$ belong to ℓ_Ψ^2. Consequently, system ($\mathcal{S}_\lambda$) has at least $n+r$ linearly independent square summable solutions, which are generated by the columns of the matrix

$$Y_k := \begin{pmatrix} Z_k(\lambda) & \widetilde{Z}_k(\lambda) \end{pmatrix} \begin{pmatrix} I & 0_{n\times r} \\ P_+(\lambda) & Q_+ \end{pmatrix} = \Phi_k(\lambda) \begin{pmatrix} I & 0_{n\times r} \\ P_+(\lambda) & Q_+ \end{pmatrix}. \tag{2.67}$$

Since $\Phi_k(\lambda)$ is invertible and $\operatorname{rank} Q_+ = r$, it follows that the matrix Y_k in (2.67) has $n+r$ linearly independent columns. Hence, we proved that system ($\mathcal{S}_\lambda$) has at least $n+r$ linearly independent square summable solutions, which completes the proof. ■

Combining the results of Theorems 2.4.6 and 2.4.8 we get the following supplement of Theorem 2.4.8.

Corollary 2.4.9. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.3.7 holds. Then*

$$0 < \lim_{k\to\infty} \mu_k^{[j]} =: \rho^{[j]} < \infty, \quad 1 \le j \le r(\lambda), \quad \text{and} \quad \lim_{k\to\infty} \mu_k^{[j]} = \infty, \quad r(\lambda)+1 \le j \le n,$$

where the numbers $\left(\rho^{[1]}\right)^{-1/2}, \dots, \left(\rho^{[r(\lambda)]}\right)^{-1/2}$ are the positive eigenvalues of $R_+(\lambda)$.

Moreover, yet another simple corollary follows from Theorem 2.4.8 as a counterpart of Theorem 2.4.3.

Corollary 2.4.10. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.3.7 holds. Then system ($\mathcal{S}_\lambda$) is in the limit circle case if and only if the matrix $R_+(\lambda)$ is invertible, i.e., $r(\lambda) = n$.*

In the remaining part of this section we present some characterizations of the two extreme cases $r(\lambda) = 0$ (i.e., the limit point case) and $r(\lambda) = n$ (i.e., the limit circle case). We will utilize the following strengthened Atkinson-type condition, which includes both Hypotheses 2.3.7 and 2.3.13. An alternative terminology is that system ($\mathcal{S}_\lambda$) is *definite* on the discrete interval $[0,\infty)_{\mathbb{Z}}$, see Section 6.2 for more details.

Hypothesis 2.4.11 (Strong Atkinson condition – infinite). There exists $N_3 \in [0,\infty)_{\mathbb{Z}}$ such that each nontrivial solution $z(\lambda)$ of system ($\mathcal{S}_\lambda$) satisfies inequality (2.25) with $N = N_3$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

The following three results represent direct generalizations of [A4, Theorems 4.13, 4.14 and Corollary 4.15] from the special linear dependence on λ in (2.7) to the general linear dependence on λ. The proofs of the statements in Theorems 2.4.12 and 2.4.13 follow exactly the same ideas as in the corresponding proofs in [A4] quoted above, which are now considered for the general linear dependence on λ. The details are therefore omitted. Note that Theorem 2.4.12 requires the strengthened Atkinson-type condition in Hypothesis 2.4.11, because it uses in its proof both the limiting Weyl disk $D_+(\lambda)$ and the $M_k(\lambda)$ functions for large k. On the other hand, in Theorem 2.4.13 we utilize the weaker condition from Hypothesis 2.3.7, because its proof uses only the limiting Weyl disk $D_+(\lambda)$.

Theorem 2.4.12. *Let $\alpha \in \Gamma$, $\lambda, \nu \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.4.11 holds. If systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) are both in the limit point or limit circle case, then*

$$\lim_{k\to\infty} \mathcal{X}_k^*(\lambda,\alpha,M_+(\lambda))\,\mathcal{J}\,\mathcal{X}_k(\nu,\alpha,M_+(\nu)) = 0, \tag{2.68}$$

where $\mathcal{X}(\lambda,\alpha,M_+(\lambda)) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ and $\mathcal{X}(\nu,\alpha,M_+(\nu)) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ mean the Weyl solutions of systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) defined as in (2.23) through the matrices $M_+(\lambda)$ and $M_+(\nu)$, respectively, which are determined by the limit in (2.50).

Theorem 2.4.13. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.3.7 holds. Systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_{\bar\lambda}$) are in the limit point case if and only if for every square summable solutions $z(\lambda)$ and $\tilde z(\bar\lambda)$ of ($\mathcal{S}_\lambda$) and ($\mathcal{S}_{\bar\lambda}$), respectively, we have*

$$z_k^*(\lambda)\,\mathcal{J}\,\tilde z_k(\bar\lambda) = 0 \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}. \tag{2.69}$$

Corollary 2.4.14. *Let $\alpha \in \Gamma$ and suppose that Hypothesis 2.4.11 holds. System ($\mathcal{S}_\lambda$) is in the limit point case for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ if and only if for every $\lambda, \nu \in \mathbb{C}\setminus\mathbb{R}$ and every square summable solutions $z(\lambda)$ and $\tilde z(\nu)$ of systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$), respectively, we have*

$$\lim_{k\to\infty} z_k^*(\lambda)\,\mathcal{J}\,\tilde z_k(\nu) = 0. \tag{2.70}$$

Proof. If system ($\mathcal{S}_\lambda$) is in the limit point case for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then the square summable solution $z(\lambda)$ must be a constant multiple of the Weyl solution $\mathcal{X}(\lambda)$, by Theorem 2.4.1.

Similarly, the square summable solution $z(\nu)$ is a constant multiple of $\mathcal{X}(\nu)$. Identity (2.70) then follows from Theorem 2.4.12. Conversely, let (2.70) be satisfied for every $\lambda,\nu \in \mathbb{C}\setminus\mathbb{R}$ and every $z(\lambda) \in \mathcal{N}(\lambda)$ and $\tilde{z}(\nu) \in \mathcal{N}(\nu)$. Fix $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and put $\nu := \bar{\lambda}$. Then from Lemma 2.1.5 we know that the value of $z_k^*(\lambda)\mathcal{J}\tilde{z}_k(\bar{\lambda})$ is constant on $k \in [0,\infty)_{\mathbb{Z}}$. Hence the limit in (2.70) implies that identity (2.69) is satisfied, and so system ($\mathcal{S}_\lambda$) is in the limit point case by Theorem 2.4.13. ∎

From Theorems 2.4.12 and 2.1.7 we also get the following statement.

Corollary 2.4.15. *Let $\alpha \in \Gamma$, $\lambda,\nu \in \mathbb{C}\setminus\mathbb{R}$, and suppose that Hypothesis 2.4.11 holds. If systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) are both in the limit point or limit circle case, then*

$$(\bar{\lambda}-\nu)\sum_{k=0}^{\infty} \mathcal{X}_{k+1}^*(\lambda,\alpha,M_+(\lambda))\,\Psi_k\,\mathcal{X}_{k+1}(\nu,\alpha,M_+(\nu)) = M_+^*(\lambda)-M_+(\nu), \tag{2.71}$$

where the Weyl solutions $\mathcal{X}(\lambda,\alpha,M_+(\lambda)) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ and $\mathcal{X}(\nu,\alpha,M_+(\nu)) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ are the same as in Theorem 2.4.12.

Proof. By Theorem 2.1.7, we get that the left-hand side of (2.71) is equal to the difference

$$\lim_{k\to\infty} \mathcal{X}_{k+1}^*(\lambda,\alpha,M_+(\lambda))\,\mathcal{J}\,\mathcal{X}_{k+1}(\nu,\alpha,M_+(\nu)) - \mathcal{X}_0^*(\lambda,\alpha,M_+(\lambda))\,\mathcal{J}\,\mathcal{X}_0(\nu,\alpha,M_+(\nu)).$$

While the limit above is zero by (2.68), the second term gives by the definition of the Weyl solution in (2.23) the equality $\mathcal{X}_0^*(\lambda,\alpha,M_+(\lambda))\,\mathcal{J}\,\mathcal{X}_0(\nu,\alpha,M_+(\nu)) = M_+(\nu)-M_+^*(\lambda)$. ∎

Remark 2.4.16. It can be shown under Hypothesis 2.3.7 that if ψ_k denotes the minimal eigenvalue of the Hermitian matrix Ψ_k for $k \in [0,\infty)_{\mathbb{Z}}$ and if

$$\sum_{k=0}^{\infty} \psi_k = \infty, \tag{2.72}$$

then system ($\mathcal{S}_\lambda$) is in the limit point case. This fact follows in a similar way as in [154, Theorem 5.1] by using Theorem 2.4.13. However, system ($\mathcal{S}_\lambda$) is such that $\psi_k = 0$ for all $k \in [0,\infty)_{\mathbb{Z}}$, because the matrices Ψ_k are singular, see Lemma 2.1.1 and the subsequent paragraph. Therefore, condition (2.72) can never be fulfilled in the present theory.

We conclude this section by a generalization of one half of the classical *Weyl alternative*, see e.g. [171, Theorem 8.27]. More precisely, we show that if system ($\mathcal{S}_\lambda$) is in the limit circle case for some $\lambda_0 \in \mathbb{C}$ (i.e., $\dim \mathcal{N}(\lambda_0) = 2n$), then it is in the limit circle case for every $\lambda \in \mathbb{C}$ (i.e., $\dim \mathcal{N}(\lambda) \equiv 2n$). In other words, we derive the *invariance of the limit circle case*, which provides a discrete analogue of the result established in [9, Theorem 9.11.2] for system (2.5). However we emphasize that in contrast to the latter result we do not need to impose any additional assumptions on the coefficient matrices of system ($\mathcal{S}_\lambda$) in the present setting. Similar statements for the second order Sturm–Liouville difference equations and linear Hamiltonian difference systems can be found, respectively, in [9, Theorem 5.6.1] and [142, Theorem 5.5]. In addition, we skip the proof, because it follows immediately from more a general result derived in Chapter 4, see Theorem 4.2.2 and Remark 4.2.4.

Theorem 2.4.17. *If there exists $\lambda_0 \in \mathbb{C}$ such that system ($\mathcal{S}_{\lambda_0}$) is in the limit circle case, then system ($\mathcal{S}_\lambda$) is in the limit circle case for every $\lambda \in \mathbb{C}$.*

Remark 2.4.18. From the discussion concerning equations (2.8)–(2.10) in the introduction of this chapter and from Theorem 2.4.17 we easily deduce the invariance of the limit circle case (i.e., of the situation when all solutions are square summable with respect to the weight $\mathcal{W}$) for any even order vector-valued Sturm–Liouville difference equation, Jacobi equation, and symmetric three term recurrence relation. Indeed, since the corresponding system ($\mathcal{S}_\lambda$) is such that $\Psi_k = \operatorname{diag}\{\mathcal{W}_k, 0, \dots, 0\}$ or $\Psi_k = \operatorname{diag}\{\mathcal{W}_k, 0\}$, we get immediately the equality $y^*_{k+1}(\lambda)\, \mathcal{W}_k\, y_{k+1}(\lambda) = z^*_k(\lambda)\, \Psi_k\, z_{k+1}(\lambda)$ for all $k \in [0,\infty)_{\mathbb{Z}}$. Therefore a solution $y(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^n$ of equations (2.8) or (2.9) or (2.10) satisfies $\sum_{k=0}^{\infty} y^*_{k+1}(\lambda)\, \mathcal{W}_k\, y_{k+1}(\lambda) < \infty$ if and only if the corresponding $z(\lambda)$ belongs to ℓ^2_Ψ with respect to the weight Ψ specified earlier, see also Remark 4.2.7.

As a direct consequence of Theorem 2.4.17 we obtain the following criterion for the limit circle case. This result corresponds to [9, Theorem 5.8.1] and [134, Theorem 6.3] for the second order Sturm–Liouville difference equations and linear Hamiltonian difference systems. Again we skip the proof, because the statement follows from Corollary 4.2.3 and Remark 4.2.4. We note that the matrix norm $\|\cdot\|_1$ used in (2.73) below can be replaced by any other matrix norm because of their equivalence.

Corollary 2.4.19. *Assume that*

$$\sum_{k=0}^{\infty} \| \mathcal{S}_k - I \|_1 < \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} \| \Psi_k \|_1 < \infty. \tag{2.73}$$

Then system ($\mathcal{S}_\lambda$) is in the limit circle case for all $\lambda \in \mathbb{C}$.

Upon combining Theorem 2.4.17 and Corollary 2.4.10 we get the following result.

Corollary 2.4.20. *Assume that Hypothesis 2.3.7 holds and that $\lambda_0 \in \mathbb{C}$ is such that ($\mathcal{S}_{\lambda_0}$) is in the limit circle case. Then $r(\lambda) = n$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

Remark 2.4.21. Under the assumptions of Corollary 2.4.20 we can deduce that the value of $r(\lambda)$ is constant and equal to n on $\mathbb{C}\setminus\mathbb{R}$. This observation gives rise to two additional and very natural questions. Are the values of $r(\lambda)$ and $r(\bar\lambda)$ constant on some subsets of $\mathbb{C}$ in general, especially on the upper and lower half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$? And if $r(\lambda)$ and $r(\bar\lambda)$ are constant on $\mathbb{C}_+$ and $\mathbb{C}_-$, do they satisfy $r(\lambda) = r(\bar\lambda)$ on $\mathbb{C}\setminus\mathbb{R}$? Of course, these questions can be formulated also for the numbers $\dim \mathcal{N}(\lambda)$ and $\dim \mathcal{N}(\bar\lambda)$. The first answer is positive as discussed in Remark 6.4.15. The answer to the second question is positive in the limit circle case under the assumptions stated in Corollary 2.4.20, as well as in the limit point case under analogous assumptions as in Theorem 7.1.1 from Chapter 7. Moreover, if the matrices $\mathcal{S}_k$ and $\mathcal{V}_k$ are real-valued for all $k \in [0,\infty)_{\mathbb{Z}}$, then it follows immediately that $z(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ solves system ($\mathcal{S}_\lambda$) if and only if $\overline{z(\lambda)}$ solves ($\mathcal{S}_{\bar\lambda}$), i.e., $z_k(\bar\lambda) = \overline{z_k(\lambda)}$ for all $k \in [0,\infty)_{\mathbb{Z}}$. Thus, in that case we have $\dim \mathcal{N}(\lambda) = \dim \mathcal{N}(\bar\lambda)$, i.e., $r(\lambda) = r(\bar\lambda)$. However, in other situations we conjecture that the answer can be negative similarly as it was shown in the continuous time case by the example from [120], which was already mentioned in Remark 2.4.4(i). More specifically, in the latter example the fourth order differential equation with three and two linearly independent square integrable solutions in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, was constructed.

In the scalar case (i.e., $n = 1$) the estimate in (2.55) implies that $\dim \mathcal{N}(\lambda) \in \{1, 2\}$. In this case we derive from Theorem 2.4.17 its limit point counterpart for $\lambda \in \mathbb{C}\setminus\mathbb{R}$, i.e., the second part of the Weyl alternative.

Corollary 2.4.22. *Let $n = 1$ and assume that Hypothesis 2.3.7 holds. If there $\lambda_0 \in \mathbb{C}$ such that system ($\mathcal{S}_{\lambda_0}$) is in the limit point case, then system ($\mathcal{S}_\lambda$) is in the limit point case for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

Proof. Let system $(\mathcal{S}_{\lambda_0})$ be in the limit point case and assume that there exists $\lambda_1 \in \mathbb{C}\setminus\mathbb{R}$ such that system $(\mathcal{S}_{\lambda_1})$ is not in the limit point case. Then, by $n = 1$ and the estimate in (2.55), we know that system $(\mathcal{S}_{\lambda_1})$ is in the limit circle case. But from Theorem 2.4.17 (applied with $\lambda_0 = \lambda_1$) we obtain that system $(\mathcal{S}_{\lambda})$ is in the limit circle case for every $\lambda \in \mathbb{C}$, which contradicts the original assumption that system $(\mathcal{S}_{\lambda_0})$ is in the limit point case. ∎

Upon combining Theorems 2.4.17 and 2.4.22 we get the scalar symplectic analogue of the Weyl alternative for system $(\mathcal{S}_{\lambda})$.

Corollary 2.4.23 (Weyl alternative). *Let $n = 1$ and assume that Hypothesis 2.3.7 holds. Then system $(\mathcal{S}_{\lambda})$ is either in the limit circle case for all $\lambda \in \mathbb{C}$, or in the limit point case for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

Although in this section we have conducted a thorough analysis of the number of linearly independent square summable solutions of system $(\mathcal{S}_{\lambda})$, we gave absolutely no information about $\dim \mathcal{N}(\lambda)$ if $\lambda \in \mathbb{R}$ (except for the limit circle case). Fortunately, a basic estimate of this value is derived in Theorem 6.4.16 by using the theory of linear relations developed in Chapter 6, see also Remark 6.4.15. Moreover, there exists an intimate connection between the Weyl solution $\mathcal{X}(\lambda)$ with $\lambda \in \mathbb{R}$ and the so-called *principal* (or recessive) solution of system $(\mathcal{S}_{\lambda})$ in the *nonoscillatory* case. This connection represents one of the goals of our current research and it will be new even in the continuous case, i.e., for system (2.5). Similar results were established for the second order Sturm–Liouville differential, difference, and dynamic equations in [37] and [A23].

2.5 Illustrating examples

Now we present several examples which illustrate the results of this chapter. In the whole section we focus on the special case of system $(\mathcal{S}_{\lambda})$ with $n = 1$ and $\mathcal{S}_k \equiv \mathcal{S} := I_2$, which can be considered as a discrete analogue of the so-called no potential case known in the theory of Sturm–Liouville differential equations and linear Hamiltonian differential systems. That is, we analyze the system

$$\Delta z_k(\lambda) = \lambda \mathcal{V}_k z_k(\lambda), \quad k \in [0,\infty)_{\mathbb{Z}}, \tag{2.74}$$

with two special choices of the matrices $\mathcal{V} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2\times 2}$ satisfying the assumptions in (2.1). In each example we determine the centers and the radii of the Weyl disk and of the limiting Weyl disk, as well as we give the corresponding limit point or limit circle classification. We note that in the limit point case we denote by $\mathcal{X}^{+}(\lambda)$ the Weyl solution defined as in (2.23) with $M = P_{+}(\lambda)$.

Example 2.5.1. Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and consider system (2.74) with the constant matrices

$$\mathcal{V}_k \equiv \mathcal{V} = \begin{pmatrix} \sqrt{ab} & a \\ -b & -\sqrt{ab} \end{pmatrix}, \quad \Psi_k \equiv \Psi = \begin{pmatrix} b & \sqrt{ab} \\ \sqrt{ab} & a \end{pmatrix} \geq 0 \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}, \tag{2.75}$$

where $a > 0$ and $b \geq 0$ are given real numbers. This choice of $\mathcal{V}$ is naturally based on the properties required in (2.1). Note that for $\Psi \geq 0$ we need only $a \geq 0$, but as we will see, the crucial Hypothesis 2.3.7 is not satisfied when $a = 0$. The fundamental matrix $\Phi_k(\lambda)$ of system (2.74) with $\alpha = (1,\ 0)$ has in this case the form

$$\Phi_k(\lambda) = (I + \lambda\mathcal{V})^k = \begin{pmatrix} 1 + k\lambda\sqrt{ab} & k\lambda a \\ -k\lambda b & 1 - k\lambda\sqrt{ab} \end{pmatrix}, \quad k \in [0,\infty)_{\mathbb{Z}}. \tag{2.76}$$

In view of (2.22), the two solutions $Z_k(\lambda)$ and $\widetilde{Z}_k(\lambda)$ of system (2.74) are equal to the first and second columns of the matrix $\Phi_k(\lambda)$ in (2.76), respectively. The sum in (2.25) with $N = N_0$ is then equal to $(N_0 + 1)a$, which shows that Hypothesis 2.3.7 is satisfied for any $N_0 \in [0, \infty)_{\mathbb{Z}}$ and only $a > 0$. From (2.36) we then get

$$\mathcal{F}_k(\lambda) = 2kb\,|\operatorname{im}(\lambda)|, \quad \mathcal{G}_k(\lambda) = -i\delta(\lambda) + 2k\sqrt{ab}\,|\operatorname{im}(\lambda)|, \quad \mathcal{H}_k(\lambda) = 2ka\,|\operatorname{im}(\lambda)|,$$

which yields through (2.42) that the center and the radius of the Weyl disk $D_k(\lambda)$ for $k \in [1, \infty)_{\mathbb{Z}}$ are

$$P_k(\lambda) = -\sqrt{b/a} + \frac{i}{2ka\operatorname{im}(\lambda)} \quad \text{and} \quad R_k(\lambda) = \frac{1}{\sqrt{2ka\,|\operatorname{im}(\lambda)|}}.$$

By taking the limit as $k \to \infty$ we can see that the center and the radius of the limiting Weyl disk $D_+(\lambda)$ are $P_+(\lambda) = -\sqrt{b/a}$ and $R_+(\lambda) = 0$, that is, $D_+(\lambda) = \big\{-\sqrt{b/a}\big\}$. This shows that system (2.74) with $\mathcal{V}_k$ given in (2.75) is in the limit point case for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$. The limiting behavior of the Weyl disks is demonstrated in Figure 2.1 below. The Weyl solution $\mathcal{X}_k^+(\lambda) \equiv \big(1, -\sqrt{b/a}\big)^\top$ satisfies $\|\mathcal{X}^+(\lambda)\|_\Psi = 0$ and it is the only square summable solution (up to a nonzero constant multiple). Note that $\|\widetilde{Z}(\lambda)\|_\Psi = \infty$ and $\|Z(\lambda)\|_\Psi = \infty$ for $b > 0$, while for $b = 0$ we have $Z(\lambda) = \mathcal{X}^+(\lambda)$.

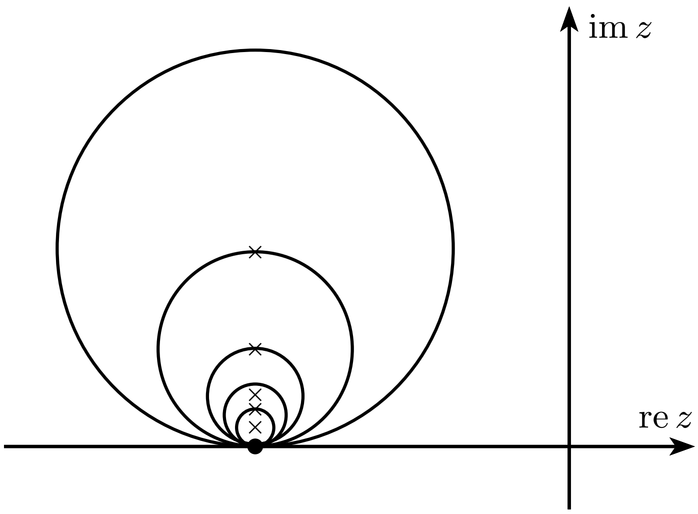

Figure 2.1: The Weyl disks $D_k(\lambda)$ for $k \in \{1, 2, 4, 6, 10\}$, their centers, and $P_+(\lambda) = -1$ from Example 2.5.1 with $a = b = 2$ and $\lambda = 0.4 + 0.4i$.

▲

Although the choice of $a = 0$ was not possible in Example 2.5.1, we show that also in this case the system from Example 2.5.1 is in the limit points case. This is done by using a suitable transformation.

Example 2.5.2. Let $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and consider the system from Example 2.5.1 with $a = 0$ and $b > 0$, i.e.,

$$z_{k+1}(\lambda) = \mathbb{S}(\lambda)z_k(\lambda), \quad \mathbb{S}(\lambda) = \begin{pmatrix} 1 & 0 \\ -\lambda b & 1 \end{pmatrix}, \quad \Psi = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \geq 0, \quad k \in [0, \infty)_{\mathbb{Z}}. \tag{2.77}$$

The dependence on λ in system (2.77) is special as in (2.7) and, as we discussed in Example 2.5.1, Hypothesis 2.3.7 is not satisfied in this case. Therefore the theory developed in Section 2.3 cannot be applied, neither can be applied the results in [26] and [A4]. On the other hand, by using the transformation

$$y_k(\lambda) := T^{-1} z_k(\lambda), \quad \text{where} \quad T := \begin{pmatrix} -\sqrt{c/b} & -1 \\ 1 & 0 \end{pmatrix}$$

is a constant symplectic matrix with $c \geq 0$, we obtain another symplectic system

$$y_{k+1}(\lambda) = \widetilde{\mathbb{S}}(\lambda)\, y_k(\lambda), \quad \widetilde{\mathbb{S}}(\lambda) = \begin{pmatrix} 1+\lambda\sqrt{bc} & \lambda b \\ -\lambda c & 1-\lambda\sqrt{bc} \end{pmatrix}, \quad \widetilde{\Psi} = \begin{pmatrix} c & \sqrt{bc} \\ \sqrt{bc} & b \end{pmatrix}, \quad k \in [0,\infty)_{\mathbb{Z}},$$

where $\widetilde{\mathbb{S}}(\lambda) := T^{-1}\mathbb{S}(\lambda)\,T$, see [20, Lemma 6]. Now, since $b > 0$ and $c \geq 0$, the results in Example 2.5.1 can be used for the above transformed system. In particular, system (2.77) is in the limit point case for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Depending on the choice of the constant $c \geq 0$ in the transformation matrix T, we obtain from the reversed transformation the two linearly independent solutions $Z_k(\lambda) = \big(-\sqrt{c/b},\ 1+k\lambda\sqrt{bc}\big)^{\top}$ and $\widetilde{Z}_k(\lambda) = (-1,\ k\lambda b)^{\top}$ of system (2.77). For these solutions we easily calculate that $\|Z(\lambda)\|_{\Psi} = 0$ when $c = 0$, $\|Z(\lambda)\|_{\Psi} = \infty$ when $c > 0$, and $\|\widetilde{Z}(\lambda)\|_{\Psi} = \infty$. The corresponding Weyl solution is $\mathcal{X}_k^{+}(\lambda) \equiv (0,\ 1)^{\top}$, which obviously satisfies $\|\mathcal{X}^{+}(\lambda)\|_{\Psi} = 0$. ▲

Finally, we present a system of the form as in (2.74) with nonconstant $\mathcal{V}_k$, which can be either in the limit point case or in the limit circle case.

Example 2.5.3. Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and $v \in \mathbb{C}([0,\infty)_{\mathbb{Z}})$ be a given sequence such that $v_0 = 0$, $\Delta v_k \geq 0$ for all $k \in [0,\infty)_{\mathbb{Z}}$, and $v_\ell > 0$ for some index $\ell \in [1,\infty)_{\mathbb{Z}}$. Define the matrix $\mathcal{V}_k = \left(\begin{smallmatrix} 0 & \Delta v_k \\ 0 & 0 \end{smallmatrix}\right)$ for all $k \in [0,\infty)_{\mathbb{Z}}$ and consider the system

$$z_{k+1}(\lambda) = \mathbb{S}_k(\lambda) z_k(\lambda), \quad \mathbb{S}_k(\lambda) = \begin{pmatrix} 1 & \lambda\,\Delta v_k \\ 0 & 1 \end{pmatrix}, \quad \Psi_k = \begin{pmatrix} 0 & 0 \\ 0 & \Delta v_k \end{pmatrix} \geq 0, \quad k \in [0,\infty)_{\mathbb{Z}}. \tag{2.78}$$

The fundamental matrix of system (2.78) with $\alpha = (1,0)$ is equal to

$$\Phi_k(\lambda) = \begin{pmatrix} 1 & \lambda v_k \\ 0 & 1 \end{pmatrix}, \quad \text{i.e.,} \quad Z_k(\lambda) \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \widetilde{Z}_k(\lambda) = \begin{pmatrix} \lambda v_k \\ 1 \end{pmatrix}, \quad k \in [0,\infty)_{\mathbb{Z}}. \tag{2.79}$$

This implies that Hypothesis 2.3.7 is satisfied for any $N_0 \in [\ell-1,\infty)_{\mathbb{Z}}$, since the sum in (2.25) with $N = \ell - 1$ is equal to $v_\ell > 0$. From (2.36) we get $\mathcal{F}_k(\lambda) = 0$, $\mathcal{G}_k(\lambda) = -i\delta(\lambda)$, and $\mathcal{H}_k(\lambda) = 2v_k\,|\operatorname{im}(\lambda)|$. The assumptions imply that $\mathcal{H}_k(\lambda) > 0$ for all $k \in [\ell,\infty)_{\mathbb{Z}}$, so that the center and the radius of the Weyl disk $D_k(\lambda)$ are well defined and equal to

$$P_k(\lambda) = \frac{i}{2v_k \operatorname{im}(\lambda)} \quad \text{and} \quad R_k(\lambda) = \frac{1}{\sqrt{2v_k\,|\operatorname{im}(\lambda)|}} \quad \text{for all } k \in [\ell,\infty)_{\mathbb{Z}}.$$

If we put $v_\infty := \lim_{k\to\infty} v_k = \sup_{k\in[0,\infty)_{\mathbb{Z}}}\{v_k\}$, then $v_\infty > 0$ and the center and the radius of the limiting Weyl disk $D_+(\lambda)$ are equal to $P_+(\lambda) = i/[2v_\infty \operatorname{im}(\lambda)]$ and $R_+(\lambda) = 1/\sqrt{2v_\infty\,|\operatorname{im}(\lambda)|}$. From this one can easily conclude that system (2.78) is in the limit point case if and only if $v_\infty = \infty$, while it is in the limit circle case if and only if $v_\infty < \infty$. In the latter case, the linearly independent solutions $Z(\lambda)$ and $\widetilde{Z}(\lambda)$ are square summable with $\|Z(\lambda)\|_{\Psi} = 0$ and $\|\widetilde{Z}(\lambda)\|_{\Psi} = \sqrt{v_\infty}$. In addition, the Weyl solution $\mathcal{X}(\lambda)$ defined by (2.23) through the fundamental matrix $\Phi(\lambda)$ from (2.79) and $M = m \in \mathbb{C}$ is also square summable with the corresponding semi-norm $\|\mathcal{X}(\lambda)\|_{\Psi} = \sqrt{v_\infty}\,|m| < \infty$. The behavior of the Weyl disks is demonstrated in Figure 2.2 below.

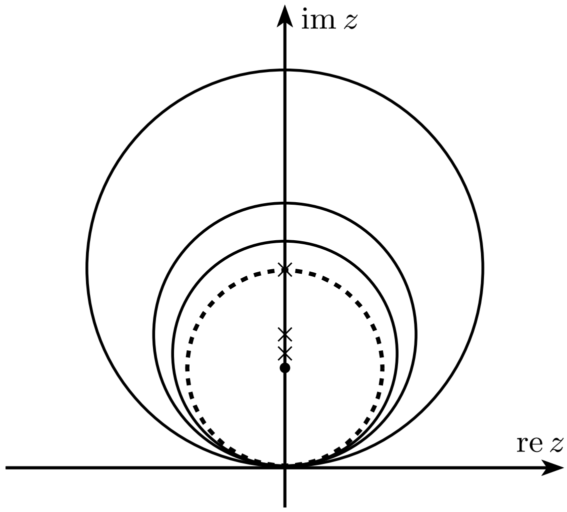

Figure 2.2: The Weyl disks $D_k(\lambda)$ for $k \in \{1,2,3\}$, their centers, and the disk $D_+(\lambda)$ with $P_+(\lambda) = 5i/2$ and $R_+(\lambda) = \sqrt{5}/2$ from Example 2.5.3 with $v_k = 1 - 2^{-k}$ and $\lambda = 0.4 + 0.4i$.

▲

The results of the previous example are summarized in the following statement.

Corollary 2.5.4. *Let $n = 1$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $v \in \mathrm{C}([0,\infty)_{\mathbb{Z}})$ be such that $v_0 = 0$, $\Delta v_k \geq 0$ for all $k \in [0,\infty)_{\mathbb{Z}}$, $v_\ell > 0$ for some index $\ell \in [1,\infty)_{\mathbb{Z}}$, and define $v_\infty := \lim_{k\to\infty} v_k$. If we put $\mathcal{V}_k := \left(\begin{smallmatrix} 0 & \Delta v_k \\ 0 & 0 \end{smallmatrix}\right)$ for all $k \in [0,\infty)_{\mathbb{Z}}$, then the following holds.*

(i) *System (2.74) is in the limit point case if and only if $v_\infty = \infty$. In this case, $P_+(\lambda) = 0$, $R_+(\lambda) = 0$, and the Weyl solution $\mathcal{X}_k^+(\lambda) \equiv (1,\ 0)^\top$ is the only (up to a nonzero constant multiple) square summable solution of system (2.74).*

(ii) *System (2.74) is in the limit circle case if and only if $v_\infty < \infty$. In this case we have $P_+(\lambda) = i/[2v_\infty \operatorname{im}(\lambda)]$ and $R_+(\lambda) = 1/\sqrt{2v_\infty |\operatorname{im}(\lambda)|}$. The solutions $Z_k(\lambda) \equiv (1,\ 0)^\top$ and $\widetilde{Z}_k(\lambda) = (\lambda v_k, 1)^\top$ are linearly independent with $\|Z(\lambda)\|_\Psi = 0$ and $\|\widetilde{Z}(\lambda)\|_\Psi = \sqrt{v_\infty}$.*

We note that one direction in part (ii) of Corollary 2.5.4 also follows from Corollary 2.4.19, because in this case we have $\sum_{k=0}^{\infty} \|\mathcal{S}_k - I\|_1 = 0$ and $\sum_{k=0}^{\infty} \|\Psi_k\|_1 = v_\infty < \infty$.

2.6 Bibliographical notes

The results of this chapter (including the direct proofs of Theorem 2.4.17 and Corollary 2.4.19) were published in [A15]. Moreover, their generalization to symplectic systems on time scales was established in [A16]. In our future research we aim to construct a particular example of system ($\mathcal{S}_\lambda$) mentioned in Remarks 2.4.4(i) and 2.4.21, i.e., such that $\dim \mathcal{N}(\lambda) \neq \dim \mathcal{N}(\bar{\lambda})$.

Perhaps the most surprising thing about mathematics is that it is so surprising. The rules which we make up at the beginning seem ordinary and inevitable, but it is impossible to foresee their consequences. These have only been found out by long study, extending over many centuries. Much of our knowledge is due to a comparatively few great mathematicians such as Newton, Euler, Gauss, Cauchy, or Riemann; few careers can have been more satisfying than theirs. They have contributed something to human thought even more lasting than great literature, since it is independent of language.

EDWARD CHARLES TITCHMARSH, SEE [1, PG. 12]

Chapter 3

JOINTLY VARYING ENDPOINTS

In the previous chapter we started with the eigenvalue problem given in (2.24), which includes the separated boundary conditions. Then, by using the fundamental matrix $\Phi(\lambda)$ determined in (2.21), we developed the theory of Weyl disks and square summable solutions for system ($\mathcal{S}_\lambda$). These results were achieved under the *weak Atkinson condition*, see Hypothesis 2.3.7, instead of the traditional *strong Atkinson condition*, see Hypothesis 2.4.11 and [A4], [26, 142, 154]. This is crucial and absolutely essential in the context of the results established in this chapter, where we will extend the results of Chapter 2 to problems with general jointly varying endpoints

$$\gamma \begin{pmatrix} z_0 \\ z_{N+1} \end{pmatrix} = 0, \qquad \gamma \in \Gamma := \left\{ \gamma \in \mathbb{C}^{2n\times 4n} \mid \gamma\gamma^* = I,\ \gamma \begin{pmatrix} -\mathcal{J} & 0 \\ 0 & \mathcal{J} \end{pmatrix} \gamma^* = 0 \right\}. \tag{3.1}$$

The boundary conditions in (3.1) include, among others, the periodic endpoints $z_0 = z_{N+1}$ or the antiperiodic endpoints $z_0 = -z_{N+1}$, which could not be treated by the previous case in (2.24), see the discussion following (2.24). The method we use is based on the augmentation of system ($\mathcal{S}_\lambda$) into double dimension, which leads to a problem with separated endpoints having the original boundary conditions from (3.1) as one of its constraints. This technique is known in the literature in principle (cf. [17, 87, 88, 92, 112, 153]), but the transformation to separated endpoints presented in this chapter is much simpler. At the same time, the transformed symplectic system no longer satisfies the corresponding strong Atkinson condition, but only its weak form. Thus, the derivation of the Weyl–Titchmarsh theory in Chapter 2 under the weak Atkinson condition is truly crucial for its further extension to jointly varying endpoints.

For this general situation, we give a characterization of eigenvalues of the eigenvalue problem determined by system ($\mathcal{S}_\lambda$) together with the boundary conditions in (3.1), we construct the Weyl disks, their centers and matrix radii, and also focus on properties of square summable solutions. More precisely, we give an exact connection between the limit point or limit circle classification of the original system (in dimension $2n$) and the augmented system (in dimension $4n$). This connection reveals an interesting fact, namely that the limiting matrix radius of the augmented system has its rank at least

n (see Theorem 3.1.11), and so it is never zero in the limit point case as one would expect from Theorem 2.4.3. The results of this chapter (see Theorem 3.1.2) also imply the existence of multiple eigenvalues for scalar symplectic eigenvalue problems with jointly varying endpoints. This is known e.g. for the second order discrete Sturm–Liouville problems with periodic endpoints in [105, Example 7.6] or [170, Theorem 2.2] and here we extend it to discrete symplectic systems. The transformation of jointly varying endpoints into separated endpoints will also find applications in the continuous time problems or time scales problems, see e.g. [153]. Finally, we remark that the results shown in this chapter were established as new even for special discrete symplectic systems, such as those with (2.7), the Jacobi equations and symmetric three term recurrence relations, i.e., equations (2.9) and (2.10), and also for linear Hamiltonian difference systems.

3.1 Weyl–Titchmarsh theory for jointly varying endpoints

Throughout this chapter we use the same notation as in Chapter 2, see Notation 2.1.2. Moreover, we emphasize the augmentation by the bold notation. For a given $\gamma \in \Gamma$ and $N \in [0, \infty)_{\mathbb{Z}}$ we consider the eigenvalue problem

$$(\mathcal{S}_\lambda), \quad k \in [0, N]_{\mathbb{Z}}, \quad \lambda \in \mathbb{C}, \quad (3.1). \tag{3.2}$$

The eigenvalues of (3.2) are defined as for (2.24). That is, a number $\lambda \in \mathbb{C}$ is an *eigenvalue* of problem (3.2) if, for this particular value λ, system $(\mathcal{S}_\lambda)$ has a nontrivial solution $z(\lambda) \in \mathbb{C}([0, N+1]_{\mathbb{Z}})^{2n}$ satisfying the boundary conditions in (3.1). In this case, $z(\lambda)$ is called an *eigenfunction* for λ and the dimension of such eigenfunctions for λ is its geometric multiplicity. As one of the main assumptions we suppose that system $(\mathcal{S}_\lambda)$ satisfies the strong Atkinson condition on a finite or infinite interval, see Hypothesis 3.1.1 below and Hypothesis 2.4.11, respectively.

Hypothesis 3.1.1 (Strong Atkinson condition – finite). The inequality in (2.25) is satisfied for every nontrivial solution $z(\lambda) \in \mathbb{C}([0, N+1]_{\mathbb{Z}})^{2n}$ of system $(\mathcal{S}_\lambda)$ on the discrete interval $[0, N]_{\mathbb{Z}}$ and every $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

The results of this chapter will be formulated with the aid of a particular fundamental matrix $\Phi(\lambda)$ of system $(\mathcal{S}_\lambda)$ starting with the initial value $\Phi_0(\lambda) = -\mathcal{J}$, which corresponds to the fundamental matrix $\Phi(\lambda)$ specified in (2.21) with the choice $\alpha = (0, I)$, i.e.,

$$\Phi_{k+1}(\lambda) = (\mathcal{S}_k + \lambda \mathcal{V}_k)\, \Phi_k(\lambda), \quad k \in [0, \infty)_{\mathbb{Z}}, \quad \Phi_0(\lambda) = -\mathcal{J}, \quad \lambda \in \mathbb{C}. \tag{3.3}$$

Our first result describes the orthogonality of the eigenfunctions and the multiplicity of the eigenvalues of problem (3.2). It generalizes Theorem 2.2.3 to jointly varying endpoints, compare also with [19, Theorem 2.2]. The proofs mostly follow by direct calculations in a similar way as the corresponding results in Chapter 2. For completeness and comparison we provide alternative proofs based on the transformation in Section 3.3.

Theorem 3.1.2. *Let $\gamma \in \Gamma$ be given. Then the following statements hold.*

(i) *A number $\lambda \in \mathbb{C}$ is an eigenvalue of problem (3.2) if and only if the matrix*

$$L(\lambda) := \gamma \begin{pmatrix} -\mathcal{J} \\ \Phi_{N+1}(\lambda) \end{pmatrix} \tag{3.4}$$

is singular. In this case, the eigenfunctions corresponding to the eigenvalue λ have the form $z(\lambda) = \Phi(\lambda) d$ on $[0, N+1]_{\mathbb{Z}}$ with a nonzero $d \in \operatorname{Ker} L(\lambda)$. Moreover, the geometric multiplicity of λ is equal to its algebraic multiplicity, i.e., to the value of $\dim \operatorname{Ker} L(\lambda)$.

(ii) *Under Hypothesis 3.1.1, the eigenvalues of problem (3.2) are real and the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the semi-inner product $\langle\cdot,\cdot\rangle_{\Psi,N}$ defined in (2.26).*

Proof. The statement follows from Theorem 3.3.5 with (3.29) and Corollary 3.3.3. ∎

By Theorem 3.1.2, the multiplicities of the eigenvalues of problem (3.2) are at most $2n$, compared to the separated endpoints case in Theorem 2.2.3 in which the multiplicities of the eigenvalues are at most n. It implies that in the scalar case (i.e., for $n = 1$) there may exist multiple eigenvalues of problem (3.2). This phenomenon was observed in [105, Example 7.6] and later justified in [170, Theorem 2.2] for the periodic discrete Sturm–Liouville eigenvalue problem, see also Example 3.2.4.

Remark 3.1.3. When system ($\mathcal{S}_\lambda$) has the special structure shown in (2.7), it can be deduced from [152, Corollary 4.6] that the total number of eigenvalues of (3.2) is equal to the dimension of the space of admissible functions for the associated discrete quadratic functional. In some even more special cases, such as for the second order Sturm–Liouville difference equations with periodic or antiperiodic endpoints, this exact number of the eigenvalues of problem (3.2) is derived in [170, Theorem 4.2] or [145, Theorem 4.1]. The result in [152, Corollary 4.6] is based on the Rayleigh principle for system ($\mathcal{S}_\lambda$) with the boundary condition from (3.1), compare also with [153, Theorem 3.2], and on the fact that the space of admissible functions is independent of λ, which follows from the special structure in (2.7). As the Rayleigh principle for eigenvalue problems (3.2) is not known and the space of admissible functions is in this case not constant in λ, the question about the total number of eigenvalues of problem (3.2) remains open for the general linear dependence on λ. On the other hand, the oscillation theorem for discrete symplectic eigenvalue problems with jointly varying endpoints in [148, Theorem 6.13] yields that the total number of the eigenvalues of problem (3.2) is less or equal to $(N+3)n$.

Next we define the *Weyl–Titchmarsh $\boldsymbol{M}(\lambda)$-function* for problem (3.2), compare with identity (2.27). For $k \in [0, N+1]_{\mathbb{Z}}$ and $\lambda \in \mathbb{C}$ we set

$$\boldsymbol{M}_k(\lambda) := -\left[\gamma\begin{pmatrix}-\mathcal{J}\\ \Phi_k(\lambda)\end{pmatrix}\right]^{-1}\gamma\begin{pmatrix}\mathcal{J}\\ \Phi_k(\lambda)\end{pmatrix}\mathcal{J}, \tag{3.5}$$

whenever the inverse above exists. In particular, we can see from Theorem 3.1.2 that $\boldsymbol{M}_k(\lambda)$ is well defined for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and $k = N+1$, when Hypothesis 3.1.1 holds. The following statement generalizes Lemma 2.2.5 to jointly varying endpoints.

Theorem 3.1.4. *Let $\gamma \in \Gamma$, $\lambda \in \mathbb{C}$, and $k \in [0,\infty)_{\mathbb{Z}}$. If $\boldsymbol{M}_k(\lambda)$ and $\boldsymbol{M}_k(\bar\lambda)$ exist, then we have $\boldsymbol{M}_k^*(\lambda) = \boldsymbol{M}_k(\bar\lambda)$. Moreover, $\boldsymbol{M}_k(\cdot)$ is an analytic function in its argument λ.*

Proof. This result follows from (3.36) and (3.33) via Lemma 2.2.5. ∎

For any $M \in \mathbb{C}^{2n\times 2n}$ we define the *Weyl-solution* $\mathfrak{X}(\lambda)$ of ($\mathcal{S}_\lambda$) with values in $\mathbb{C}^{4n\times 2n}$ by

$$\mathfrak{X}_k(\lambda) := \frac{1}{\sqrt{2}}\begin{pmatrix}\mathcal{J} & -\mathcal{J}\\ \Phi_k(\lambda) & \Phi_k(\lambda)\end{pmatrix}\begin{pmatrix}\mathcal{J}\\ M\end{pmatrix}. \tag{3.6}$$

It then follows, compare with Remark 2.2.6(i), that $\gamma\,\mathfrak{X}_k(\lambda) = 0$ if and only if the matrix M equals to $\boldsymbol{M}_k(\lambda)$ defined in (3.5).

One of the central concepts of this chapter is the $\mathcal{E}(M)$-*function* with values in $\mathbb{C}^{2n\times 2n}$, through which we later on define the Weyl disks. For $M \in \mathbb{C}^{2n\times 2n}$ we put

$$\mathcal{E}_k(M) := i\delta(\lambda)\mathfrak{X}_k^*(\lambda)\begin{pmatrix}-\mathcal{J} & 0\\ 0 & \mathcal{J}\end{pmatrix}\mathfrak{X}_k(\lambda) = \begin{pmatrix}\mathcal{J}\\ M\end{pmatrix}^*\begin{pmatrix}\mathcal{H}_k(\lambda) & \mathcal{G}_k(\lambda)\\ \mathcal{G}_k(\lambda) & \mathcal{H}_k(\lambda)\end{pmatrix}\begin{pmatrix}\mathcal{J}\\ M\end{pmatrix}, \tag{3.7}$$

$$\mathcal{H}_k(\lambda) := \tfrac{1}{2}\, i\delta(\lambda)\,[\Phi_k^*(\lambda)\,\mathcal{J}\,\Phi_k(\lambda) - \mathcal{J}], \qquad \mathcal{G}_k(\lambda) := \mathcal{H}_k(\lambda) + i\delta(\lambda)\mathcal{J}. \tag{3.8}$$

From $(i\mathcal{J})^* = i\mathcal{J}$ one can easily see that $\mathcal{E}_k(M)$, $\mathcal{H}_k(\lambda)$, and $\mathcal{G}_k(\lambda)$ are Hermitian matrices. Moreover, we have $\mathcal{H}_0(\lambda) = 0$ and the Lagrange identity in Theorem 2.1.7 implies the following crucial equalities

$$\mathcal{E}_k(M) = -2\delta(\lambda)\,\mathrm{im}(M) + |\,\mathrm{im}(\lambda)|(M^* - \mathcal{J})\left[\sum_{j=0}^{k-1}\Phi_{j+1}^*(\lambda)\,\Psi_j\,\Phi_{j+1}(\lambda)\right](M + \mathcal{J}), \tag{3.9}$$

$$\mathcal{H}_k(\lambda) = |\,\mathrm{im}(\lambda)|\sum_{j=0}^{k-1}\Phi_{j+1}^*(\lambda)\,\Psi_j\,\Phi_{j+1}(\lambda), \tag{3.10}$$

compare with (2.32), (2.36), (2.33), and (2.37), respectively. Since $\Phi(\lambda)$ represents a fundamental matrix of system $(\mathcal{S}_\lambda)$, equality (3.10) justifies the following result.

Theorem 3.1.5. *If Hypothesis 2.4.11 holds, then the matrix $\mathcal{H}_k(\lambda)$ is positive definite for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and $k \in [N_3 + 1, \infty)_{\mathbb{Z}}$. In addition, for such k we have (suppressing the argument λ)*

$$\mathcal{E}_k(M) = -\mathcal{J}\big(\mathcal{H}_k - \mathcal{G}_k\mathcal{H}_k^{-1}\mathcal{G}_k\big)\mathcal{J} + \big(M^* - \mathcal{J}\,\mathcal{G}_k\mathcal{H}_k^{-1}\big)\mathcal{H}_k\big(M + \mathcal{H}_k^{-1}\mathcal{G}_k\,\mathcal{J}\big). \tag{3.11}$$

Proof. The invertibility of $\mathcal{H}_k(\lambda)$ for all $k \in [N_3 + 1, \infty)_{\mathbb{Z}}$ follows from (3.10) and Hypothesis 2.4.11. Moreover, identity (3.11) is a consequence of (3.37) and (3.39). ∎

For any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ we now define the *Weyl disk* $\boldsymbol{D}_k(\lambda)$ and the *Weyl circle* $\boldsymbol{C}_k(\lambda)$ as

$$\boldsymbol{D}_k(\lambda) := \left\{M \in \mathbb{C}^{2n\times 2n} \mid \mathcal{E}_k(M) \le 0\right\} \quad \text{and} \quad \boldsymbol{C}_k(\lambda) := \left\{M \in \mathbb{C}^{2n\times 2n} \mid \mathcal{E}_k(M) = 0\right\},$$

compare with Definition 2.3.1. The following result provides some properties of the elements in $\boldsymbol{D}_k(\lambda)$ and $\boldsymbol{C}_k(\lambda)$. It is a generalization of Theorems 2.3.2 and 2.3.3 to jointly varying endpoints.

Theorem 3.1.6. *Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $k \in [0, \infty)_{\mathbb{Z}}$, and $M \in \mathbb{C}^{2n\times 2n}$. Then the following hold.*

(i) *The matrix $M \in \boldsymbol{C}_k(\lambda)$ if and only if there exists $\gamma \in \Gamma$ such that $\gamma\mathfrak{X}_k(\lambda) = 0$. In this case, we have with such a matrix γ that $M = \boldsymbol{M}_k(\lambda)$, whenever the matrix $\boldsymbol{M}_k(\lambda)$ exists.*

(ii) *The matrix M satisfies $\mathcal{E}_k(M) < 0$ if and only if there exists $\gamma \in \mathbb{C}^{2n\times 4n}$ such that $i\delta(\lambda)\gamma\left(\begin{smallmatrix}-\mathcal{J} & 0\\ 0 & \mathcal{J}\end{smallmatrix}\right)\gamma^* > 0$ and $\gamma\mathfrak{X}_k(\lambda) = 0$. In this case, we have with such a matrix γ that $M = \boldsymbol{M}_k(\lambda)$, whenever the matrix $\boldsymbol{M}_k(\lambda)$ exists, and γ can be chosen so that $\gamma\gamma^* = I$.*

(iii) *We have $\mathcal{E}_k(-\mathcal{J}) = -2\delta(\lambda)i\mathcal{J}$, i.e., $-\mathcal{J} \notin \boldsymbol{D}_k(\lambda)$.*

Proof. Statements (i) and (ii) follow by Theorems 2.3.2 and 2.3.3 from the facts that the sets $\boldsymbol{D}_k(\lambda)$ and $\boldsymbol{C}_k(\lambda)$ coincide respectively with the Weyl disk and Weyl circle in (3.38). Statement (iii) is verified by direct calculation from (3.7), because the matrix $i\mathcal{J}$ is indefinite. ∎

The *center* $\boldsymbol{P}_k(\lambda)$ and the *matrix radius* $\boldsymbol{R}_k(\lambda)$ of the Weyl disk $\boldsymbol{D}_k(\lambda)$ are defined as the $2n \times 2n$ matrices

$$\boldsymbol{P}_k(\lambda) := -\mathcal{H}_k^{-1}(\lambda)\,\mathcal{G}_k(\lambda)\,\mathcal{J} = -\mathcal{J} + i\delta(\lambda)\,\mathcal{H}_k^{-1}(\lambda) \quad \text{and} \quad \boldsymbol{R}_k(\lambda) := \mathcal{H}_k^{-1/2}(\lambda), \tag{3.12}$$

whenever $\mathcal{H}_k(\lambda)$ is invertible, i.e., whenever $\mathcal{H}_k(\lambda) > 0$, compare with (2.42). Note that from (3.8) and (1.5) we get the series expansion

$$\boldsymbol{P}_k(\lambda) = -\mathcal{J} + 2\mathcal{J}\sum_{j=0}^{\infty}\Big[-\Phi_k^*(\lambda)\,\mathcal{J}\,\Phi_k(\lambda)\,\mathcal{J}\Big]^j, \quad \text{when spread } \Phi_k^*(\lambda)\,\mathcal{J}\,\Phi_k(\lambda)\,\mathcal{J} < 1.$$

The following theorem provides the most important geometric properties of the Weyl disks, including their nested property, closedness, and convexity. It is a generalization of Theorems 2.3.6 and 2.3.8 to the case of jointly varying endpoints. Hence we denote by $\mathbb{U}$ and $\mathbb{V}$ the sets of all unitary and contractive $2n \times 2n$ complex matrices, respectively, i.e.,

$$\mathbb{U} := \{U \in \mathbb{C}^{2n\times 2n} \mid U^*U = I\} \quad \text{and} \quad \mathbb{V} := \{V \in \mathbb{C}^{2n\times 2n} \mid V^*V \leq I\}. \tag{3.13}$$

Theorem 3.1.7. *Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then $\boldsymbol{D}_k(\lambda) \subseteq \boldsymbol{D}_j(\lambda)$ for every $k, j \in [0,\infty)_{\mathbb{Z}}$ with $k \geq j$. In addition, under Hypothesis 2.4.11 we have for every $k \in [N_3+1,\infty)_{\mathbb{Z}}$ the representations*

$$\boldsymbol{D}_k(\lambda) = \Big\{\boldsymbol{P}_k(\lambda) + \boldsymbol{R}_k(\lambda)\,V\,\boldsymbol{R}_k(\bar{\lambda}) \mid V \in \mathbb{V}\Big\},$$
$$\boldsymbol{C}_k(\lambda) = \Big\{\boldsymbol{P}_k(\lambda) + \boldsymbol{R}_k(\lambda)\,U\,\boldsymbol{R}_k(\bar{\lambda}) \mid U \in \mathbb{U}\Big\}.$$

Consequently, the Weyl disks $\boldsymbol{D}_k(\lambda)$ are closed and convex for every $k \in [N_3+1,\infty)_{\mathbb{Z}}$.

Proof. The result follows from (3.40), (3.41), and (3.42) combined with Corollary 3.3.4. ∎

The latter theorem implies that the intersection of all Weyl disks $\boldsymbol{D}_k(\lambda)$ over the discrete interval $[N_3+1,\infty)_{\mathbb{Z}}$ is a nonempty, closed, and convex set. This yields that the *limiting Weyl disk* has the form

$$\boldsymbol{D}_+(\lambda) := \bigcap\nolimits_{k\in[N_3+1,\infty)_{\mathbb{Z}}} \boldsymbol{D}_k(\lambda) = \Big\{\boldsymbol{P}_+(\lambda) + \boldsymbol{R}_+(\lambda)\,V\,\boldsymbol{R}_+(\bar{\lambda}) \mid V \in \mathbb{V}\Big\},$$

where $\boldsymbol{P}_+(\lambda)$ and $\boldsymbol{R}_+(\lambda)$ are the $2n \times 2n$ matrices defined by

$$\boldsymbol{P}_+(\lambda) := \lim_{k\to\infty} \boldsymbol{P}_k(\lambda), \qquad \boldsymbol{R}_+(\lambda) := \lim_{k\to\infty} \boldsymbol{R}_k(\lambda) \geq 0. \tag{3.14}$$

They are called the *center* and the *matrix radius* of the limiting Weyl disk $\boldsymbol{D}_+(\lambda)$, compare with Definition 2.3.10. Note that the convergence of $\boldsymbol{P}_k(\lambda)$ and $\boldsymbol{R}_k(\lambda)$ can be seen from their definitions given in (3.12) and equality (3.10).

Remark 3.1.8. If $\mathcal{H}_+^{\mathrm{inv}}(\lambda)$ denotes the limit of $\mathcal{H}_k^{-1}(\lambda)$ as $k \to \infty$, which exists by (3.10), then the formulas in (3.14) for the center and matrix radius of the limiting Weyl disk reduce to

$$\boldsymbol{P}_+(\lambda) = -\mathcal{J} + i\delta(\lambda)\,\mathcal{H}_+^{\mathrm{inv}}(\lambda), \qquad \boldsymbol{R}_+(\lambda) = \Big[\mathcal{H}_+^{\mathrm{inv}}(\lambda)\Big]^{1/2}. \tag{3.15}$$

The next result is a generalization of Corollary 2.3.12 to jointly varying endpoints. Note that as in Theorem 3.1.6(iii) we have $-\mathcal{J} \notin \boldsymbol{D}_+(\lambda)$.

Theorem 3.1.9. *Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $M \in \mathbb{C}^{2n\times 2n}$, and suppose that Hypothesis 2.4.11 holds. Then M belongs to the limiting Weyl disk $\boldsymbol{D}_+(\lambda)$ if and only if*

$$(M^* - \mathcal{J})\left[\sum_{k=0}^{\infty} \Phi_{k+1}^*(\lambda)\,\Psi_k\,\Phi_{k+1}(\lambda)\right](M + \mathcal{J}) \le \frac{2\,\mathrm{im}(M)}{\mathrm{im}(\lambda)}.$$

Proof. This statement follows from (3.9), or alternatively by Corollary 2.3.12 from (3.43) and the definition of the Weyl solution in (3.36) and (3.33). ∎

Remark 3.1.10. The *limiting Weyl circle* $\boldsymbol{C}_+(\lambda)$ can be introduced as the boundary of the limiting Weyl disk $\boldsymbol{D}_+(\lambda)$. Then $M \in \boldsymbol{C}_+(\lambda)$ if and only if any of the following two equivalent conditions hold, compare with Remark 2.3.17(ii),

$$(M^* - \mathcal{J})\left[\sum_{k=0}^{\infty} \Phi_{k+1}^*(\lambda)\,\Psi_k\,\Phi_{k+1}(\lambda)\right](M + \mathcal{J}) = \frac{2\,\mathrm{im}(M)}{\mathrm{im}(\lambda)},$$

$$\lim_{k\to\infty} \mathcal{X}_k^*(\lambda)\begin{pmatrix} -\mathcal{J} & 0 \\ 0 & \mathcal{J} \end{pmatrix}\mathcal{X}_k(\lambda) = 0.$$

Finally, let us discuss some properties of square summable solutions of system ($\mathcal{S}_\lambda$). The following result is quite surprising in the sense that one would expect to have $\boldsymbol{R}_+(\lambda) = 0$ in the limit point case, see Theorem 2.4.3. To the contrary, due to the augmented structure of the matrix $\boldsymbol{R}_+(\lambda)$, which has dimension $2n$, it is the rank of $\boldsymbol{R}_+(\lambda)$ alone which determines the number of linearly independent square summable solutions of system ($\mathcal{S}_\lambda$), compare with Theorem 2.4.8. In the result below we show that $\operatorname{rank} \boldsymbol{R}_+(\lambda) \ge n$, so that the equality $\operatorname{rank} \boldsymbol{R}_+(\lambda) = n$ must necessarily hold in the limit point case. This fact is stated in Corollary 3.1.12 below and also illustrated in Example 3.2.6.

Theorem 3.1.11. *Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and suppose that Hypothesis 2.4.11 holds. Then system ($\mathcal{S}_\lambda$) has exactly* $\operatorname{rank} \boldsymbol{R}_+(\lambda)$ *linearly independent square summable solutions, i.e.,*

$$n \le \dim \mathcal{N}(\lambda) = \operatorname{rank} \boldsymbol{R}_+(\lambda) \le 2n. \tag{3.16}$$

Proof. The statement in (3.16) is proven in Theorem 3.3.6 and (3.44). ∎

The meaning of Theorem 3.1.11 can be explained also directly from (3.10) and Remark 3.1.8. In particular, by (3.15), the rank of $\boldsymbol{R}_+(\lambda)$ is equal to the number of positive eigenvalues of the matrix $\mathcal{H}_+^{\mathrm{inv}}(\lambda)$ from Remark 3.1.8 and this number is the same as the number of the eigenvalues of $\mathcal{H}_k(\lambda)$, which tend to a finite limit as $k \to \infty$. Consequently, equality (3.10) shows that it is equal to the number of linearly independent square summable solutions of system ($\mathcal{S}_\lambda$).

Corollary 3.1.12. *Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and suppose that Hypothesis 2.4.11 holds. Then system ($\mathcal{S}_\lambda$) is in the limit point case if and only if* $\operatorname{rank} \boldsymbol{R}_+(\lambda) = n$*, while ($\mathcal{S}_\lambda$) is in the limit circle case if and only if* $\operatorname{rank} \boldsymbol{R}_+(\lambda) = 2n$.

3.2 Examples

Now we examine several examples which illustrate the theory presented in the previous section. In particular, we consider the periodic and antiperiodic boundary conditions as in [153, Remark 6.17] and the corresponding $\boldsymbol{M}(\lambda)$-function.

Example 3.2.1. For the periodic endpoints $z_0 = z_{N+1}$ we take $\gamma = \frac{1}{\sqrt{2}}(\mathcal{J}, -\mathcal{J}) \in \Gamma$. In this case, the matrix in (3.4) is

$$L(\lambda) = \tfrac{1}{\sqrt{2}}\,\mathcal{J}\,[\Phi_{N+1}(\lambda) + \mathcal{J}]$$

Then, by Theorem 3.1.2, a number $\lambda \in \mathbb{C}$ is an eigenvalue of problem (3.2) if and only if the matrix $\Phi_{N+1}(\lambda) + \mathcal{J}$ is singular, and the number $\dim\operatorname{Ker}[\Phi_{N+1}(\lambda) + \mathcal{J}]$ is its multiplicity. Moreover, the $\boldsymbol{M}(\lambda)$-function in (3.5) reduces to

$$\boldsymbol{M}_k^{[p]}(\lambda) = -[\Phi_k(\lambda) + \mathcal{J}]^{-1}\,[\Phi_k(\lambda) - \mathcal{J}]\,\mathcal{J}. \tag{3.17}$$

▲

Example 3.2.2. For the antiperiodic endpoints $z_0 = -z_{N+1}$ we take $\gamma = \frac{1}{\sqrt{2}}(\mathcal{J}, \mathcal{J}) \in \Gamma$. In this case we have

$$L(\lambda) = \tfrac{1}{\sqrt{2}}\,\mathcal{J}\,[\Phi_{N+1}(\lambda) - \mathcal{J}]$$

and, by Theorem 3.1.2, a number $\lambda \in \mathbb{C}$ is an eigenvalue of problem (3.2) if and only if the matrix $\Phi_{N+1}(\lambda) - \mathcal{J}$ is singular. The multiplicity of λ is then $\dim\operatorname{Ker}[\Phi_{N+1}(\lambda) - \mathcal{J}]$. In addition, the $\boldsymbol{M}(\lambda)$-function in (3.5) now has the form

$$\boldsymbol{M}_k^{[ap]}(\lambda) = -[\Phi_k(\lambda) - \mathcal{J}]^{-1}\,[\Phi_k(\lambda) + \mathcal{J}]\,\mathcal{J}. \tag{3.18}$$

▲

The $\boldsymbol{M}(\lambda)$-functions $\boldsymbol{M}_k^{[p]}(\lambda)$ and $\boldsymbol{M}_k^{[ap]}(\lambda)$ from (3.17) and (3.18) for the periodic and antiperiodic endpoints are closely related, as we show in the next interesting statement. Let Λ_p be the set of all eigenvalues of the periodic problem (3.2) with $\gamma = \gamma_p$ from Example 3.2.1. Similarly, let Λ_{ap} be the set of all eigenvalues of the antiperiodic problem (3.2) with $\gamma = \gamma_{ap}$ from Example 3.2.2. Then by Theorem 3.1.2(ii) we have $\Lambda_p \cup \Lambda_{ap} \subseteq \mathbb{R}$ under Hypothesis 3.1.1.

Corollary 3.2.3. *Let $k \in [0, N+1]_{\mathbb{Z}}$ and $\lambda \in \mathbb{C}$ be fixed. The matrices $\boldsymbol{M}_k^{[p]}(\lambda)$ and $\boldsymbol{M}_k^{[ap]}(\lambda)$ given in (3.17) and (3.18) satisfy the following conditions.*

(i) *If $\Phi_k(\lambda) + \mathcal{J}$ is invertible, then $\operatorname{rank}\boldsymbol{M}_k^{[p]}(\lambda) = \operatorname{rank}[\Phi_k(\lambda) - \mathcal{J}]$. In particular, this equality holds at $k = N+1$ for every $\lambda \notin \Lambda_p$.*

(ii) *If $\Phi_k(\lambda) - \mathcal{J}$ is invertible, then $\operatorname{rank}\boldsymbol{M}_k^{[ap]}(\lambda) = \operatorname{rank}[\Phi_k(\lambda) + \mathcal{J}]$. In particular, this equality holds at $k = N+1$ for every $\lambda \notin \Lambda_{ap}$.*

(iii) *If $\Phi_k(\lambda) + \mathcal{J}$ and $\Phi_k(\lambda) - \mathcal{J}$ are invertible, then $\boldsymbol{M}_k^{[p]}(\lambda)$ and $\boldsymbol{M}_k^{[ap]}(\lambda)$ are also invertible and satisfy the equalities*

$$\Big[\boldsymbol{M}_k^{[p]}(\lambda)\,\mathcal{J}\Big]^{-1} = \boldsymbol{M}_k^{[ap]}(\lambda)\,\mathcal{J} \quad \text{and} \quad \det\boldsymbol{M}_k^{[p]}(\lambda) \times \det\boldsymbol{M}_k^{[ap]}(\lambda) = 1.$$

In particular, these equalities hold at $k = N+1$ for every $\lambda \notin \Lambda_p \cup \Lambda_{ap}$.

As an addendum to Corollary 3.2.3 we derive the series representations for $\boldsymbol{M}_k^{[p]}(\lambda)$ and $\boldsymbol{M}_k^{[ap]}(\lambda)$. If we expand the inverses in (3.17) and (3.18) by (1.5), then under the condition $\operatorname{sprad}\Phi_k(\lambda)\,\mathcal{J} < 1$ we obtain

$$\boldsymbol{M}_k^{[p]}(\lambda) = 2\mathcal{J}\bigg(\sum_{j=0}^{\infty}\Big[\Phi_k(\lambda)\,\mathcal{J}\Big]^j\bigg) - \mathcal{J} \quad \text{and} \quad \boldsymbol{M}_k^{[ap]}(\lambda) = 2\mathcal{J}\bigg(\sum_{j=0}^{\infty}\Big[-\Phi_k(\lambda)\,\mathcal{J}\Big]^j\bigg) - \mathcal{J}.$$

Now, we illustrate our results on the scalar symplectic system

$$z_{k+1}(\lambda) = \begin{pmatrix} 1 & 1 \\ -\lambda & 1-\lambda \end{pmatrix} z_k(\lambda) \quad \text{with} \quad \Psi_k \equiv \Psi := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.19}$$

i.e., $\mathcal{S}_k \equiv \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathcal{V}_k \equiv \left(\begin{smallmatrix} 0 & 0 \\ -1 & -1 \end{smallmatrix}\right)$. This system corresponds to the second order Sturm–Liouville difference equation (2.8) with $m = n = 1$, $P_k^{[1]} = \mathcal{W}_k \equiv 1$, and $P_k^{[0]} \equiv 0$, i.e.,

$$-\Delta(p_k \Delta y_k) + q_k y_{k+1} = \lambda w_k y_{k+1}, \quad \text{where } p_k = w_k \equiv 1,\ q_k \equiv 0. \tag{3.20}$$

System (3.19) satisfies the strong Atkinson condition in Hypothesis 3.1.1 or 2.4.11 with $N_3 = 1$, as can be easily verified.

Example 3.2.4. Let us consider the scalar eigenvalue problem with periodic endpoints

$$(3.19), \quad k \in [0,3]_{\mathbb{Z}}, \quad z_0 = z_4, \tag{3.21}$$

i.e., we look for the solutions of system (3.19) with period 4. This problem corresponds to the periodic Sturm–Liouville eigenvalue problem (3.20) with $k \in [0,3]_{\mathbb{Z}}$ and $y_0 = y_4$, $\Delta y_0 = \Delta y_4$, which was studied in [105, Example 7.6]. It was shown in the latter reference that $\lambda = 2$ is a double eigenvalue of problem (3.21) by finding two linearly independent eigenfunctions. The results in Theorem 3.1.2 and Example 3.2.1 confirm this conclusion. The fundamental matrix $\Phi(\lambda)$ from (3.3) now satisfies

$$\Phi_2(\lambda) = \begin{pmatrix} -\lambda+2 & \lambda-1 \\ \lambda^2-3\lambda+1 & -\lambda^2+2\lambda \end{pmatrix}, \tag{3.22}$$

$$\Phi_4(\lambda) = \begin{pmatrix} -\lambda^3+6\lambda^2-10\lambda+4 & \lambda^3-5\lambda^2+6\lambda-1 \\ \lambda^4-7\lambda^3+15\lambda^2-10\lambda+1 & -\lambda^4+6\lambda^3-10\lambda^2+4\lambda \end{pmatrix}. \tag{3.23}$$

This yields that $\det[\Phi_4(\lambda) + \mathcal{I}] = -\lambda(\lambda-4)(\lambda-2)^2$. Thus, by Theorem 3.1.2 and Example 3.2.1, $\lambda = 2$ is indeed a double eigenvalue of problem (3.21), and the columns of $\Phi(2)$ are the two linearly independent eigenfunctions. Note that it holds $\Phi_4(2) = -\mathcal{I} = \Phi_0(2)$. The other eigenvalues of problem (3.21) are $\lambda = 0$ with the eigenfunction $\Phi(0)\,(0,\ 1)^\top$ and $\lambda = 4$ with the eigenfunction $\Phi(4)\,(2,\ 1)^\top$. ▲

Example 3.2.5. Let us consider again system (3.19), but now only on the interval $[0,2]_{\mathbb{Z}}$ and with the antiperiodic boundary conditions $z_0 = -z_2$. From equality (3.22) we see that $\det[\Phi_2(\lambda) - \mathcal{I}] = (\lambda-2)^2$. Hence, by Theorem 3.1.2 and Example 3.2.2, $\lambda = 2$ is a double eigenvalue of this problem with the columns of $\Phi(2)$ as the two linearly independent eigenfunctions. Note that it holds $\Phi_2(2) = \mathcal{I} = -\Phi_0(2)$. This problem then does not have any other eigenvalues. ▲

In the last example we calculate the rank of the limiting radius $\boldsymbol{R}_+(\lambda)$ and compare it with the corresponding number of linearly independent square summable solutions.

Example 3.2.6. We examine system (3.19) on the discrete interval $[0,\infty)_{\mathbb{Z}}$ with a particular choice of $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$. We show that system (3.19) with $\lambda = \lambda_0$ is in the limit point case, so that by Corollary 2.4.22 it is in the limit point case for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Let $\lambda_0 = 2 + 2i\sqrt{3}$, i.e., system (3.19) reduces to the second order difference equation $y_{k+2} + 2i\sqrt{3}\,y_{k+1} + y_k = 0$ on $[0,\infty)_{\mathbb{Z}}$. The roots of the corresponding characteristic polynomial are $\nu_\pm := (\pm 2 - \sqrt{3})\,i$, so that the fundamental matrix $\Phi(\lambda)$ of system (3.19) satisfying (3.3) has the form

$$\Phi_k(\lambda_0) = \begin{pmatrix} \nu_+^k & \nu_-^k \\ \nu_+^k(\nu_+-1) & \nu_-^k(\nu_--1) \end{pmatrix} T, \quad \text{where} \quad T := \tfrac{1}{4} \begin{pmatrix} -i & -2-\sqrt{3}+i \\ i & -2+\sqrt{3}-i \end{pmatrix}. \tag{3.24}$$

By (3.10), we obtain with $\mu_\pm := |\nu_\pm|^2 = 7 \mp 4\sqrt{3}$ that

$$\mathcal{H}_k(\lambda_0) = 2\sqrt{3}\, T^* \sum_{j=0}^{k-1} \begin{pmatrix} |\nu_+|^{2j+2} & (\bar{\nu}_+\nu_-)^{j+1} \\ (\nu_+\bar{\nu}_-)^{j+1} & |\nu_-|^{2j+2} \end{pmatrix} T = 2\sqrt{3}\, T^* \sum_{j=0}^{k-1} \begin{pmatrix} (\mu_+)^{j+1} & (-1)^{j+1} \\ (-1)^{j+1} & (\mu_-)^{j+1} \end{pmatrix} T.$$

Since each entry of $\mathcal{H}_k(\lambda_0)$ represents a geometric series, it can be evaluated explicitly as

$$\mathcal{H}_k(\lambda_0) = 2\sqrt{3}\, T^* \begin{pmatrix} (\mu_+)[1-(\mu_+)^k]/(1-\mu_+) & [(-1)^k - 1]/2 \\ [(-1)^k - 1]/2 & (\mu_-)[1-(\mu_-)^k]/(1-\mu_-) \end{pmatrix} T.$$

Therefore, the matrix $\mathcal{H}_k(\lambda_0)$ is indeed invertible (and positive definite) on the discrete interval $[N_3+1,\infty)_{\mathbb{Z}} = [2,\infty)_{\mathbb{Z}}$ and, by Remark 3.1.8, we have

$$\mathcal{H}_+^{\mathrm{inv}}(\lambda_0) = \lim_{k\to\infty} \mathcal{H}_k^{-1}(\lambda_0) = \begin{pmatrix} 4 & 2+\sqrt{3}-i \\ 2+\sqrt{3}+i & 2+\sqrt{3} \end{pmatrix},$$

$$\boldsymbol{P}_+(\lambda_0) = -\mathcal{J} + i\mathcal{H}_+^{\mathrm{inv}}(\lambda_0) = \begin{pmatrix} 4i & (2+\sqrt{3})i \\ (2+\sqrt{3})i & (2+\sqrt{3})i \end{pmatrix}.$$

The matrix $\boldsymbol{R}_+(\lambda_0)$ can also be calculated explicitly by (3.15), but it is not really important. We can find the eigenvalues of $\boldsymbol{R}_+(\lambda_0)$ as the nonnegative square roots of the eigenvalues of $\mathcal{H}_+^{\mathrm{inv}}(\lambda_0)$. Namely, since the eigenvalues of the matrix $\mathcal{H}_+^{\mathrm{inv}}(\lambda_0)$ are 0 and $6+\sqrt{3}$, we obtain $\operatorname{rank} \boldsymbol{R}_+(\lambda_0) = 1$ and system (3.19) is in the limit point case, by Corollary 3.1.12. The square summable solution of (3.19) is then given as the second component of the columns of the Weyl solution $\mathcal{X}_+(\lambda_0)$ from (3.6) with $M = \boldsymbol{P}_+(\lambda_0)$. That is, the columns of the matrix $\Phi(\lambda_0)\times\mathcal{H}_+^{\mathrm{inv}}(\lambda_0)$ are square summable. But since $\mathcal{H}_+^{\mathrm{inv}}(\lambda_0)$ is singular, it follows that the square summable solutions of (3.19) are generated by exactly one column of the matrix $\Phi(\lambda_0)\times\mathcal{H}_+^{\mathrm{inv}}(\lambda_0)$. On the other hand, since $|\nu_+|<1$, one can identify the first column of $\Phi(\lambda_0)\times T^{-1}$ in (3.24) as the square summable solution of (3.19). ▲

3.3 Augmented symplectic system

We will show that problem (3.2) is equivalent to a certain eigenvalue problem in dimension $4n$ with separated endpoints. At first, let us define the $4n\times 4n$ matrices

$$\boldsymbol{\mathcal{S}}_k := \begin{pmatrix} I & 0 \\ 0 & \mathcal{S}_k \end{pmatrix}, \quad \boldsymbol{\mathcal{V}}_k := \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{V}_k \end{pmatrix}, \quad \boldsymbol{\Psi}_k := \begin{pmatrix} 0 & 0 \\ 0 & \Psi_k \end{pmatrix} \geq 0, \tag{3.25}$$

$$\boldsymbol{\Phi}_k(\lambda) := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{J} & -\mathcal{J} \\ \Phi_k(\lambda) & \Phi_k(\lambda) \end{pmatrix}, \quad \boldsymbol{\mathcal{J}} := \begin{pmatrix} -\mathcal{J} & 0 \\ 0 & \mathcal{J} \end{pmatrix}, \quad \boldsymbol{\mathcal{K}} := \begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}, \tag{3.26}$$

where $\Phi(\lambda)$ is the fundamental matrix of system $(\mathcal{S}_\lambda)$ specified in (3.3). Then one can easily verify that $\boldsymbol{\mathcal{J}}^* = -\boldsymbol{\mathcal{J}} = \boldsymbol{\mathcal{J}}^{-1}$, $\boldsymbol{\mathcal{K}}^* = -\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{K}}^{-1}$, and

$$\boldsymbol{\mathcal{S}}_k^* \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{S}}_k = \boldsymbol{\mathcal{J}}, \quad \boldsymbol{\mathcal{S}}_k^* \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{V}}_k \text{ is Hermitian}, \quad \boldsymbol{\mathcal{V}}_k^* \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{V}}_k = 0, \quad \boldsymbol{\Psi}_k = \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{V}}_k \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{S}}_k^* \boldsymbol{\mathcal{J}}.$$

With this setting, we introduce the augmented symplectic system

$$\boldsymbol{z}_{k+1}(\lambda) = (\boldsymbol{\mathcal{S}}_k + \lambda \boldsymbol{\mathcal{V}}_k)\, \boldsymbol{z}_k(\lambda). \tag{$\boldsymbol{\mathcal{S}}_\lambda$}$$

It follows that $\boldsymbol{\Phi}(\lambda)$ is a fundamental matrix of system $(\boldsymbol{\mathcal{S}}_\lambda)$, because

$$\boldsymbol{\Phi}_k(\lambda) = \begin{pmatrix} I & 0 \\ 0 & \Phi_k(\lambda)\mathcal{J} \end{pmatrix} \boldsymbol{Q} \quad \text{with} \quad \boldsymbol{Q} := \boldsymbol{\Phi}_0(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{J} & -\mathcal{J} \\ -\mathcal{J} & -\mathcal{J} \end{pmatrix} \quad \text{and} \quad \det \boldsymbol{Q} = 1,$$

Moreover, by analogy with Lemma 2.1.6 we obtain on $[0,\infty)_{\mathbb{Z}}$ the identities

$$\boldsymbol{\Phi}_k^*(\lambda)\,\mathfrak{J}\,\boldsymbol{\Phi}_k(\bar\lambda) = \mathfrak{K}, \quad \boldsymbol{\Phi}_k(\bar\lambda)\,\mathfrak{K}\,\boldsymbol{\Phi}_k^*(\lambda) = \mathfrak{J}, \quad \text{and} \quad \boldsymbol{\Phi}_0^*(\lambda)\,\boldsymbol{\Phi}_0(\lambda) = I.$$

Note that the latter equality means that the matrix $\boldsymbol{\Phi}_0(\lambda)$ is unitary. The double size of system ($\boldsymbol{\mathcal{S}}_\lambda$) implies that we consider all vector solutions $\boldsymbol{z}(\lambda)$ of system ($\boldsymbol{\mathcal{S}}_\lambda$) in dimension $4n$ and matrix-valued solutions $\boldsymbol{Z}(\lambda)$ of system ($\boldsymbol{\mathcal{S}}_\lambda$) in dimension $4n\times 2n$. It then follows that they have the form

$$\boldsymbol{z}_k(\lambda) = \begin{pmatrix} d \\ z_k(\lambda) \end{pmatrix} \quad \text{and} \quad \boldsymbol{Z}_k(\lambda) = \begin{pmatrix} D & E \\ Z_k^{[1]}(\lambda) & Z_k^{[2]}(\lambda) \end{pmatrix}, \tag{3.27}$$

where $d \in \mathbb{C}^n$ and $D, E \in \mathbb{C}^{2n\times n}$ are constant, while the sequences $z(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ and $Z^{[1]}(\lambda), Z^{[2]}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ solve system ($\mathcal{S}_\lambda$). It turns out that the main properties of system ($\mathcal{S}_\lambda$) and its solutions, such as those in Section 2.1 or [A4, Section 2], are preserved for the augmented system ($\boldsymbol{\mathcal{S}}_\lambda$). In particular, the coefficients of system ($\boldsymbol{\mathcal{S}}_\lambda$) satisfy the following identities

$$(\boldsymbol{\mathcal{S}}_k + \lambda\boldsymbol{\mathcal{V}}_k)^*\,\mathfrak{J}\,(\boldsymbol{\mathcal{S}}_k + \bar\lambda\boldsymbol{\mathcal{V}}_k) = \mathfrak{J}, \quad (\boldsymbol{\mathcal{S}}_k + \lambda\boldsymbol{\mathcal{V}}_k)^{-1} = -\mathfrak{J}\,(\boldsymbol{\mathcal{S}}_k^* + \lambda\boldsymbol{\mathcal{V}}_k^*)\,\mathfrak{J},$$

and we have also the Lagrange identity for two $4n\times m$ matrix solutions of systems ($\boldsymbol{\mathcal{S}}_\lambda$) and ($\boldsymbol{\mathcal{S}}_\nu$), i.e., for all $\lambda,\nu \in \mathbb{C}$ and $k \in [0,\infty)_{\mathbb{Z}}$ it holds

$$\boldsymbol{Z}_{k+1}^*(\lambda)\,\mathfrak{J}\,\boldsymbol{Z}_{k+1}(\nu) = \boldsymbol{Z}_0^*(\lambda)\,\mathfrak{J}\,\boldsymbol{Z}_0(\nu) + (\bar\lambda - \nu)\sum_{j=0}^{k} \boldsymbol{Z}_{j+1}^*(\lambda)\,\boldsymbol{\Psi}_j\,\boldsymbol{Z}_{j+1}(\nu). \tag{3.28}$$

Given $\gamma \in \Gamma$ from (3.1), we define the $2n\times 4n$ matrices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{\Gamma} := \Gamma$ by

$$\boldsymbol{\alpha} := \tfrac{1}{\sqrt{2}}\begin{pmatrix} -I & I \end{pmatrix} \quad \text{and} \quad \boldsymbol{\beta} := \gamma. \tag{3.29}$$

Since vector solutions of system ($\boldsymbol{\mathcal{S}}_\lambda$) can be written as in (3.27), the choice of $\boldsymbol{\alpha}$ in (3.29) implies that the solutions of ($\boldsymbol{\mathcal{S}}_\lambda$) satisfying $\boldsymbol{\alpha}\,\boldsymbol{z}_0(\lambda) = 0$ have necessarily the form

$$\boldsymbol{z}_k(\lambda) = \begin{pmatrix} z_0(\lambda) \\ z_k(\lambda) \end{pmatrix}, \tag{3.30}$$

where $z(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ solves system ($\mathcal{S}_\lambda$). Conversely, every solution $z(\lambda)$ of system ($\mathcal{S}_\lambda$) yields through formula (3.30) a solution $\boldsymbol{z}(\lambda)$ of system ($\boldsymbol{\mathcal{S}}_\lambda$) such that $\boldsymbol{\alpha}\,\boldsymbol{z}_0(\lambda) = 0$. Therefore, the original eigenvalue problem (3.2) is equivalent to the augmented eigenvalue problem with separated endpoints

$$(\boldsymbol{\mathcal{S}}_\lambda), \quad k \in [0,N]_{\mathbb{Z}}, \quad \lambda \in \mathbb{C}, \quad \boldsymbol{\alpha}\,\boldsymbol{z}_0 = 0, \quad \boldsymbol{\beta}\,\boldsymbol{z}_{N+1} = 0. \tag{3.31}$$

In addition, the form of $\boldsymbol{\Psi}_k$ in (3.25) implies that the semi-norm of any augmented solution $\boldsymbol{z}(\lambda)$ is the same as the semi-norm of the corresponding solution $z(\lambda)$, because

$$\langle \boldsymbol{z}, \tilde{\boldsymbol{z}} \rangle_{\boldsymbol{\Psi},N} := \sum_{k=0}^{N} \boldsymbol{z}_{k+1}^*\,\boldsymbol{\Psi}_k\,\tilde{\boldsymbol{z}}_{k+1} = \sum_{k=0}^{N} z_{k+1}^*\,\Psi_k \tilde{z}_{k+1} = \langle z, \tilde{z} \rangle_{\Psi,N} \text{ with } \boldsymbol{z}_k = \begin{pmatrix} z_0 \\ z_k \end{pmatrix}, \ \tilde{\boldsymbol{z}}_k = \begin{pmatrix} \tilde{z}_0 \\ \tilde{z}_k \end{pmatrix}. \tag{3.32}$$

In order to apply the theory from Sections 2.2 and 2.3, we need to find the fundamental matrix $\widetilde{\mathbf{\Phi}}(\lambda)$ of the augmented system $(\boldsymbol{\mathcal{S}}_\lambda)$ such that $\widetilde{\mathbf{\Phi}}_0(\lambda) = (\boldsymbol{\alpha}^*, -\boldsymbol{\mathcal{J}}\boldsymbol{\alpha}^*)$, see (2.21) and (2.22). That is,

$$\widetilde{\mathbf{\Phi}}_k(\lambda) = \big(\mathbf{Z}_k(\lambda), \widetilde{\mathbf{Z}}_k(\lambda)\big), \quad \mathbf{Z}_k(\lambda) := \tfrac{1}{\sqrt{2}}\begin{pmatrix} -I \\ \Phi_k(\lambda)\mathcal{J} \end{pmatrix}, \quad \widetilde{\mathbf{Z}}_k(\lambda) := \tfrac{1}{\sqrt{2}}\begin{pmatrix} -\mathcal{J} \\ \Phi_k(\lambda) \end{pmatrix}. \tag{3.33}$$

with $\Phi(\lambda)$ being the fundamental matrix of $(\mathcal{S}_\lambda)$ used in (3.26). The above transformation then yields the results for the eigenvalue problem (3.31) in terms of $\widetilde{\mathbf{\Phi}}(\lambda)$. When translating these results to the data of the original problem (3.2) we use the fundamental matrix $\mathbf{\Phi}(\lambda)$ in (3.26). Its relationship with $\widetilde{\mathbf{\Phi}}(\lambda)$ is given by the equality

$$\widetilde{\mathbf{\Phi}}_k(\lambda) = \mathbf{\Phi}_k(\lambda)\boldsymbol{\mathcal{L}}, \quad \text{where} \quad \boldsymbol{\mathcal{L}} := \begin{pmatrix} \mathcal{J} & 0 \\ 0 & I \end{pmatrix}. \tag{3.34}$$

From (3.34) we can see that the second column $\widetilde{\mathbf{Z}}(\lambda)$ of $\widetilde{\mathbf{\Phi}}(\lambda)$ and $\mathbf{\Phi}(\lambda)$ is the same. Finally, the theory also requires the following weak Atkinson-type conditions for system $(\boldsymbol{\mathcal{S}}_\lambda)$, compare with Hypotheses 2.2.2 and 2.3.7.

Hypothesis 3.3.1 (Weak augmented Atkinson condition – finite). For any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ every column $\boldsymbol{z}(\lambda)$ of the solution $\widetilde{\mathbf{Z}}(\lambda)$ satisfies

$$\sum_{k=0}^{N} \boldsymbol{z}_{k+1}^*(\lambda)\,\mathbf{\Psi}_k\,\boldsymbol{z}_{k+1}(\lambda) > 0. \tag{3.35}$$

Hypothesis 3.3.2 (Weak augmented Atkinson condition – infinite). There exists a number $N_3 \in [0,\infty)_{\mathbb{Z}}$ such that each column $\boldsymbol{z}(\lambda)$ of $\widetilde{\mathbf{Z}}(\lambda)$ satisfies inequality (3.35) with $N = N_3$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

When we write the weak Atkinson condition in (3.35) in terms of the data of the original problem (3.2), we get by (3.25) and (3.33) that

$$\sum_{k=0}^{N} \widetilde{\mathbf{Z}}_{k+1}^*(\lambda)\,\mathbf{\Psi}_k\,\widetilde{\mathbf{Z}}_{k+1}(\lambda) = \sum_{k=0}^{N} \Phi_{k+1}^*(\lambda)\,\Psi_k\,\Phi_{k+1}(\lambda) > 0.$$

This shows that the conditions in Hypotheses 3.1.1 and 3.3.1 are intimately connected as stated in the following corollary.

Corollary 3.3.3. *System $(\mathcal{S}_\lambda)$ satisfies the strong Atkinson condition on the discrete interval $[0,N]_{\mathbb{Z}}$ (Hypothesis 3.1.1) if and only if the augmented system $(\boldsymbol{\mathcal{S}}_\lambda)$ satisfies the corresponding weak Atkinson condition on $[0,N]_{\mathbb{Z}}$ (Hypothesis 3.3.1).*

Similar connection is true also for Hypotheses 2.4.11 and 3.3.2. This fact justifies the use of the same index N_3 in both conditions.

Corollary 3.3.4. *System $(\mathcal{S}_\lambda)$ satisfies the strong Atkinson condition on the discrete interval $[0,\infty)_{\mathbb{Z}}$ (Hypothesis 2.4.11) if and only if the augmented system $(\boldsymbol{\mathcal{S}}_\lambda)$ satisfies the corresponding weak Atkinson condition on $[0,\infty)_{\mathbb{Z}}$ (Hypothesis 3.3.2).*

In particular, we can see why assuming the weak Atkinson condition in Chapter 2 is really essential – the transformation of the problem (3.2) with jointly varying endpoints, which satisfies the strong Atkinson condition, leads to the augmented problem (3.31) satisfying the corresponding weak Atkinson condition. Therefore, one can simply apply the previous results on separated endpoints to the augmented problem and then transform the obtained results back to the data of the original problem (3.2).

The next theorem provides basic properties of problem (3.31).

Theorem 3.3.5. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{\Gamma}$ be given. Then the following statements hold.*

(i) *A number $\lambda \in \mathbb{C}$ is an eigenvalue of (3.31) if and only if the matrix $\boldsymbol{L}(\lambda) := \boldsymbol{\beta} \widetilde{\boldsymbol{Z}}_{N+1}(\lambda)$ is singular. In this case, the eigenfunctions of problem (3.31) corresponding to the eigenvalue λ have the form $z = \widetilde{\boldsymbol{Z}}_k(\lambda) d$ on $[0, N+1]_{\mathbb{Z}}$ with a nonzero $d \in \operatorname{Ker} \boldsymbol{L}(\lambda)$. Moreover, the geometric and algebraic multiplicities of λ coincide and are equal to $\dim \operatorname{Ker} \boldsymbol{L}(\lambda)$.*

(ii) *Under Hypothesis 3.3.1, the eigenvalues of problem (3.31) are real and the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the semi-inner product $\langle \cdot, \cdot \rangle_{\boldsymbol{\Psi}, N}$ defined in (3.32).*

Proof. The statement follows from Theorem 2.2.3, when it is applied to the augmented eigenvalue problem (3.31). ∎

Following Definitions 2.2.1 and 2.2.4, we define the Weyl solution for system ($\mathcal{S}_\lambda$) corresponding to $M \in \mathbb{C}^{2n\times 2n}$ and the $M(\lambda)$-function associated with problem (3.31) as

$$\mathcal{X}_k(\lambda) := \widetilde{\boldsymbol{\Phi}}_k(\lambda) \begin{pmatrix} I \\ M \end{pmatrix} = \boldsymbol{\Phi}_k(\lambda) \begin{pmatrix} \mathcal{J} \\ M \end{pmatrix} \quad \text{and} \quad \boldsymbol{M}_k(\lambda) := -\left[\boldsymbol{\beta} \widetilde{\boldsymbol{Z}}_k(\lambda)\right]^{-1} \boldsymbol{\beta} \boldsymbol{Z}_k(\lambda), \tag{3.36}$$

where $\widetilde{\boldsymbol{\Phi}}(\lambda)$, $\boldsymbol{Z}(\lambda)$, and $\widetilde{\boldsymbol{Z}}(\lambda)$ are given in (3.33). In addition, for $M \in \mathbb{C}^{2n\times 2n}$ we define the $\mathcal{E}(M)$-function by

$$\mathcal{E}_k(M) := i\delta(\lambda) \mathcal{X}_k^*(\lambda) \boldsymbol{\mathcal{J}} \mathcal{X}_k(\lambda) = \begin{pmatrix} I \\ M \end{pmatrix}^* \begin{pmatrix} \mathcal{F}_k(\lambda) & \mathcal{G}_k^*(\lambda) \\ \mathcal{G}_k(\lambda) & \mathcal{H}_k(\lambda) \end{pmatrix} \begin{pmatrix} I \\ M \end{pmatrix}, \tag{3.37}$$

where $\mathcal{F}_k(\lambda)$, $\mathcal{G}_k(\lambda)$, and $\mathcal{H}_k(\lambda)$ are the $2n \times 2n$ matrices

$$\begin{aligned}
\mathcal{F}_k(\lambda) &:= i\delta(\lambda) \boldsymbol{Z}_k^*(\lambda) \boldsymbol{\mathcal{J}} \boldsymbol{Z}_k(\lambda) = \mathcal{J}^* \mathcal{H}_k(\lambda) \mathcal{J}, \\
\mathcal{G}_k(\lambda) &:= i\delta(\lambda) \widetilde{\boldsymbol{Z}}_k^*(\lambda) \boldsymbol{\mathcal{J}} \boldsymbol{Z}_k(\lambda) = \mathcal{H}_k(\lambda) \mathcal{J} - i\delta(\lambda), \\
\mathcal{H}_k(\lambda) &:= i\delta(\lambda) \widetilde{\boldsymbol{Z}}_k^*(\lambda) \boldsymbol{\mathcal{J}} \widetilde{\boldsymbol{Z}}_k(\lambda) = \tfrac{1}{2} i\delta(\lambda) [\Phi_k^*(\lambda) \mathcal{J} \Phi_k(\lambda) - \mathcal{J}].
\end{aligned}$$

As in Definition 2.3.1, the Weyl disk $\boldsymbol{D}_k(\lambda)$ and the Weyl circle $\boldsymbol{C}_k(\lambda)$ are defined by

$$\boldsymbol{D}_k(\lambda) := \left\{ M \in \mathbb{C}^{2n\times 2n} \mid \mathcal{E}_k(M) \le 0 \right\}, \qquad \boldsymbol{C}_k(\lambda) := \left\{ M \in \mathbb{C}^{2n\times 2n} \mid \mathcal{E}_k(M) = 0 \right\}. \tag{3.38}$$

Note that under Hypothesis 3.3.2 the matrices $\mathcal{H}_k(\lambda)$ are positive definite (and hence invertible) for all $k \in [N_3 + 1, \infty)_{\mathbb{Z}}$, because by (3.28) we have

$$\mathcal{H}_k(\lambda) = 2 |\operatorname{im}(\lambda)| \sum_{j=0}^{k-1} \widetilde{\boldsymbol{Z}}_{j+1}^*(\lambda) \boldsymbol{\Psi}_j \widetilde{\boldsymbol{Z}}_{j+1}(\lambda). \tag{3.39}$$

By Theorem 2.3.8, the Weyl disk and the Weyl circle possess the representations

$$\boldsymbol{D}_k(\lambda) = \left\{ \boldsymbol{P}_k(\lambda) + \boldsymbol{R}_k(\lambda) V \boldsymbol{R}_k(\bar{\lambda}) \mid V \in \mathbb{V} \right\}, \tag{3.40}$$

$$\boldsymbol{C}_k(\lambda) = \left\{ \boldsymbol{P}_k(\lambda) + \boldsymbol{R}_k(\lambda) U \boldsymbol{R}_k(\bar{\lambda}) \mid U \in \mathbb{U} \right\}, \tag{3.41}$$

where $\mathbb{V}$ and $\mathbb{U}$ are, respectively, the sets of all complex contractive and unitary $2n \times 2n$ matrices introduced in (3.13) and where the center $\boldsymbol{P}_k(\lambda)$ and the matrix radius $\boldsymbol{R}_k(\lambda)$ are defined by

$$\boldsymbol{P}_k(\lambda) := -\mathcal{H}_k^{-1}(\lambda) \mathcal{G}_k(\lambda) = -\mathcal{J} + i\delta(\lambda) \mathcal{H}_k^{-1}(\lambda), \qquad \boldsymbol{R}_k(\lambda) := \mathcal{H}_k^{-1/2}(\lambda). \tag{3.42}$$

Therefore, the Weyl disks $\boldsymbol{D}_k(\lambda)$ are closed, convex, and nested, which implies that the limiting Weyl disk

$$\boldsymbol{D}_+(\lambda) := \bigcap\nolimits_{k\in[N_3+1,\infty)_{\mathbb{Z}}} \boldsymbol{D}_k(\lambda) = \left\{\boldsymbol{P}_+(\lambda) + \boldsymbol{R}_+(\lambda)\, V\, \boldsymbol{R}_+(\bar{\lambda}) \mid V \in \mathbb{V}\right\} \tag{3.43}$$

exists and is nonempty, closed, and convex as well. By using Theorem 2.3.9 and the monotonicity of $\mathcal{H}_k(\lambda)$ shown in (3.39), the center and the matrix radius of $\boldsymbol{D}_+(\lambda)$ are

$$\boldsymbol{P}_+(\lambda) := \lim_{k\to\infty} \boldsymbol{P}_k(\lambda) \quad \text{and} \quad \boldsymbol{R}_+(\lambda) := \lim_{k\to\infty} \boldsymbol{R}_k(\lambda) \geq 0.$$

The final results of this section are devoted to the square integrable solutions of the augmented system ($\mathcal{S}_\lambda$). Let $\boldsymbol{\ell}^2_{\boldsymbol{\Psi}}$ be the space of all square summable sequences $\boldsymbol{z} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{4n}$ with the corresponding semi-norm defined as

$$\|\boldsymbol{z}\|_{\boldsymbol{\Psi}} < \infty, \quad \text{where} \quad \|\boldsymbol{z}\|_{\boldsymbol{\Psi}} := \left(\sum_{k=0}^{\infty} \boldsymbol{z}^*_{k+1} \boldsymbol{\Psi}_k\, \boldsymbol{z}_{k+1}\right)^{1/2} = \lim_{N\to\infty} \sqrt{\langle \boldsymbol{z}, \boldsymbol{z}\rangle_{\boldsymbol{\Psi},N}}.$$

For every $\lambda \in \mathbb{C}$ we denote the space of all square summable solutions of system ($\mathcal{S}_\lambda$) by

$$\boldsymbol{\mathcal{N}}(\lambda) := \left\{\boldsymbol{z} \in \boldsymbol{\ell}^2_{\boldsymbol{\Psi}} \mid \boldsymbol{z} \text{ solves } (\mathcal{S}_\lambda)\right\}.$$

If Hypothesis 3.3.2 is satisfied, we know from Theorem 2.4.1 that the dimension of $\boldsymbol{\mathcal{N}}(\lambda)$ is at least $2n$, or more precisely

$$\dim \boldsymbol{\mathcal{N}}(\lambda) = 2n + \operatorname{rank} \boldsymbol{R}_+(\lambda), \tag{3.44}$$

by Theorem 2.4.8. On the other hand, the analysis of the structure of the square summable solutions of the augmented system ($\mathcal{S}_\lambda$) yields the following result.

Theorem 3.3.6. *Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and suppose that Hypothesis 3.3.2 holds. Then*

$$3n \leq 2n + \dim \mathcal{N}(\lambda) = \dim \boldsymbol{\mathcal{N}}(\lambda) \leq 4n \tag{3.45}$$
$$n \leq \dim \mathcal{N}(\lambda) = n + \operatorname{rank} R_+(\lambda) = \operatorname{rank} \boldsymbol{R}_+(\lambda) \leq 2n. \tag{3.46}$$

Proof. Let $e_j \in \mathbb{C}^{2n}$ be the j-th canonical unit vector for $j \in \{1,\dots,2n\}$. Then system ($\mathcal{S}_\lambda$) possesses constant solutions $\boldsymbol{z}^{[j]} := (e_j^*,\ 0^*)^* \in \mathbb{C}^{4n}$ for all $j \in \{1,\dots,2n\}$, which certainly belong to $\boldsymbol{\mathcal{N}}(\lambda)$, because they satisfy $\|\boldsymbol{z}^{[j]}\|_{\boldsymbol{\Psi}} = 0$. In addition, any square summable solution $z \in \mathcal{N}(\lambda)$ naturally generates a square summable solution $\boldsymbol{z} = (0^*,\ z^*)^* \in \boldsymbol{\mathcal{N}}(\lambda)$, which is linearly independent with the above defined solutions $\boldsymbol{z}^{[1]},\dots,\boldsymbol{z}^{[2n]}$. This yields that $\dim \boldsymbol{\mathcal{N}}(\lambda) = 2n + \dim \mathcal{N}(\lambda)$. Hence identity (3.45) follows from Corollary 3.3.4 and the inequality $\dim \mathcal{N}(\lambda) \geq n$ in Theorem 2.4.1. Identity (3.46) is then only a direct consequence of equality (3.44). ∎

Combining Theorem 3.3.6 and Corollaries 3.3.4 and 2.4.20 then yields the next result.

Corollary 3.3.7. *Let $\lambda_0 \in \mathbb{C}\setminus\mathbb{R}$ and suppose that Hypothesis 2.4.11 is satisfied. Then we have* $\dim \boldsymbol{\mathcal{N}}(\lambda_0) = 3n$ *if and only if system* ($\mathcal{S}_{\lambda_0}$) *is in the limit point case. Similarly,* $\dim \boldsymbol{\mathcal{N}}(\lambda_0) = 4n$ *if and only if system* ($\mathcal{S}_{\lambda_0}$) *is in the limit circle case. Moreover, if it holds* $\dim \boldsymbol{\mathcal{N}}(\lambda_1) = 4n$ *for some* $\lambda_1 \in \mathbb{C}$*, then* $\dim \boldsymbol{\mathcal{N}}(\lambda) = 4n$ *for all* $\lambda \in \mathbb{C}$.

We can now see that the rank of the limiting matrix radius $\boldsymbol{R}_+(\lambda)$ can never be zero, so that the "limit point" behavior of ($\boldsymbol{\mathcal{S}}_\lambda$), i.e., $\dim \boldsymbol{\mathcal{N}}(\lambda) = 3n$, should not be determined by the equality $\boldsymbol{R}_+(\lambda) = 0$, as one would expect from the separated endpoints case in Theorem 2.4.3. However, we can read Theorems 2.4.3, 2.4.8 and Corollary 2.4.10 also in a different way. Namely, the number of linearly independent solutions of system ($\mathcal{S}_\lambda$), which *are not* square summable, is equal to $\dim \operatorname{Ker} R_+(\lambda)$. For the augmented system ($\boldsymbol{\mathcal{S}}_\lambda$) we now have exactly the same statement, i.e., the number of linearly independent solutions of system ($\boldsymbol{\mathcal{S}}_\lambda$), which are not square summable, is equal to $\dim \operatorname{Ker} \boldsymbol{R}_+(\lambda)$.

Remark 3.3.8. The augmentation of system ($\mathcal{S}_\lambda$) into the double dimension is a known technique for studying the problems with jointly varying endpoints, see e.g. [17, 87, 88, 92, 112, 153]. The transformation introduced in this chapter has the advantage that it uses the solutions z or Z of system ($\mathcal{S}_\lambda$) rather than their components x, u or X, U as in the above references. This yields a direct connection between the original system ($\mathcal{S}_\lambda$) and the augmented system ($\boldsymbol{\mathcal{S}}_\lambda$). For example, the boundary conditions in [153, Section 6] are of the form

$$\mathcal{P}_1 \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + \mathcal{P}_2 \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} = 0 \tag{3.47}$$

with certain $2n \times 2n$ matrices $\mathcal{P}_1$ and $\mathcal{P}_2$. One can see that the approach via (3.1) is much easier and more transparent. The relationship between the transformation in the above mentioned references and the transformation, which is utilized in this section, is determined by the multiplication of the data (from one side or from both sides) by the following $4n \times 4n$ matrix

$$\mathfrak{T} := \begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} = \mathfrak{T}^{-1}.$$

In particular, the equality

$$\mathfrak{T} \begin{pmatrix} -x_0 \\ x_{N+1} \\ u_0 \\ u_{N+1} \end{pmatrix} = \begin{pmatrix} x_0 \\ u_0 \\ x_{N+1} \\ u_{N+1} \end{pmatrix} = \begin{pmatrix} z_0 \\ z_{N+1} \end{pmatrix}$$

gives a direct connection between the boundary conditions in (3.47) and (3.1).

3.4 Bibliographical notes

The results of this chapter were established in [A13] and their generalization to symplectic systems on time scales was given in [A16, Section 8]. In addition, Corollary 3.2.3 is published for the first time in the present setting and it is derived as a special case of [A16, Corollary 8.5].

The question of the ultimate foundations and the ultimate meaning of mathematics remains open; we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. "Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization.

HERMANN WEYL, SEE [106, PG. 319]

Chapter 4

INVARIANCE OF LIMIT CIRCLE CASE FOR TWO DISCRETE SYSTEMS

In this chapter we derive an invariance of the situation, when all solutions are square summable, i.e., of the limit circle case for system ($\mathcal{S}_\lambda$) as it was already stated in Theorem 2.4.17, see Remark 4.2.4. However, instead of system ($\mathcal{S}_\lambda$) we consider two discrete systems of the first order in the form

$$\hat{z}_{k+1}(\lambda) = \left(\widehat{\mathcal{S}}_k + \lambda \widehat{\mathcal{V}}_k\right) \hat{z}_k(\lambda), \tag{$\widehat{\mathcal{S}}_\lambda$}$$

$$\tilde{z}_{k+1}(\lambda) = \left(\widetilde{\mathcal{S}}_k + \lambda \widetilde{\mathcal{V}}_k\right) \tilde{z}_k(\lambda), \tag{$\widetilde{\mathcal{S}}_\lambda$}$$

where $k \in [0, \infty)_{\mathbb{Z}}$, $\lambda \in \mathbb{C}$, and $\widehat{\mathcal{S}}, \widehat{\mathcal{V}}, \widetilde{\mathcal{S}}, \widetilde{\mathcal{V}} \in \mathbb{C}([0, \infty)_{\mathbb{Z}})^{2n\times 2n}$. Moreover, the coefficients of systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$) satisfy for every $k \in [0, \infty)_{\mathbb{Z}}$ the relations

$$\widetilde{\mathcal{S}}_k^* \mathcal{J} \widehat{\mathcal{S}}_k = \mathcal{J}, \quad \widetilde{\mathcal{V}}_k^* \mathcal{J} \widehat{\mathcal{S}}_k + \widetilde{\mathcal{S}}_k^* \mathcal{J} \widehat{\mathcal{V}}_k = 0, \quad \widetilde{\mathcal{V}}_k^* \mathcal{J} \widehat{\mathcal{V}}_k = 0, \tag{4.1}$$

$$\widehat{\Psi}_k := \mathcal{J} \widehat{\mathcal{V}}_k \mathcal{J} \widetilde{\mathcal{S}}_k^* \mathcal{J} \geq 0, \tag{4.2}$$

i.e., the matrix $\widehat{\Psi}_k$ is Hermitian and positive semidefinite for all $k \in [0, \infty)_{\mathbb{Z}}$, compare with Remark 4.1.4(i). Despite the analogy between the conditions in (4.1), (4.2) and the assumptions for system ($\mathcal{S}_\lambda$) displayed in (2.1), we emphasize that systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$) are generally non-symplectic in the sense of the terminology introduced in Chapter 2, see also Remark 4.1.4(ii).

Our investigation is motivated by Walker's results in [168], where an analogous problem was studied for a pair of the non-Hermitian linear Hamiltonian differential systems

$$\hat{z}'(t, \lambda) = [H(t) + \lambda W(t)]\, \hat{z}(t, \lambda), \tag{$\widehat{\mathrm{H}}_\lambda^{\mathbb{R}}$}$$

$$\tilde{z}'(t, \lambda) = [\mathcal{J} H^*(t) \mathcal{J} + \lambda W(t)]\, \tilde{z}(t, \lambda) \tag{$\widetilde{\mathrm{H}}_\lambda^{\mathbb{R}}$}$$

with $t \in [a, \infty)$ and $H(t), W(t)$ being locally integrable $2n \times 2n$ complex matrix-valued functions such that the matrix $W(t)$ is Hamiltonian and $-\mathcal{J} W(t) \geq 0$ on $[a, \infty)$. More specifically, let us denote by L^2_A the space of all functions $z:[a,\infty)\to\mathbb{C}^{2n}$, which are square integrable with respect to the weight $A(t):=-\mathcal{J}W(t)$, i.e., $-\int_a^\infty z^*(t)\mathcal{J}W(t)z(t)\,\mathrm{d}t<\infty$. Then it was proven in [168, Theorem 2] that if

$$\int_a^\infty |\operatorname{tr} W(t)|\,\mathrm{d}t<\infty \tag{4.3}$$

and all solutions of systems ($\widehat{\mathrm{H}}^{\mathbb{R}}_{\lambda_0}$) and ($\widetilde{\mathrm{H}}^{\mathbb{R}}_{\lambda_0}$) belong to L^2_A for some $\lambda_0\in\mathbb{C}$, then this property holds for all solutions of systems ($\widehat{\mathrm{H}}^{\mathbb{R}}_{\lambda}$) and ($\widetilde{\mathrm{H}}^{\mathbb{R}}_{\lambda}$) with an arbitrary $\lambda\in\mathbb{C}$. This statement extends the invariance of the limit circle case for one system ($\widehat{\mathrm{H}}^{\mathbb{R}}_{\lambda}$) with the coefficient matrix $H(t)$ being Hamiltonian on $[a,\infty)$, i.e., for the situation ($\widehat{\mathrm{H}}^{\mathbb{R}}_{\lambda}$)=($\widetilde{\mathrm{H}}^{\mathbb{R}}_{\lambda}$)=(2.5), established by Atkinson in [9, Theorem 9.11.2]. Although the main result of this chapter (Theorem 4.2.2) yields a discrete counterpart of [168, Theorem 2], we point out that, surprisingly, it does not require any analogue of condition (4.3).

4.1 Preliminaries

In this section we collect some auxiliary results about the coefficients of systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$). Similarly as in Chapter 2, let us define

$$\widehat{\mathbb{S}}_k(\lambda):=\widehat{\mathcal{S}}_k+\lambda\widehat{\mathcal{V}}_k,\qquad \widetilde{\mathbb{S}}_k(\lambda):=\widetilde{\mathcal{S}}_k+\lambda\widetilde{\mathcal{V}}_k. \tag{4.4}$$

Then the identities in (4.1) imply for all $k\in[0,\infty)_{\mathbb{Z}}$ and $\lambda\in\mathbb{C}$ that

$$\widetilde{\mathbb{S}}^*_k(\lambda)\mathcal{J}\widehat{\mathbb{S}}_k(\bar{\lambda})=\mathcal{J}. \tag{4.5}$$

Thus from the invertibility of $\mathcal{J}$ it follows immediately that the matrices $\widehat{\mathbb{S}}_k(\lambda)$ and $\widetilde{\mathbb{S}}_k(\lambda)$ are invertible for any $k\in[0,\infty)_{\mathbb{Z}}$ and $\lambda\in\mathbb{C}$ with

$$\widehat{\mathbb{S}}^{-1}_k(\lambda)=-\mathcal{J}\widetilde{\mathbb{S}}^*_k(\bar{\lambda})\mathcal{J},\qquad \widetilde{\mathbb{S}}^{-1}_k(\lambda)=-\mathcal{J}\widehat{\mathbb{S}}^*_k(\bar{\lambda})\mathcal{J}. \tag{4.6}$$

Hence any initial value problem associated with system ($\widehat{\mathcal{S}}_\lambda$) or ($\widetilde{\mathcal{S}}_\lambda$) possesses a unique solution on $[0,\infty)_{\mathbb{Z}}$. Moreover, the fundamental matrices of systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$) are invertible on the whole discrete interval $[0,\infty)_{\mathbb{Z}}$.

In the following lemma we give several conditions which are equivalent to (4.1). Namely, we show that any variation of the superscripts star, hat, and tilde is possible.

Lemma 4.1.1. *Let $n\in\mathbb{N}$ be given. For any $k\in[0,\infty)_{\mathbb{Z}}$ the following conditions are equivalent.*

(i) *The matrices $\widehat{\mathcal{S}}(t)$, $\widetilde{\mathcal{S}}(t)$, $\widehat{\mathcal{V}}(t)$, $\widetilde{\mathcal{V}}(t)$ satisfy (4.1).*

(ii) *The matrices $\widehat{\mathbb{S}}_k(\lambda)$ and $\widetilde{\mathbb{S}}_k(\lambda)$ satisfy (4.5) for all $\lambda\in\mathbb{C}$.*

(iii) *The matrices $\widehat{\mathcal{S}}(t)$, $\widetilde{\mathcal{S}}(t)$, $\widehat{\mathcal{V}}(t)$, $\widetilde{\mathcal{V}}(t)$ satisfy*

$$\widetilde{\mathcal{S}}_k\mathcal{J}\widehat{\mathcal{S}}^*_k=\mathcal{J},\quad \widetilde{\mathcal{V}}_k\mathcal{J}\widehat{\mathcal{S}}^*_k+\widetilde{\mathcal{S}}_k\mathcal{J}\widehat{\mathcal{V}}^*_k=0,\quad \widetilde{\mathcal{V}}_k\mathcal{J}\widehat{\mathcal{V}}^*_k=0. \tag{4.7}$$

(iv) *The matrices $\widehat{\mathbb{S}}_k(\lambda)$ and $\widetilde{\mathbb{S}}_k(\lambda)$ satisfy for all $\lambda\in\mathbb{C}$ that*

$$\widetilde{\mathbb{S}}_k(\lambda)\mathcal{J}\widehat{\mathbb{S}}^*_k(\bar{\lambda})=\mathcal{J}. \tag{4.8}$$

Proof. The equivalence of (i) and (ii), and of (iii) and (iv), follows by direct calculations with the notation introduced in (4.4). The equivalence of (ii) and (iv) is a consequence of the relations in (4.6) and of the fact $\widehat{\mathbb{S}}_k(\lambda)\widehat{\mathbb{S}}^{-1}_k(\lambda)=I$. ∎

Further conditions which are equivalent to (i)–(iv) in Lemma 4.1.1 can be obtained by the conjugate transpose of the identities in (4.1), (4.5), (4.7), and (4.8).

Now we focus on the coefficients of systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$) with different values of the spectral parameter. Together with $\widehat{\Psi}_k$ defined in (4.2) we consider also the matrix

$$\widetilde{\Psi}_k := \mathcal{J}\,\widetilde{\mathcal{V}}_k\,\mathcal{J}\,\widehat{\mathcal{S}}_k^*\,\mathcal{J}. \tag{4.9}$$

Lemma 4.1.2. *Let $n \in \mathbb{N}$ be fixed and $k \in [0,\infty)_{\mathbb{Z}}$ be such that the conditions in (4.1) hold. Then the matrices $\widehat{\Psi}_k$, $\widetilde{\Psi}_k$ defined in (4.2) and (4.9), respectively, satisfy*

$$\widetilde{\Psi}_k = \widehat{\Psi}_k^*, \quad \widehat{\Psi}_k\,\mathcal{J}\,\widehat{\Psi}_k = 0, \quad \widetilde{\Psi}_k\,\mathcal{J}\,\widetilde{\Psi}_k = 0, \tag{4.10}$$

and for all $\lambda, \nu \in \mathbb{C}$ we have

$$\mathcal{J}\,\widehat{\mathbb{S}}_k(\lambda)\,\mathcal{J}\,\widetilde{\mathbb{S}}_k^*(\nu)\,\mathcal{J} = (\lambda - \bar{\nu})\,\widehat{\Psi}_k - \mathcal{J}, \tag{4.11}$$

$$\mathcal{J}\,\widetilde{\mathbb{S}}_k(\lambda)\,\mathcal{J}\,\widehat{\mathbb{S}}^*(\nu)\,\mathcal{J} = (\lambda - \bar{\nu})\,\widetilde{\Psi}_k - \mathcal{J}. \tag{4.12}$$

Proof. The above identities follow by direct calculations. ∎

Identities (4.11) and (4.12) play a crucial role in the proof of the following generalization of the Lagrange identity for two systems, compare with Theorem 2.1.7.

Theorem 4.1.3 (Generalized Lagrange identity). *Let $n, m \in \mathbb{N}$ and $\lambda, \nu \in \mathbb{C}$ be fixed and assume that the conditions in (4.1) hold for all $k \in [0,\infty)_{\mathbb{Z}}$. If $\widehat{Z}(\lambda), \widetilde{Z}(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times m}$ solve systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\nu$) on $[0,\infty)_{\mathbb{Z}}$, respectively, then for any $k \in [0,\infty)_{\mathbb{Z}}$ we have*

$$\Delta\Big[\widetilde{Z}_k^*(\lambda)\,\mathcal{J}\,\widehat{Z}_k(\nu)\Big] = (\bar{\lambda} - \nu)\,\widetilde{Z}_{k+1}^*(\lambda)\,\widehat{\Psi}_k\,\widehat{Z}_{k+1}(\nu), \tag{4.13}$$

$$\Delta\Big[\widehat{Z}_k^*(\lambda)\,\mathcal{J}\,\widetilde{Z}_k(\nu)\Big] = (\bar{\lambda} - \nu)\,\widehat{Z}_{k+1}^*(\lambda)\,\widetilde{\Psi}_k\,\widetilde{Z}_{k+1}(\nu). \tag{4.14}$$

Proof. Identity (4.13) follows from the first equality in (4.6) and from (4.11), because

$$\begin{aligned}\Delta\Big[\widetilde{Z}_k^*(\lambda)\,\mathcal{J}\,\widehat{Z}_k(\nu)\Big] &= \widetilde{Z}_{k+1}^*(\lambda)\Big[\mathcal{J} - \widetilde{\mathbb{S}}_k^{*-1}(\lambda)\,\mathcal{J}\,\widehat{\mathbb{S}}_k^{-1}(\nu)\Big]\widehat{Z}_{k+1}(\nu) \\ &\overset{(4.6)}{=} \widetilde{Z}_{k+1}^*(\lambda)\Big[\mathcal{J} + \mathcal{J}\,\widehat{\mathbb{S}}_k(\bar{\lambda})\,\mathcal{J}\,\widetilde{\mathbb{S}}_k^*(\bar{\nu})\,\mathcal{J}\Big]\widehat{Z}_{k+1}(\nu) \overset{(4.11)}{=} (\bar{\lambda} - \nu)\,\widetilde{Z}_{k+1}^*(\lambda)\,\widehat{\Psi}_k\,\widehat{Z}_{k+1}(\nu).\end{aligned}$$

Similarly we get identity (4.14) from the second equality in (4.6) and from (4.12). ∎

Remark 4.1.4.

(i) The results in Theorem 4.1.3 imply that in order to have a single weight matrix for the semi-inner product and the semi-norm in the associated space of square summable solutions, we must necessarily assume that $\widetilde{\Psi}_k = \widehat{\Psi}_k$. This means, in view of Lemma 4.1.2, that we need to have $\widehat{\Psi}_k$ Hermitian. This is, in fact, the original motivation for our assumption (4.2).

(ii) In the continuous case it is obvious that systems ($\widehat{\mathrm{H}}_\lambda^{\mathbb{R}}$) and ($\widetilde{\mathrm{H}}_\lambda^{\mathbb{R}}$) coincide if and only if $H(t)$ is Hamiltonian on $[a,\infty)$. Now we give the answer to the same question for systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$). From the first equality in (4.1) (or from (4.6) with $\lambda = 0$) and from the first and the second conditions in (4.1) we obtain, respectively,

$$\widetilde{\mathcal{S}}_k = -\mathcal{J}\,\widehat{\mathcal{S}}_k^{*-1}\,\mathcal{J} \quad \text{and} \quad \widetilde{\mathcal{V}}_k = \mathcal{J}\,\widehat{\mathcal{S}}_k^{*-1}\,\widehat{\mathcal{V}}_k^*\,\widehat{\mathcal{S}}_k^{*-1}\,\mathcal{J}. \tag{4.15}$$

Therefore, $\widetilde{\mathcal{S}}(t) = \widehat{\mathcal{S}}(t)$ in (4.15) if and only if

$$\widehat{\mathcal{S}}_k^* \mathcal{J} \widehat{\mathcal{S}}_k = \mathcal{J},$$

i.e., $\widehat{\mathcal{S}}_k$ is symplectic for all $k \in [0,\infty)_{\mathbb{Z}}$. In this case, $\widetilde{\mathcal{V}}_k = \widehat{\mathcal{V}}_k$ in (4.15) if and only if the matrix $\widetilde{\mathcal{V}}_k \mathcal{J} \widehat{\mathcal{S}}_k^* = \widehat{\mathcal{V}}_k \mathcal{J} \widehat{\mathcal{S}}_k^*$ is Hermitian for all $k \in [0,\infty)_{\mathbb{Z}}$, i.e., the matrix $\widehat{\Psi}_k$ is Hermitian on $[0,\infty)_{\mathbb{Z}}$. In other words, systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$) satisfying (4.1) and (4.2) coincide if and only if they represent the discrete symplectic system studied in Chapter 2, i.e., system ($\mathcal{S}_\lambda$) with the coefficient matrices satisfying (2.1).

Finally, we calculate the determinant of the matrices $\widehat{\mathbb{S}}_k(\lambda)$, $\widetilde{\mathbb{S}}_k(\lambda)$ and of their product. Let $k \in [0,\infty)_{\mathbb{Z}}$ be such that the conditions in (4.1) hold. From the first equality in (4.1) we obtain for any $\lambda \in \mathbb{C}$ that

$$\widehat{\mathbb{S}}_k(\lambda) = (I + \lambda \mathcal{J} \widehat{\Psi}_k) \widehat{\mathcal{S}}_k \quad \text{and} \quad \widetilde{\mathbb{S}}_k(\lambda) = (I + \lambda \mathcal{J} \widetilde{\Psi}_k) \widetilde{\mathcal{S}}_k.$$

Moreover, by the second and third identities in (4.10) the matrices $\lambda \mathcal{J} \widehat{\Psi}_k$ and $\lambda \mathcal{J} \widetilde{\Psi}_k$ are nilpotent of degree two, which with the aid of Proposition 1.1.3 yields

$$\det(I + \lambda \mathcal{J} \widehat{\Psi}_k) = 1 = \det(I + \lambda \mathcal{J} \widetilde{\Psi}_k).$$

Therefore $\det \widehat{\mathbb{S}}_k(\lambda) = \det \widehat{\mathcal{S}}_k$ and $\det \widetilde{\mathbb{S}}_k(\lambda) = \det \widetilde{\mathcal{S}}_k$, i.e., the determinants do not depend on λ. Consequently from the first condition in (4.1) we get

$$\det \widetilde{\mathbb{S}}_k^*(\lambda) \mathcal{J} \widehat{\mathbb{S}}_k(\lambda) = \det \widetilde{\mathcal{S}}_k^* \mathcal{J} \widehat{\mathcal{S}}_k \overset{(4.1)}{=} \det \mathcal{J} = 1, \quad \text{i.e.,} \quad \det \widetilde{\mathbb{S}}_k^*(\lambda) \times \det \widehat{\mathbb{S}}_k(\lambda) = 1. \tag{4.16}$$

In addition, from the latter equality and Remark 4.1.4(ii) one concludes that when systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$) coincide, then the absolute value of $\det \widehat{\mathbb{S}}_k(\lambda)$ is equal to one, as we claim in Lemma 2.1.3.

4.2 Main result

In this section we establish the main result of this chapter (Theorem 4.2.2). We also provide sufficient conditions for the invariance, present an illustrative example, and discuss some special cases. For convenience, we summarize the used notation.

Notation 4.2.1. The number $n \in \mathbb{N}$ is fixed and $\widehat{\mathcal{S}}, \widetilde{\mathcal{S}}, \widehat{\mathcal{V}}, \widetilde{\mathcal{V}}, \widehat{\Psi} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ are such that the conditions in (4.1) and (4.2) are satisfied for all $k \in [0,\infty)_{\mathbb{Z}}$.

It follows from Remark 4.1.4(i) that with Notation 4.2.1 we have just one weight matrix $\widehat{\Psi}_k = \widetilde{\Psi}_k$, which is Hermitian and positive semidefinite on $[0,\infty)_{\mathbb{Z}}$. Therefore, we denote by $\ell^2_{\widehat{\Psi}}$ the space of all sequences defined on $[0,\infty)_{\mathbb{Z}}$, which are square summable with respect to the weight matrix $\widehat{\Psi}_k$, i.e.,

$$\ell^2_{\widehat{\Psi}}[0,\infty)_{\mathbb{Z}} = \ell^2_{\widehat{\Psi}} := \Big\{ z \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n} \;\Big|\; \sum_{k=0}^{\infty} z_k^* \widehat{\Psi}_k z_k < \infty \Big\}.$$

Let us note that the statement in Theorem 4.2.2 below remains the same for other types of unbounded discrete intervals, such as for $(-\infty, b]_{\mathbb{Z}}$ or $(-\infty,\infty)_{\mathbb{Z}}$, if the corresponding space $\ell^2_{\widehat{\Psi}}$ is defined over that interval.

Theorem 4.2.2. *Let us assume that there exists $\lambda_0 \in \mathbb{C}$ such that all solutions of systems ($\widehat{\mathcal{S}}_{\lambda_0}$) and ($\widetilde{\mathcal{S}}_{\lambda_0}$) belong to $\ell^2_{\widehat{\Psi}}$. Then all solutions of systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$) belong to $\ell^2_{\widehat{\Psi}}$ for any $\lambda \in \mathbb{C}$.*

Proof. Let $\lambda_0 \in \mathbb{C}$ be as stated in the theorem and $\lambda \in \mathbb{C}\setminus\{\lambda_0\}$ be fixed. For $\nu \in \{\lambda, \lambda_0\}$ we denote by $\widehat{\Phi}_k(\nu)$ and $\widetilde{\Phi}_k(\nu)$ the fundamental matrices of systems $(\widehat{\mathbb{S}}_\nu)$ and $(\widetilde{\mathbb{S}}_\nu)$, respectively, such that $\widehat{\Phi}_0(\nu) = I = \widetilde{\Phi}_0(\nu)$. In the first part of the proof we show that all solutions of system $(\widehat{\mathbb{S}}_\lambda)$ belong to $\ell^2_{\widehat{\Psi}}$. Since the matrices $\widehat{\Phi}_k(\lambda)$ and $\widehat{\Phi}_k(\lambda_0)$ are obviously invertible on $[0,\infty)_{\mathbb{Z}}$, it follows that for every $k \in [0,\infty)_{\mathbb{Z}}$ we have

$$\widehat{\Phi}_k(\lambda) = \widehat{\Phi}_k(\lambda_0)\,\widehat{\Omega}_k \tag{4.17}$$

for some invertible matrix $\widehat{\Omega}_k \in \mathbb{C}^{2n\times 2n}$, i.e., $\widehat{\Omega}_k = \widehat{\Phi}_k^{-1}(\lambda_0)\,\widehat{\Phi}_k(\lambda)$. Hence by straightforward calculations with using (4.1), (4.4), (4.6), and (4.17) we get

$$\begin{aligned}
\Delta\,\widehat{\Omega}_k &\overset{(4.17)}{=} \widehat{\Phi}_{k+1}^{-1}(\lambda_0)\Big[\widehat{\mathbb{S}}_k(\lambda) - \widehat{\mathbb{S}}_k(\lambda_0)\Big]\widehat{\Phi}_k(\lambda) \overset{(4.4)}{=} (\lambda-\lambda_0)\,\widehat{\Phi}_{k+1}^{-1}(\lambda_0)\,\widehat{\mathcal{V}}_k\,\widehat{\Phi}_k(\lambda_0)\,\widehat{\Omega}_k \\
&= (\lambda-\lambda_0)\,\widehat{\Phi}_{k+1}^{-1}(\lambda_0)\,\widehat{\mathcal{V}}_k\,\widehat{\mathbb{S}}_k^{-1}(\lambda_0)\,\widehat{\Phi}_{k+1}(\lambda_0)\,\widehat{\Omega}_k \\
&\overset{(4.6)}{=} -(\lambda-\lambda_0)\,\widehat{\Phi}_{k+1}^{-1}(\lambda_0)\,\widehat{\mathcal{V}}_k\,\mathcal{J}\,\widehat{\mathbb{S}}_k^{*}(\bar{\lambda}_0)\,\mathcal{J}\,\widehat{\Phi}_{k+1}(\lambda_0)\,\widehat{\Omega}_k \\
&\overset{(4.1)}{=} -(\lambda-\lambda_0)\,\widehat{\Phi}_{k+1}^{-1}(\lambda_0)\,\widehat{\mathcal{V}}_k\,\mathcal{J}\,\widehat{\mathcal{S}}_k^{*}\,\mathcal{J}\,\widehat{\Phi}_{k+1}(\lambda_0)\,\widehat{\Omega}_k \\
&= (\lambda-\lambda_0)\,\widehat{\Phi}_{k+1}^{-1}(\lambda_0)\,\mathcal{J}\,\widehat{\Psi}_k\,\widehat{\Phi}_{k+1}(\lambda_0)\,\widehat{\Omega}_k.
\end{aligned} \tag{4.18}$$

It means that $\widehat{\Omega}_k$ satisfies the recurrence relation

$$\widehat{\Omega}_{k+1} = \Big[I + (\lambda-\lambda_0)\,\Upsilon_k\Big]\widehat{\Omega}_k, \quad k \in [0,\infty)_{\mathbb{Z}}, \tag{4.19}$$

where $\Upsilon \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ is given by the formula

$$\Upsilon_k := \widehat{\Phi}_{k+1}^{-1}(\lambda_0)\,\mathcal{J}\,\widehat{\Psi}_k\,\widehat{\Phi}_{k+1}(\lambda_0) = -\Big[\widetilde{\Phi}_{k+1}^{*}(\lambda_0)\,\mathcal{J}\,\widehat{\Phi}_{k+1}(\lambda_0)\Big]^{-1}\widetilde{\Phi}_{k+1}^{*}(\lambda_0)\,\widehat{\Psi}_k\,\widehat{\Phi}_{k+1}(\lambda_0). \tag{4.20}$$

Identity (4.17) implies that for the required conclusion it suffices to prove the boundedness of $\|\widehat{\Omega}_k\|_\sigma$ on $[0,\infty)_{\mathbb{Z}}$. However equality (4.19) and the submultiplicative property of the norm $\|\cdot\|_\sigma$ yield

$$\begin{aligned}
\big\|\widehat{\Omega}_{k+1}\big\|_\sigma &= \Big\|\Big[I + (\lambda-\lambda_0)\,\Upsilon_k\Big]\widehat{\Omega}_k\Big\|_\sigma \le \Big(\|I\|_\sigma + |\lambda-\lambda_0| \times \|\Upsilon_k\|_\sigma\Big) \times \big\|\widehat{\Omega}_k\big\|_\sigma \\
&= \Big(1 + |\lambda-\lambda_0| \times \|\Upsilon_k\|_\sigma\Big) \times \Big\|\Big[I + (\lambda-\lambda_0)\,\Upsilon_{k-1}\Big]\widehat{\Omega}_{k-1}\Big\|_\sigma \\
&\le \Big(1 + |\lambda-\lambda_0| \times \|\Upsilon_k\|_\sigma\Big) \times \Big(1 + |\lambda-\lambda_0| \times \|\Upsilon_{k-1}\|_\sigma\Big) \times \big\|\widehat{\Omega}_{k-1}\big\|_\sigma \\
&\le \cdots \le \Big(1 + |\lambda-\lambda_0| \times \|\Upsilon_k\|_\sigma\Big) \times \cdots \times \Big(1 + |\lambda-\lambda_0| \times \|\Upsilon_0\|_\sigma\Big) \times \big\|\widehat{\Omega}_0\big\|_\sigma \\
&\le \mathrm{e}^{|\lambda-\lambda_0|\,\omega_k} \quad \text{with} \quad \omega_k := \sum_{j=0}^{k} \|\Upsilon_j\|_\sigma,
\end{aligned} \tag{4.21}$$

where in the last step we used the inequality $1 + x \le \mathrm{e}^x$ and the fact $\widehat{\Omega}_0 = I$; cf. Proposition 1.1.4. Therefore we need to show that $\lim_{k\to\infty} \omega_k < \infty$.

Since $\widehat{\Psi}_k \ge 0$ on $[0,\infty)_{\mathbb{Z}}$, we obtain from the Cauchy–Schwarz inequality and the arithmetic-geometric mean inequality for any $\xi, \zeta \in \mathbb{C}^{2n}$ that

$$\big|\xi^*\widehat{\Psi}_k\zeta\big| \le \big(\xi^*\widehat{\Psi}_k\xi\big)^{1/2}\big(\zeta^*\widehat{\Psi}_k\zeta\big)^{1/2} \le \tfrac{1}{2}\big(\xi^*\widehat{\Psi}_k\xi + \zeta^*\widehat{\Psi}_k\zeta\big),$$

which for any $\tilde{z}, \hat{z} \in \ell^2_{\widehat{\Psi}}$ implies

$$\Big|\sum_{k=0}^{\infty} \tilde{z}_k^*\,\widehat{\Psi}_k\,\hat{z}_k\Big| \le \sum_{k=0}^{\infty} \big|\tilde{z}_k^*\,\widehat{\Psi}_k\,\hat{z}_k\big| \le \tfrac{1}{2}\sum_{k=0}^{\infty} \big(\tilde{z}_k^*\,\widehat{\Psi}_k\,\tilde{z}_k + \hat{z}_k^*\,\widehat{\Psi}_k\,\hat{z}_k\big) < \infty.$$

Hence inequality (1.7) and the assumption that all solutions of systems ($\widehat{\mathbb{S}}_{\lambda_0}$) and ($\widetilde{\mathbb{S}}_{\lambda_0}$) belong to $\ell^2_{\widehat{\Psi}}$, yield

$$\sum_{k=0}^{\infty} \left\| \widetilde{\Phi}^*_{k+1}(\lambda_0)\, \widehat{\Psi}_k\, \widehat{\Phi}_{k+1}(\lambda_0) \right\|_\sigma \overset{(1.7)}{\leq} \sum_{k=0}^{\infty} \left\| \widetilde{\Phi}^*_{k+1}(\lambda_0)\, \widehat{\Psi}_k\, \widehat{\Phi}_{k+1}(\lambda_0) \right\|_1 \leq \varepsilon < \infty \tag{4.22}$$

for some $\varepsilon > 0$. Now we put

$$Q_k := \widetilde{\Phi}^*_{k+1}(\lambda_0)\, \mathcal{J}\, \widehat{\Phi}_{k+1}(\lambda_0)$$

and show that the value of $\left\| Q_k^{-1} \right\|_\sigma$ is bounded on $[0,\infty)_{\mathbb{Z}}$. By the generalized Lagrange identity in Theorem 4.1.3 we get

$$Q_k = \mathcal{J} - 2i\,\mathrm{im}(\lambda_0) \sum_{k=0}^{\infty} \widetilde{\Phi}^*_{j+1}(\lambda_0)\, \widehat{\Psi}_j\, \widehat{\Phi}_{j+1}(\lambda_0),$$

and so inequality (4.22) implies that the limit $\lim_{k\to\infty} Q_k$ exists and is bounded in the spectral norm, i.e., $\|Q_k\|_\sigma$ is bounded on $[0,\infty)_{\mathbb{Z}}$. Therefore also the adjugate matrix Q_k^{adj} is bounded on $[0,\infty)_{\mathbb{Z}}$ in the spectral norm. Moreover, it holds

$$\begin{aligned}
\det Q_k &= \det \widetilde{\Phi}^*_{k+1}(\lambda_0) \times \det \widehat{\Phi}_{k+1}(\lambda_0) = \det \widetilde{\Phi}^*_k(\lambda_0) \times \det \widehat{\Phi}_k(\lambda_0) \times \det \widetilde{\mathbb{S}}^*_k(\lambda_0) \times \det \widehat{\mathbb{S}}_k(\lambda_0) \\
&\overset{(4.16)}{=} \det \widetilde{\Phi}^*_{k-1}(\lambda_0) \times \det \widehat{\Phi}_{k-1}(\lambda_0) \times \det \widetilde{\mathbb{S}}^*_{k-1}(\lambda_0) \times \det \widehat{\mathbb{S}}_{k-1}(\lambda_0) \\
&= \cdots = \det \widetilde{\Phi}^*_0(\lambda_0) \times \det \widehat{\Phi}_0(\lambda_0) = 1,
\end{aligned}$$

which yields $Q_k^{-1} = Q_k^{\mathrm{adj}}$ and consequently the matrices Q_k^{-1} are bounded in the spectral norm, i.e., $\left\| Q_k^{-1} \right\|_\sigma \leq \kappa$ for all $k \in [0,\infty)_{\mathbb{Z}}$ and some $\kappa > 0$. By combining the submultiplicative property of the spectral norm, (4.20), and (4.22) we get

$$\sum_{k=0}^{\infty} \|\Upsilon_k\|_\sigma \leq \sum_{k=0}^{\infty} \left\| Q_k^{-1} \right\|_\sigma \times \left\| \widetilde{\Phi}^*_{k+1}(\lambda_0)\, \widehat{\Psi}_k\, \widehat{\Phi}_{k+1}(\lambda_0) \right\|_\sigma \leq \kappa\varepsilon < \infty,$$

i.e., $\lim_{k\to\infty} \omega_k < \infty$ by (4.21). Thus $\left\| \widehat{\Omega}_k \right\|_\sigma \leq \tau$ on $[0,\infty)_{\mathbb{Z}}$ for some $\tau > 0$. In turn, the definition of $\widehat{\Omega}_k$, the second inequality in (1.7), and the submultiplicativity and self-adjointness of the spectral norm imply

$$\begin{aligned}
(2n)^{-3/2} \sum_{k=0}^{\infty} \left\| \widehat{\Phi}^*_{k+1}(\lambda)\, \widehat{\Psi}_k\, \widehat{\Phi}_{k+1}(\lambda) \right\|_1 &\overset{(1.7)}{\leq} \sum_{k=0}^{\infty} \left\| \widehat{\Phi}^*_{k+1}(\lambda)\, \widehat{\Psi}_k\, \widehat{\Phi}_{k+1}(\lambda) \right\|_\sigma \\
\overset{(4.17)}{=} \sum_{k=0}^{\infty} \left\| \widehat{\Omega}^*_{k+1}\, \widehat{\Phi}^*_{k+1}(\lambda_0)\, \widehat{\Psi}_k\, \widehat{\Phi}_{k+1}(\lambda_0)\, \widehat{\Omega}_{k+1} \right\|_\sigma &\leq \sum_{k=0}^{\infty} \left\| \widehat{\Omega}_{k+1} \right\|^2_\sigma \times \left\| \widehat{\Phi}^*_{k+1}(\lambda_0)\, \widehat{\Psi}_k\, \widehat{\Phi}_{k+1}(\lambda_0) \right\|_\sigma \\
&\leq \tau^2 \sum_{k=0}^{\infty} \left\| \widehat{\Phi}^*_{k+1}(\lambda_0)\, \widehat{\Psi}_k\, \widehat{\Phi}_{k+1}(\lambda_0) \right\|_\sigma < \infty,
\end{aligned}$$

because all columns of the fundamental matrix $\widehat{\Phi}(\lambda_0)$ belong to $\ell^2_{\widehat{\Psi}}$. This shows that all columns of $\widehat{\Phi}(\lambda)$ belong to $\ell^2_{\widehat{\Psi}}$ and consequently any solution of system ($\widehat{\mathbb{S}}_\lambda$) is square summable with respect to $\widehat{\Psi}_k$. For the proof of the fact that all solutions of system ($\widetilde{\mathbb{S}}_\lambda$)

are also in $\ell^2_{\widetilde\Psi}$ we only switch the roles of systems $(\widehat{\mathbb{S}}_\lambda)$ and $(\widetilde{\mathbb{S}}_\lambda)$. Namely, we define $\widetilde\Omega_k := \widetilde\Phi_k^{-1}(\lambda_0)\,\widehat\Phi_k(\lambda)$ and similarly as in (4.18) we derive

$$\Delta\,\widetilde\Omega_k = (\lambda-\lambda_0)\,\widetilde\Phi_{k+1}^{-1}(\lambda_0)\,\mathcal{J}\,\widetilde\Psi_k\,\widetilde\Phi_{k+1}(\lambda_0)\,\widetilde\Omega_k.$$

But since we have $\widetilde\Psi_k = \widehat\Psi_k$ for all $k \in [0,\infty)_{\mathbb{Z}}$, the rest of the proof is the same as in the previous part. ■

In the following result we give sufficient conditions in terms of the coefficient matrices, which guarantee that all solutions of systems $(\widehat{\mathbb{S}}_\lambda)$ and $(\widetilde{\mathbb{S}}_\lambda)$ belong to $\ell^2_{\widetilde\Psi}$ for any $\lambda\in\mathbb{C}$. Let us note that the matrix norm $\|\cdot\|_1$ used in (4.23) below can be replaced by any other matrix norm because of their equivalence.

Corollary 4.2.3. *Let us assume that*

$$\sum_{k=0}^{\infty}\big\|\widehat{\mathbb{S}}_k - I\big\|_1 < \infty,\quad \sum_{k=0}^{\infty}\big\|\widetilde{\mathbb{S}}_k - I\big\|_1 < \infty,\quad \textit{and}\quad \sum_{k=0}^{\infty}\big\|\widehat\Psi_k\big\|_1 < \infty. \tag{4.23}$$

Then all solutions of systems $(\widehat{\mathbb{S}}_\lambda)$ and $(\widetilde{\mathbb{S}}_\lambda)$ belong to $\ell^2_{\widetilde\Psi}$ for any $\lambda\in\mathbb{C}$.

Proof. It suffices to show that the assumptions of Theorem 4.2.2 are satisfied for $\lambda_0 = 0$. The equality $\widehat{\mathbb{S}}_k(0) = \widehat{\mathbb{S}}_k$ for all $k\in[0,\infty)_{\mathbb{Z}}$ and the first condition in (4.23) imply by Proposition 1.1.4 that for a fundamental matrix $\widehat\Phi(0)\in\mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ of system $(\widehat{\mathbb{S}}_0)$ there exists $\kappa>0$ such that $\big\|\widehat\Phi_k(0)\big\|_1 \le \kappa < \infty$ for all $k\in[0,\infty)_{\mathbb{Z}}$. Hence by the submultiplicativity and self-adjointness of the matrix norm $\|\cdot\|_1$ and the third condition in (4.23) we have

$$\sum_{k=0}^{\infty}\big\|\widehat\Phi^*_{k+1}(0)\,\widehat\Psi_k\,\widehat\Phi_{k+1}(0)\big\|_1 \le \kappa^2\sum_{k=0}^{\infty}\big\|\widehat\Psi_k\big\|_1 < \infty,$$

i.e., all solutions of system $(\widehat{\mathbb{S}}_0)$ belong to $\ell^2_{\widehat\Psi}$. In a similar way we prove that any solution of system $(\widetilde{\mathbb{S}}_0)$ also belongs to $\ell^2_{\widetilde\Psi}$. Thus Theorem 4.2.2 implies that all solutions of systems $(\widehat{\mathbb{S}}_\lambda)$ and $(\widetilde{\mathbb{S}}_\lambda)$ are in $\ell^2_{\widetilde\Psi}$ for any $\lambda\in\mathbb{C}$. ■

Remark 4.2.4. In accordance with Remark 4.1.4(ii), if both systems $(\widehat{\mathbb{S}}_\lambda)$ and $(\widetilde{\mathbb{S}}_\lambda)$ coincide, Theorem 4.2.2 reduces to Theorem 2.4.17 and Corollary 4.2.3 to Corollary 2.4.22. Therefore, with a slight abuse in the terminology, Theorem 4.2.2 can be interpreted as the invariance of the limit circle case for systems $(\widehat{\mathbb{S}}_\lambda)$ and $(\widetilde{\mathbb{S}}_\lambda)$, i.e., the situation when all solutions of systems $(\widehat{\mathbb{S}}_\lambda)$ and $(\widetilde{\mathbb{S}}_\lambda)$ belong to $\ell^2_{\widetilde\Psi}$ for any $\lambda\in\mathbb{C}$.

Now we provide an illustrative example of the established invariance, i.e., the application of Theorem 4.2.2. This example also shows that all solutions of systems $(\widehat{\mathbb{S}}_\lambda)$ and $(\widetilde{\mathbb{S}}_\lambda)$ may be in $\ell^2_{\widetilde\Psi}$ for any $\lambda\in\mathbb{C}$ even when conditions (4.23) in Corollary 4.2.3 are not satisfied.

Example 4.2.5. Let $n=1$ and fix $\varepsilon\in\mathbb{R}$, $\varepsilon\ge 1$. Let $\{v_k\}_{k=0}^{\infty}$ be a real sequence such that $v_k\ge 0$ for all $k\in[0,\infty)_{\mathbb{Z}}$ and $\sum_{k=0}^{\infty}\varepsilon^{2k}v_k<\infty$. We note that then the series $\sum_{k=0}^{\infty}v_k<\infty$ and $\sum_{k=0}^{\infty}v_k/\varepsilon^{2k}<\infty$ are convergent as well, because $0\le v_k/\varepsilon^{2k}\le v_k\le\varepsilon^{2k}v_k$. Consider systems $(\widehat{\mathbb{S}}_\lambda)$ and $(\widetilde{\mathbb{S}}_\lambda)$ with

$$\widehat{\mathbb{S}}_k := (1/\varepsilon)I,\quad \widehat{\mathcal{V}}_k := \begin{pmatrix}0 & v_k\\ 0 & 0\end{pmatrix},\quad \widetilde{\mathbb{S}}_k := \varepsilon I,\quad \widetilde{\mathcal{V}}_k := \begin{pmatrix}0 & \varepsilon^2 v_k\\ 0 & 0\end{pmatrix},\quad \widehat\Psi_k := \begin{pmatrix}0 & 0\\ 0 & \varepsilon v_k\end{pmatrix} \tag{4.24}$$

for all $k \in [0,\infty)_{\mathbb{Z}}$. Then all the conditions (4.1) and (4.2) are satisfied. The fundamental matrices $\widehat{\Phi}_k(0) = \big(\hat{z}_k^{[1]}(0),\ \hat{z}_k^{[2]}(0)\big)$ and $\widetilde{\Phi}_k(0) = \big(\tilde{z}_k^{[1]}(0),\ \tilde{z}_k^{[2]}(0)\big)$ of systems ($\widehat{\mathrm{S}}_0$) and ($\widetilde{\mathrm{S}}_0$) with (4.24) satisfying $\widehat{\Phi}_0(0) = I = \widetilde{\Phi}_0(0)$ are equal to $\widehat{\Phi}_k(0) = (1/\varepsilon^k) I$ and $\widetilde{\Phi}_k(0) = \varepsilon^k I$, so that $\hat{z}_k^{[1]}(0) = (1/\varepsilon^k,\ 0)^\top$, $\hat{z}_k^{[2]}(0) = (0,\ 1/\varepsilon^k)^\top$, $\tilde{z}_k^{[1]}(0) = (\varepsilon^k,\ 0)^\top$, and $\tilde{z}_k^{[2]}(0) = (0,\ \varepsilon^k)^\top$ for all $k \in [0,\infty)_{\mathbb{Z}}$. Then with the notation $\|z\|^2_{\widehat{\Psi}} := \sum_{k=0}^{\infty} z_k^* \widehat{\Psi}_k z_k$ we have

$$\big\|\hat{z}^{[1]}(0)\big\|^2_{\widehat{\Psi}} = 0, \quad \big\|\hat{z}^{[2]}(0)\big\|^2_{\widehat{\Psi}} = \varepsilon \sum_{k=0}^{\infty} v_k/\varepsilon^{2k} < \infty,$$
$$\big\|\tilde{z}^{[1]}(0)\big\|^2_{\widetilde{\Psi}} = 0, \quad \big\|\tilde{z}^{[2]}(0)\big\|^2_{\widetilde{\Psi}} = \varepsilon \sum_{k=0}^{\infty} \varepsilon^{2k} v_k < \infty,$$

i.e., the solutions $\hat{z}^{[1]}(0)$, $\hat{z}^{[2]}(0)$, $\tilde{z}^{[1]}(0)$, $\tilde{z}^{[2]}(0)$ belong to $\ell^2_{\widetilde{\Psi}}$. Thus the assumptions of Theorem 4.2.2 are satisfied for $\lambda_0 = 0$, which implies that all solutions of systems ($\widehat{\mathrm{S}}_\lambda$) and ($\widetilde{\mathrm{S}}_\lambda$) with the coefficients specified in (4.24) belong to $\ell^2_{\widetilde{\Psi}}$ for any $\lambda \in \mathbb{C}$. Indeed, for any $\lambda \in \mathbb{C}$ the fundamental matrices $\widehat{\Phi}_k(\lambda)$ and $\widetilde{\Phi}_k(\lambda)$ of systems ($\widehat{\mathrm{S}}_\lambda$) and ($\widetilde{\mathrm{S}}_\lambda$) with (4.24) satisfying $\widehat{\Phi}_0(\lambda) = I = \widetilde{\Phi}_0(\lambda)$ are given by

$$\widehat{\Phi}_k(\lambda) = \begin{pmatrix} 1/\varepsilon^k & (\lambda/\varepsilon^{k-1}) \sum_{j=0}^{k-1} v_j \\ 0 & 1/\varepsilon^k \end{pmatrix}, \quad \widetilde{\Phi}_k(\lambda) = \begin{pmatrix} \varepsilon^k & \lambda\, \varepsilon^{k+1} \sum_{j=0}^{k-1} v_j \\ 0 & \varepsilon^k \end{pmatrix},$$

from which we obtain again

$$\big\|\hat{z}^{[1]}(\lambda)\big\|^2_{\widehat{\Psi}} = 0 = \big\|\tilde{z}^{[1]}(\lambda)\big\|^2_{\widetilde{\Psi}} \quad \text{and} \quad \big\|\hat{z}^{[2]}(\lambda)\big\|^2_{\widehat{\Psi}} < \infty, \quad \big\|\tilde{z}^{[2]}(\lambda)\big\|^2_{\widetilde{\Psi}} < \infty.$$

One also easily observe that the first two conditions in (4.23) are not satisfied (since the corresponding series are divergent), but still all solutions of systems ($\widehat{\mathrm{S}}_\lambda$) and ($\widetilde{\mathrm{S}}_\lambda$) with (4.24) do belong to $\ell^2_{\widetilde{\Psi}}$. In addition, we note that for $\varepsilon = 1$ both systems coincide and reduce to the system investigated in Example 2.5.3, see (2.78). ▲

Finally, let us consider systems ($\widehat{\mathrm{S}}_\lambda$) and ($\widetilde{\mathrm{S}}_\lambda$), which correspond to the following n-vector-valued difference equations of order $2m$ and of the Sturm–Liouville type

$$\sum_{s=0}^{m} (-1)^s \Delta^s \Big[\widehat{P}_k^{[s]}\, \Delta^s \hat{y}_{k+1-s}(\lambda)\Big] = \lambda\, \widehat{W}_k\, \hat{y}_{k+1}(\lambda), \tag{$\widehat{\mathrm{E}}_\lambda$}$$
$$\sum_{s=0}^{m} (-1)^s \Delta^s \Big[\widetilde{P}_k^{[s]}\, \Delta^s \tilde{y}_{k+1-s}(\lambda)\Big] = \lambda\, \widetilde{W}_k\, \tilde{y}_{k+1}(\lambda), \tag{$\widetilde{\mathrm{E}}_\lambda$}$$

on $[0,\infty)_{\mathbb{Z}}$, where $\widehat{P}^{[0]},\dots,\widehat{P}^{[m]},\widetilde{P}^{[0]},\dots,\widetilde{P}^{[m]},\widehat{W},\widetilde{W} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{n\times n}$ with $\det \widehat{P}_k^{[m]} \neq 0$ and $\det \widetilde{P}_k^{[m]} \neq 0$ for all $k \in [0,\infty)_{\mathbb{Z}}$. In particular, if $m = n = 1$ we obtain the pair of scalar difference equations

$$-\Delta\Big(\hat{p}_k^{[1]}\, \Delta \hat{y}_k(\lambda)\Big) + \hat{p}_k^{[0]}\, \hat{y}_{k+1}(\lambda) = \lambda\, \hat{w}_k\, \hat{y}_{k+1}(\lambda),$$
$$-\Delta\Big(\tilde{p}_k^{[1]}\, \Delta \tilde{y}_k(\lambda)\Big) + \tilde{p}_k^{[0]}\, \tilde{y}_{k+1}(\lambda) = \lambda\, \tilde{w}_k\, \tilde{y}_{k+1}(\lambda),$$

where $\hat{p}^{[0]},\hat{p}^{[1]},\tilde{p}^{[0]},\tilde{p}^{[1]},\hat{w},\tilde{w} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})$ with $\hat{p}^{[1]} \neq 0$ and $\tilde{p}^{[1]} \neq 0$ for all $k \in [0,\infty)_{\mathbb{Z}}$. Then equations ($\widehat{\mathrm{E}}_\lambda$) and ($\widetilde{\mathrm{E}}_\lambda$) can be written, respectively, as systems ($\widehat{\mathrm{S}}_\lambda$) and ($\widetilde{\mathrm{S}}_\lambda$) with the coefficients given similarly as in (1.20)–(1.22) and (2.11), which yield

$$\widehat{\Psi}_k = \operatorname{diag}\Big\{\widehat{W}_k, 0, \dots, 0\Big\}, \qquad \widetilde{\Psi}_k = \operatorname{diag}\Big\{\widetilde{W}_k, 0, \dots, 0\Big\}. \tag{4.25}$$

The first and second conditions in (4.1) imply $\widetilde{P}_k^{[j]} = \widehat{P}_k^{[j]*}$ and $\widetilde{\mathcal{W}}_k = \widehat{\mathcal{W}}_k^*$ for all $j = 0, \dots, m$ and $k \in [0,\infty)_{\mathbb{Z}}$, while the third condition is satisfied trivially. Moreover, assumption (4.2) forces that $\widetilde{\mathcal{W}}_k = \widehat{\mathcal{W}}_k \geq 0$ on $[0,\infty)_{\mathbb{Z}}$, i.e., the weight matrices $\widetilde{\mathcal{W}}_k$ and $\widehat{\mathcal{W}}_k$ are Hermitian matrices and coincide on the interval $[0,\infty)_{\mathbb{Z}}$. With the vectors $\hat{z}_k(\lambda), \tilde{z}_k(\lambda)$ defined similarly as in (1.20) and the matrices $\widehat{\Psi}_k$, $\widetilde{\Psi}_k$ from (4.25) we have

$$\hat{z}_{k+1}^*(\lambda)\,\widehat{\Psi}_k\,\hat{z}_{k+1}(\lambda) = \hat{y}_{k+1}^*(\lambda)\,\widehat{\mathcal{W}}_k\,\hat{y}_{k+1}(\lambda) \quad \text{and} \quad \tilde{z}_{k+1}^*(\lambda)\,\widetilde{\Psi}_k\,\tilde{z}_{k+1}(\lambda) = \tilde{y}_{k+1}^*(\lambda)\,\widetilde{\mathcal{W}}_k\,\tilde{y}_{k+1}(\lambda).$$

This shows that the associated space of square summable sequences has the form

$$\ell^2_{\widehat{\mathcal{W}}} := \Big\{ y \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^n \;\Big|\; \sum_{k=0}^{\infty} y_{k+1}^*(\lambda)\,\widehat{\mathcal{W}}_k\,y_{k+1}(\lambda) < \infty \Big\}.$$

Then from Theorem 4.2.2 we get the following result, which in the scalar case provides a discrete analogue of [168, Theorem 1].

Corollary 4.2.6. *Let the numbers $m, n \in \mathbb{N}$ be given and $\widehat{P}^{[0]}, \dots, \widehat{P}^{[m]}, \widehat{\mathcal{W}} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{n\times n}$ be such that $\widehat{\mathcal{W}}_k = \widehat{\mathcal{W}}_k^* \geq 0$ on $[0,\infty)_{\mathbb{Z}}$. Consider equations ($\widehat{\mathrm{E}}_\lambda$) and ($\widetilde{\mathrm{E}}_\lambda$) with $\widetilde{P}_k^{[j]} := \widehat{P}_k^{[j]*}$ and $\widetilde{\mathcal{W}}_k := \widehat{\mathcal{W}}_k$ for all $j \in \{0, \dots, m\}$ and $k \in [0,\infty)_{\mathbb{Z}}$. If there exists $\lambda_0 \in \mathbb{C}$ such that all solutions of equations ($\widehat{\mathrm{E}}_{\lambda_0}$) and ($\widetilde{\mathrm{E}}_{\lambda_0}$) belong to $\ell^2_{\widehat{\mathcal{W}}}$, then all solutions of equations ($\widehat{\mathrm{E}}_\lambda$) and ($\widetilde{\mathrm{E}}_\lambda$) belong to the space $\ell^2_{\widehat{\mathcal{W}}}$ for an arbitrary $\lambda \in \mathbb{C}$.*

Remark 4.2.7. If, in addition, the coefficient matrices $\widehat{P}^{[0]}, \dots, \widehat{P}^{[m]}$ are Hermitian, then from Corollary 4.2.6 one easily concludes the invariance of the limit circle case for any even order vector-valued Sturm–Liouville difference equation discussed in Remark 2.4.18. Similarly we can derive also the invariance for pairs of difference/discrete equations of the type as in (2.9) or (2.10).

4.3 Bibliographical notes

The results of this chapter are a special case of the invariance of the limit circle case for two differential systems on time scales established in [A19] but without the shift in the definition of the space $\ell^2_{\widehat{\Psi}}$. The present reformulation and the proof are published for the first time in the setting of systems ($\widehat{\mathcal{S}}_\lambda$) and ($\widetilde{\mathcal{S}}_\lambda$). The statement of Corollary 4.2.6 is new in the case $n > 1$ or $m > 1$. Moreover, the presence of the shift in the definition of $\ell^2_{\widehat{\Psi}}$ produces less restrictive assumptions on the coefficients of equations ($\widehat{\mathrm{E}}_\lambda$) and ($\widetilde{\mathrm{E}}_\lambda$), compare Corollary 4.2.6 with $m = n = 1$ and [A19, Corollary 4.4] with $\mathbb{T} = \mathbb{Z}$. Finally, Example 4.2.5 corresponds to [A19, Example 4.6].

It can be of no practical use to know that π is irrational, but if we can know, it surely would be intolerable not to know.

Edward Charles Titchmarsh, see [136, pg. 113]

Chapter 5

Polynomial and analytic dependence on spectral parameter

In this chapter we extend some of the previous results to systems with polynomial or analytic dependence on the spectral parameter. More specifically, we consider the discrete symplectic system

$$z_{k+1}(\lambda) = \mathbb{S}_k(\lambda) z_k(\lambda), \tag{$\mathcal{S}_\lambda$}$$

whose coefficient matrix $\mathbb{S}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ is analytic (or in a special case only polynomial) in the spectral parameter $\lambda \in \mathbb{C}$ in a neighborhood of 0, i.e.,

$$\mathbb{S}_k(\lambda) = \sum_{j=0}^{\infty} \lambda^j \mathbb{S}_k^{[j]}, \tag{5.1}$$

and it satisfies the symplectic-type identity

$$\mathbb{S}_k^*(\bar{\lambda}) \mathcal{J} \mathbb{S}_k(\lambda) = \mathcal{J}. \tag{5.2}$$

System ($\mathcal{S}_\lambda$) includes several significant special cases known in the literature. Obviously, system ($\mathbb{S}_\lambda$) from Chapter 2 is a special case of ($\mathcal{S}_\lambda$), see (2.2). Furthermore, we will see that the linear Hamiltonian difference system in (2.6) leads to system ($\mathcal{S}_\lambda$) with polynomial dependence on λ. Therefore, in order to unify the known theory of systems (2.6) and ($\mathbb{S}_\lambda$), it is necessary to study the systems with polynomial dependence on λ. In fact, our interest in system ($\mathcal{S}_\lambda$) is motivated by the latter observation and also by [49], where system ($\mathcal{S}_\lambda$) with $\mathbb{S}_k^{[0]} \equiv I$ was investigated.

We discuss an eigenvalue problem associated with system ($\mathcal{S}_\lambda$) and develop the theory of Weyl disks and square summable solutions (including the limit point and limit circle cases) for system ($\mathcal{S}_\lambda$). However, we point out that in this treatment we encounter several "problems", which did not appear in Chapters 2–4, i.e., when the dependence on λ was only linear. For example, the weight matrix is no longer constant in λ, which implies that the validity of the crucial weak Atkinson condition may now depend on λ, see

Hypotheses 2.3.7 and 5.3.2. Also the maximal number of linearly independent square summable solutions (i.e., the limit circle case) is not any more invariant with respect to $\lambda \in \mathbb{C}$, see Theorem 2.4.17 and Example 5.3.9. On the other hand, we prove in Theorems 5.4.1 and 5.4.5 that for system ($\mathcal{S}_\lambda$) with a special quadratic dependence on λ the invariance of the limit circle case holds true as in Chapter 4.

This chapter is organized as follows. In the next section we derive several preliminary results on system ($\mathcal{S}_\lambda$) and its coefficient matrix $\mathbb{S}(\lambda)$. We also prove a general form of the Lagrange identity for system ($\mathcal{S}_\lambda$) including the explicit calculation of the corresponding weight matrix in terms of the coefficients of ($\mathcal{S}_\lambda$), see Theorem 5.1.6. As a consequence we obtain the $\mathcal{J}$-monotonicity of a fundamental matrix of system ($\mathcal{S}_\lambda$), which is used in [49] for proving the Krein traffic rules for the eigenvalues of the fundamental matrix. In Section 5.2 we discuss in more details some special cases of system ($\mathcal{S}_\lambda$), which are known in the literature. In Section 5.3 we show that under appropriate Atkinson-type conditions involving the weight matrix, the theory of eigenvalues, Weyl disks, and square summable solutions developed in Chapter 2 remains valid without any change also for system ($\mathcal{S}_\lambda$). Finally, in Section 5.4 we establish the invariance of the limit circle case for system ($\mathcal{S}_\lambda$) with a special quadratic dependence on the spectral parameter, which includes also system (2.6) with $E_k \equiv 0$.

5.1 Preliminaries and Lagrange identity

Throughout this chapter we assume that $\mathbb{S}_k(\lambda)$ has a positive radius of convergence as a power series with respect to λ uniformly in $k \in [0,\infty)_\mathbb{Z}$. It means that there exists $\varepsilon > 0$ such that $\mathbb{S}_k(\lambda)$ is absolute convergent for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| < \varepsilon$ and all $k \in [0,\infty)_\mathbb{Z}$. We denote this region of convergence as $\mathbb{C}_\mathbb{S}$, i.e., we have $\mathbb{C}_\mathbb{S} := \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\}$. Moreover, we say that $\mathbb{S}_k(\lambda)$ is a *polynomial matrix* (of degree p) with respect to λ, if there exists $p \in \mathbb{N}_0$ such that $\mathbb{S}_k(\lambda) = \sum_{j=0}^{p} \lambda^j \mathbb{S}_k^{[j]}$ with $\mathbb{S}_k^{[p]} \not\equiv 0$. Obviously, in the latter case we can take $\varepsilon = \infty$, i.e., $\mathbb{C}_\mathbb{S} = \mathbb{C}$. If $\mathbb{S}_k^{[j]} \not\equiv 0$ for infinitely many $j \in \mathbb{N}_0$, we say that the matrix $\mathbb{S}_k(\lambda)$ is *analytic* with respect to λ.

Using the absolute convergence of the matrices $\mathbb{S}_k(\lambda)$ for all $\lambda \in \mathbb{C}_\mathbb{S}$, identity (5.2) can be equivalently written as

$$\mathbb{S}_k^{[0]*} \mathcal{J} \mathbb{S}_k^{[0]} = \mathcal{J} \quad \text{and} \quad \sum_{j=0}^{m} \mathbb{S}_k^{[j]*} \mathcal{J} \mathbb{S}_k^{[m-j]} = 0 \quad \text{for all } m \in \mathbb{N}. \tag{5.3}$$

If $\mathbb{S}_k(\lambda)$ is a polynomial matrix of degree p in λ, then the sum in (5.3) is nontrivial only for $m = 0, \dots, 2p$. Moreover, identity (5.2) also implies that $\mathbb{S}_k(\lambda)$ is invertible with

$$\mathbb{S}_k^{-1}(\lambda) = -\mathcal{J} \mathbb{S}_k^*(\bar{\lambda}) \mathcal{J} = -\sum_{j=0}^{\infty} \lambda^j \mathcal{J} \mathbb{S}_k^{[j]*} \mathcal{J}. \tag{5.4}$$

The following lemma provides several equivalent formulations of assumption (5.2), compare with Lemma 2.1.1. The proof follows (again) by direct calculations and from (5.4).

Lemma 5.1.1. *Let $n \in \mathbb{N}$ be given. For any $k \in [0,\infty)_\mathbb{Z}$ the following conditions are equivalent.*

(i) *The matrix $\mathbb{S}_k(\lambda)$ in (5.1) satisfies identity (5.2) for all $\lambda \in \mathbb{C}_\mathbb{S}$.*

(ii) *The matrices $\mathbb{S}_k^{[0]}, \mathbb{S}_k^{[1]}, \dots$ satisfy the equalities in (5.3).*

(iii) *The matrix* $\mathbb{S}_k(\lambda)$ *satisfies*

$$\mathbb{S}_k(\lambda)\mathcal{J}\mathbb{S}_k^*(\bar{\lambda}) = \mathcal{J} \quad \text{for all } \lambda \in \mathbb{C}_{\mathbb{S}}. \tag{5.5}$$

(iv) *The matrices* $\mathcal{S}_k^{[0]}, \mathcal{S}_k^{[1]}, \dots$ *satisfy*

$$\mathcal{S}_k^{[0]}\mathcal{J}\mathcal{S}_k^{[0]*} = \mathcal{J} \quad \text{and} \quad \sum_{j=0}^{m} \mathcal{S}_k^{[j]}\mathcal{J}\mathcal{S}_k^{[m-j]*} = 0 \quad \text{for all } m \in \mathbb{N}. \tag{5.6}$$

Given Lemma 5.1.1 we can summarize the basic notation used in this chapter.

Notation 5.1.2. The number $n \in \mathbb{N}$ is fixed and $\mathcal{S}^{[0]}, \mathcal{S}^{[1]}, \dots \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ are such that

(i) the equalities in (5.3) are satisfied for all $k \in [0,\infty)_{\mathbb{Z}}$ and

(ii) the radius of convergence of $\mathbb{S}_k(\lambda)$ in λ is equal to $\varepsilon > 0$ uniformly in $k \in [0,\infty)_{\mathbb{Z}}$.

The invertibility of $\mathbb{S}_k(\lambda)$ guarantees that system ($\mathcal{S}_\lambda$) is uniquely solvable on $[0,\infty)_{\mathbb{Z}}$ for any initial value at any $k_0 \in [0,\infty)_{\mathbb{Z}}$. Moreover, when $\mathbb{S}_k(\lambda)$ is polynomial of degree p in λ, we obtain an additional information about its determinant, which generalizes the result in Theorem 2.1.3. However, when $\mathbb{S}_k(\lambda)$ is analytic and not polynomial in λ, then the following statement may be violated as we will demonstrate in Example 5.2.4.

Theorem 5.1.3. *Let $k \in [0,\infty)_{\mathbb{Z}}$ and $\lambda \in \mathbb{C}_{\mathbb{S}}$ be such that the matrix $\mathbb{S}_k(\lambda)$ is polynomial in λ. Then* $|\det \mathbb{S}_k(\lambda)| = \left|\det \mathcal{S}_k^{[0]}\right| = 1$.

Proof. If $\mathbb{S}_k(\lambda)$ is polynomial in λ, then identity (5.4) implies that $\mathbb{S}_k(\lambda)$ is even an unimodular polynomial matrix. Thus, its determinant is constant in λ and we have

$$|\det \mathbb{S}_k(\lambda)| = |\det \mathbb{S}_k(0)| = \left|\det \mathcal{S}_k^{[0]}\right| = 1,$$

because $\mathcal{S}_k^{[0]}$ is a symplectic matrix by the first equality in (5.3). ■

Now we return to a general case of the analytic dependence on λ. The following lemma provides an extension of identity (5.5) and it is a main tool for the proof of the Lagrange identity given below.

Lemma 5.1.4. *For all $k \in [0,\infty)_{\mathbb{Z}}$ and any $\lambda, \nu \in \mathbb{C}_{\mathbb{S}}$ we have*

$$\mathbb{S}_k(\lambda)\mathcal{J}\mathbb{S}_k^*(\nu) = \mathcal{J} + (\lambda - \bar{\nu})\Lambda_k(\lambda,\bar{\nu}), \tag{5.7}$$

where the matrix $\Lambda(\lambda,\bar{\nu}) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ is defined by

$$\Lambda_k(\lambda,\bar{\nu}) := \sum_{m=0}^{\infty}\sum_{j=0}^{m} \lambda^{m-j}\bar{\nu}^j \sum_{\ell=0}^{j} \mathcal{S}_k^{[m-\ell+1]}\mathcal{J}\mathcal{S}_k^{[\ell]*}. \tag{5.8}$$

Moreover, for $\nu = \lambda$ the matrix $\Lambda_k(\lambda,\bar{\lambda})$ is Hermitian for all $k \in [0,\infty)_{\mathbb{Z}}$.

Remark 5.1.5. If $\mathbb{S}_k(\lambda)$ is a polynomial matrix of degree p in λ, then the infinite sum in (5.8) is in fact a finite sum for $m = 0, \dots, 2p-1$. Observe also that identity (5.8) reduces to (5.5) when $\nu = \bar{\lambda}$. Moreover, we point out that the Hermitian property of $\Lambda_k(\lambda,\bar{\lambda})$ was already shown in [49, Proposition 1].

Proof of Lemma 5.1.4. Let $k \in [0,\infty)_{\mathbb{Z}}$ and $\lambda, \nu \in \mathbb{C}_{\mathbb{S}}$ be fixed. The power series for $\mathbb{S}_k(\lambda)$ and $\mathbb{S}_k^*(\nu)$ converge absolutely, so that the terms in the product $\mathbb{S}_k(\lambda)\mathcal{J}\mathbb{S}_k^*(\nu)$ can be re-arranged to the separate powers of $\lambda^{m-j}\bar{\nu}^j$, that is,

$$\mathbb{S}_k(\lambda)\mathcal{J}\mathbb{S}_k^*(\nu) = \sum_{m=0}^{\infty}\sum_{j=0}^{m} \lambda^{m-j}\bar{\nu}^j\, \mathcal{S}_k^{[m-j]}\mathcal{J}\mathcal{S}_k^{[j]*}.$$

By using identity (5.6) for each $m \in \mathbb{N}$, we replace the term $\bar{\nu}^m \mathcal{S}_k^{[0]}\mathcal{J}\mathcal{S}_k^{[m]*}$ by

$$-\bar{\nu}^m\Big(\mathcal{S}_k^{[m]}\mathcal{J}\mathcal{S}_k^{[0]*} + \mathcal{S}_k^{[m-1]}\mathcal{J}\mathcal{S}_k^{[1]*} + \cdots + \mathcal{S}_k^{[1]}\mathcal{J}\mathcal{S}_k^{[m-1]*}\Big).$$

Thus, with the aid of the first identity in (5.6), we get

$$\mathbb{S}_k(\lambda)\mathcal{J}\mathbb{S}_k^*(\nu) = \mathcal{J} + \sum_{m=1}^{\infty}\sum_{j=1}^{m}(\lambda^j - \bar{\nu}^j)\,\bar{\nu}^{m-j}\mathcal{S}_k^{[j]}\mathcal{J}\mathcal{S}_k^{[m-j]*}.$$

Upon factoring $\lambda - \bar{\nu}$ out of each term $\lambda^j - \bar{\nu}^j = (\lambda - \bar{\nu})\sum_{\ell=1}^{j}\lambda^{j-\ell}\bar{\nu}^{\ell-1}$ and collecting the remaining products with the same powers of λ and $\bar{\nu}$, we obtain

$$\begin{aligned}
\mathbb{S}_k(\lambda)\mathcal{J}\mathbb{S}_k^*(\nu) &= \mathcal{J} + (\lambda - \bar{\nu})\sum_{m=1}^{\infty}\sum_{j=1}^{m}\Big(\sum_{\ell=1}^{j}\lambda^{j-\ell}\bar{\nu}^{\ell-1}\Big)\bar{\nu}^{m-j}\mathcal{S}_k^{[j]}\mathcal{J}\mathcal{S}_k^{[m-j]*} \\
&= \mathcal{J} + (\lambda - \bar{\nu})\sum_{m=0}^{\infty}\sum_{j=0}^{m}\Big(\sum_{\ell=0}^{j}\lambda^{j-\ell}\bar{\nu}^{m+\ell-j}\Big)\mathcal{S}_k^{[j+1]}\mathcal{J}\mathcal{S}_k^{[m-j]*} \\
&= \mathcal{J} + (\lambda - \bar{\nu})\sum_{m=0}^{\infty}\sum_{j=0}^{m}\lambda^{m-j}\bar{\nu}^j\sum_{\ell=0}^{j}\mathcal{S}_k^{[m-\ell+1]}\mathcal{J}\mathcal{S}_k^{[\ell]*} = \mathcal{J} + (\lambda - \bar{\nu})\Lambda_k(\lambda,\bar{\nu}),
\end{aligned}$$

where we used also the formula $\sum_{j=0}^{m}\sum_{\ell=0}^{j} a_{j,\ell} = \sum_{\ell=0}^{m}\sum_{j=\ell}^{m} a_{j,\ell}$. Finally, for $\nu := \lambda$ we get from the fact $\mathcal{J}^* = -\mathcal{J}$ and identities (5.6) that the matrix $\Lambda_k(\lambda,\bar{\lambda})$ is Hermitian. ■

The following theorem represents the main result of this section. Its relationship to known discrete Lagrange identities in the literature is discussed in Section 5.2.

Theorem 5.1.6 (Lagrange identity). *Let the numbers $m \in \mathbb{N}$ and $\lambda, \nu \in \mathbb{C}_{\mathbb{S}}$ be given. If the sequences $Z(\lambda), Z(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times m}$ solve systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) on $[0,\infty)_{\mathbb{Z}}$, respectively, then for any $k \in [0,\infty)_{\mathbb{Z}}$ we have*

$$\Delta\Big[Z_k^*(\lambda)\mathcal{J}Z_k(\nu)\Big] = (\bar{\lambda} - \nu)Z_{k+1}^*(\lambda)\mathcal{J}\Lambda_j(\bar{\lambda},\nu)\mathcal{J}Z_{k+1}(\nu), \tag{5.9}$$

$$Z_{k+1}^*(\lambda)\mathcal{J}Z_{k+1}(\nu) = Z_0^*(\lambda)\mathcal{J}Z_0(\nu) + (\bar{\lambda} - \nu)\sum_{j=0}^{k} Z_{j+1}^*(\lambda)\mathcal{J}\Lambda_j(\bar{\lambda},\nu)\mathcal{J}Z_{j+1}(\nu). \tag{5.10}$$

In particular, for $\nu = \bar{\lambda}$ and $\nu = \lambda$ we have on $[0,\infty)_{\mathbb{Z}}$, respectively,

$$Z_k^*(\lambda)\mathcal{J}Z_k(\bar{\lambda}) \equiv Z_0^*(\lambda)\mathcal{J}Z_0(\bar{\lambda}), \tag{5.11}$$

$$Z_{k+1}^*(\lambda)\mathcal{J}Z_{k+1}(\lambda) = Z_0^*(\lambda)\mathcal{J}Z_0(\lambda) - 2i\,\mathrm{im}(\lambda)\sum_{j=0}^{k} Z_{j+1}^*(\lambda)\mathcal{J}\Lambda_j(\bar{\lambda},\lambda)\mathcal{J}Z_{j+1}(\lambda). \tag{5.12}$$

Proof. Given that $Z_k(\lambda)$ and $Z_k(\nu)$ satisfy systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) for all $k \in [0,\infty)_{\mathbb{Z}}$, respectively, we obtain from formula (5.4) and Lemma 5.1.4 that

$$\begin{aligned}\Delta\Big[Z_k^*(\lambda)\mathcal{J}Z_k(\nu)\Big] &= Z_{k+1}^*(\lambda)\Big[\mathcal{J}-\mathbb{S}_k^{*-1}(\lambda)\mathcal{J}\mathbb{S}_k^{-1}(\nu)\Big]Z_{k+1}(\nu)\\ &\overset{(5.4)}{=} Z_{k+1}^*(\lambda)\Big[\mathcal{J}+\mathcal{J}\mathbb{S}_k(\bar\lambda)\mathcal{J}\mathbb{S}_k^*(\bar\nu)\mathcal{J}\Big]Z_{k+1}(\nu) \overset{(5.7)}{=} (\bar\lambda-\nu)Z_{k+1}^*(\lambda)\mathcal{J}\Lambda_k(\bar\lambda,\nu)\mathcal{J}Z_k(\nu).\end{aligned}$$

Identities (5.9)–(5.12) are only direct consequences of (5.9). ■

Identity (5.12) indicates that the matrix $\mathcal{J}\Lambda_k(\bar\lambda,\lambda)\mathcal{J}$ will play an important role in the study of square summable solutions of system ($\mathcal{S}_\lambda$), especially in the definition of the semi-inner product associated with system ($\mathcal{S}_\lambda$), see Section 5.3. Hence we define the Hermitian $2n\times 2n$ matrix

$$\Psi_k(\lambda) := \mathcal{J}\Lambda_k(\bar\lambda,\lambda)\mathcal{J} = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\bar\lambda^{m-j}\lambda^j\sum_{\ell=0}^{j}\mathcal{J}\mathcal{S}_k^{[m-\ell+1]}\mathcal{J}\mathcal{S}_k^{[\ell]*}\mathcal{J}. \tag{5.13}$$

Remark 5.1.7. The expression of the matrix $\Psi_k(\lambda)$ given in (5.13) can be significantly simplified for real λ, i.e., for $\lambda \in \mathbb{C}_{\mathbb{S}}\cap\mathbb{R}$. In particular, if we denote by $\dot{\mathbb{S}}_k(\lambda) := \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{S}_k(\lambda)$ the derivative of $\mathbb{S}_k(\lambda)$ with respect to λ, then

$$\Psi_k(\lambda) = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\lambda^m\sum_{\ell=0}^{j}\mathcal{J}\mathcal{S}_k^{[m-\ell+1]}\mathcal{J}\mathcal{S}_k^{[\ell]*}\mathcal{J} = \mathcal{J}\dot{\mathbb{S}}_k(\lambda)\mathcal{J}\mathbb{S}_k^*(\lambda)\mathcal{J} = -\mathcal{J}\mathbb{S}_k(\lambda)\mathcal{J}\dot{\mathbb{S}}_k^*(\lambda)\mathcal{J}, \tag{5.14}$$

where the last equality follows from the fact $\Psi_k^*(\lambda) = \Psi_k(\lambda)$. The matrix $\mathcal{J}\dot{\mathbb{S}}_k(\lambda)\mathcal{J}\mathbb{S}_k^*(\lambda)\mathcal{J}$ was used in [113,149] in the oscillation theory of discrete symplectic systems with general nonlinear dependence on $\lambda \in \mathbb{R}$.

The Lagrange identity established in (5.12) has many applications in the qualitative theory of difference equations. Apart from the results in the following sections, it yields for example the $\mathcal{J}$*-monotonicity* of the fundamental matrix of system ($\mathcal{S}_\lambda$). Following the terminology from [114, pg. 7], a matrix $M \in \mathbb{C}^{2n\times 2n}$ is called $\mathcal{J}$*-nondecreasing* if $iM^*\mathcal{J}M \ge i\mathcal{J}$, and M is $\mathcal{J}$*-nonincreasing* if $iM^*\mathcal{J}M \le i\mathcal{J}$. Similarly we define the corresponding notions of a $\mathcal{J}$*-increasing* or $\mathcal{J}$*-decreasing* matrix. These concepts were used in [114] to study the stability zones for periodic linear Hamiltonian differential systems. In a similar way, such stability zones were studied in [129,130] for the linear Hamiltonian difference systems given in (2.6) with $H_k \equiv 0$ and in [49] for system ($\mathcal{S}_\lambda$) with $\mathbb{S}_k^{[0]} = I$.

Corollary 5.1.8. *Let $\lambda \in \mathbb{C}_{\mathbb{S}}$ be fixed, $\Psi_k(\lambda) \ge 0$ on $[0,\infty)_{\mathbb{Z}}$, and $\Phi(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ be a fundamental matrix of system ($\mathcal{S}_\lambda$) such that the matrix $\Phi_0(\lambda)$ is symplectic, i.e., it satisfies $\Phi_0^*(\lambda)\mathcal{J}\Phi_0(\lambda) = \mathcal{J}$. Then for every $k \in [0,\infty)_{\mathbb{Z}}$ the matrix $\Phi_k(\lambda)$ is $\mathcal{J}$-nondecreasing when* $\mathrm{im}(\lambda) > 0$, *or $\mathcal{J}$-nonincreasing when* $\mathrm{im}(\lambda) < 0$. *If, in addition, there exists $N \in [0,\infty)_{\mathbb{Z}}$ such that every nontrivial solution $z(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ of system ($\mathcal{S}_\lambda$) satisfies*

$$\sum_{k=0}^{N} z_{k+1}^*(\lambda)\Psi_{k+1}(\lambda)z_k(\lambda) > 0, \tag{5.15}$$

then the $\mathcal{J}$-monotonicity of $\Phi_k(\lambda)$ is strict for all $k \in [N+1,\infty)_{\mathbb{Z}}$, i.e., $\Phi_k(\lambda)$ is $\mathcal{J}$-increasing when $\mathrm{im}(\lambda) > 0$, *or $\mathcal{J}$-decreasing when* $\mathrm{im}(\lambda) < 0$.

Proof. By applying (5.12) to the fundamental matrix $\Phi(\lambda)$ we get

$$i\,\Phi_k^*(\lambda)\,\mathcal{J}\,\Phi_k(\lambda)-i\mathcal{J}=2\,\mathrm{im}(\lambda)\sum_{j=0}^{k-1}\Phi_{j+1}^*(\lambda)\,\Psi_j(\lambda)\,\Phi_{j+1}(\lambda). \tag{5.16}$$

Since $\Psi_k(\lambda)\geq 0$ for all $k\in[0,\infty)_{\mathbb{Z}}$, the sum on the right-hand side of (5.16) is zero for $k=0$ and nonnegative for $k\in[1,\infty)_{\mathbb{Z}}$, so that $\Phi_k(\lambda)$ is $\mathcal{J}$-nondecreasing when $\mathrm{im}(\lambda)>0$, and it is $\mathcal{J}$-nonincreasing when $\mathrm{im}(\lambda)<0$. Moreover, the additional assumption concerning inequality (5.15) guarantees that the sum in (5.16) is positive definite for all $k\in[N+1,\infty)_{\mathbb{Z}}$, so that $\Phi_k(\lambda)$ is $\mathcal{J}$-increasing when $\mathrm{im}(\lambda)>0$, and it is $\mathcal{J}$-decreasing when $\mathrm{im}(\lambda)<0$. ■

5.2 Special examples

In this section we show the connection of the generalized Lagrange identity from Theorem 5.1.6 with several special cases known in the literature. We also demonstrate that a nonsingular weight matrix $\Psi_k(\lambda)$ can be obtained when $\mathbb{S}_k(\lambda)$ is quadratic in λ, compare with the weight matrix Ψ_k defined in (2.1).

Example 5.2.1. The simplest example of system ($\mathcal{S}_\lambda$) provides system ($\mathcal{S}_\lambda$) with general linear dependence on the spectral parameter studied in Chapter 2. Indeed, if the matrix $\mathbb{S}_k(\lambda)$ is linear in λ, i.e., $\mathbb{S}_k(\lambda)=\mathcal{S}_k^{[0]}+\lambda\,\mathcal{S}_k^{[1]}$ and $\mathcal{S}_k^{[j]}:=0$ for $j=2,3,\dots$, then Lemma 5.1.1 implies that

$$\mathcal{S}_k^{[0]*}\mathcal{J}\,\mathcal{S}_k^{[1]}=\mathcal{J},\quad \mathcal{S}_k^{[0]*}\mathcal{J}\,\mathcal{S}_k^{[1]}+\mathcal{S}_k^{[1]*}\mathcal{J}\,\mathcal{S}_k^{[0]}=0,\quad\text{and}\quad \mathcal{S}_k^{[1]*}\mathcal{J}\,\mathcal{S}_k^{[1]}=0.$$

In other words, the matrices $\mathcal{S}_k:=\mathcal{S}_k^{[0]}$ and $\mathcal{V}_k=\mathcal{S}_k^{[1]}$ satisfy the first three conditions in (2.1) and system ($\mathcal{S}_\lambda$) reduces to ($\mathcal{S}_\lambda$) with $\mathbb{S}_k(\lambda):=\mathbb{S}_k(\lambda)$. In this case $\varepsilon=\infty$, i.e., $\mathbb{C}_{\mathbb{S}}=\mathbb{C}$, and $\Lambda_k(\lambda,\bar{\nu})=\mathcal{S}_k^{[1]}\mathcal{J}\,\mathcal{S}_k^{[0]*}$. Hence $\Psi_k(\lambda)=\mathcal{J}\,\mathcal{S}_k^{[1]}\mathcal{J}\,\mathcal{S}_k^{[0]*}\mathcal{J}$, i.e., $\Psi_k(\lambda)=\Psi_k$ as defined in (2.1), which shows that Theorem 5.1.6 generalizes Theorem 2.1.7 and Lemma 2.1.5. Consequently, system ($\mathcal{S}_\lambda$) includes all special cases of system ($\mathcal{S}_\lambda$) mentioned in the introduction of Chapter 2, see (2.7)–(2.10). ▲

Example 5.2.2. Now, let the dependence on λ in system ($\mathcal{S}_\lambda$) be quadratic, i.e., $\mathcal{S}_k^{[j]}\equiv 0$ for $j=3,4,\dots$ and

$$z_{k+1}(\lambda)=\Big[\mathcal{S}_k^{[0]}+\lambda\,\mathcal{S}_k^{[1]}+\lambda^2\,\mathcal{S}_k^{[2]}\Big]z_k(\lambda), \tag{5.17}$$

where $\mathcal{S}_k^{[0]}$ satisfies the first identity in (5.3), the matrices $\mathcal{S}_k^{[0]*}\mathcal{J}\,\mathcal{S}_k^{[1]}$ and $\mathcal{S}_k^{[1]*}\mathcal{J}\,\mathcal{S}_k^{[2]}$ are Hermitian, $\mathcal{S}_k^{[2]*}\mathcal{J}\,\mathcal{S}_k^{[2]}=0$, and

$$\mathcal{S}_k^{[0]*}\mathcal{J}\,\mathcal{S}_k^{[2]}+\mathcal{S}_k^{[1]*}\mathcal{J}\,\mathcal{S}_k^{[1]}+\mathcal{S}_k^{[2]*}\mathcal{J}\,\mathcal{S}_k^{[0]}=0.$$

These conditions represent identity (5.3) with $m=0,\dots,4$, while for $m=5,6,\dots$ the sum in identity (5.3) is trivial. In particular, we consider system (5.17) with the special quadratic dependence on λ given by

$$\mathcal{S}_k^{[0]}=\begin{pmatrix}\mathcal{A}_k & \mathcal{B}_k\\ \mathcal{C}_k & \mathcal{D}_k\end{pmatrix},\quad \mathcal{S}_k^{[1]}=\begin{pmatrix}0 & \mathcal{A}_k\,\mathcal{W}_k^{[2]}\\ -\mathcal{W}_k^{[1]}\mathcal{A}_k & \mathcal{C}_k\,\mathcal{W}_k^{[2]}-\mathcal{W}_k^{[1]}\mathcal{B}_k\end{pmatrix},\quad \mathcal{S}_k^{[2]}=\begin{pmatrix}0 & 0\\ 0 & -\mathcal{W}_k^{[1]}\mathcal{A}_k\,\mathcal{W}_k^{[2]}\end{pmatrix}, \tag{5.18}$$

where $\mathcal{W}_k^{[1]}$ and $\mathcal{W}_k^{[2]}$ are Hermitian $n\times n$ matrices, see also system ($\mathcal{Q}_\lambda$) in Section 5.4. Note that the coefficients in (5.18) corresponds to (2.7) when $\mathcal{W}_k^{[2]}\equiv 0$. In this case the matrix $\mathbb{S}_k(\lambda)$ can be factorized as

$$\mathbb{S}_k(\lambda)=\begin{pmatrix}I & 0\\ -\lambda\,\mathcal{W}_k^{[1]} & I\end{pmatrix}\mathcal{S}_k^{[0]}\begin{pmatrix}I & \lambda\,\mathcal{W}_k^{[2]}\\ 0 & I\end{pmatrix}, \tag{5.19}$$

compare with (2.13). This shows that $|\det \mathcal{S}_k(\lambda)| = \big|\det \mathcal{S}_k^{[0]}(\lambda)\big| = 1$ on $[0,\infty)_{\mathbb{Z}}$ for all $\lambda \in \mathbb{C}$ as claimed in Theorem 5.1.3. If we put

$$\mathcal{T}_k(\lambda) := \begin{pmatrix} 0 & \mathcal{A}_k \\ -I & \mathcal{C}_k - \lambda \mathcal{W}_k^{[1]} \mathcal{A}_k \end{pmatrix} \quad \text{and} \quad \mathcal{W}_k := \operatorname{diag}\big\{\mathcal{W}_k^{[1]}, \mathcal{W}_k^{[2]}\big\}, \tag{5.20}$$

then by (5.8) we get

$$\Lambda(\lambda,\bar\nu) = \mathcal{S}_k^{[1]} \mathcal{J} \mathcal{S}_k^{[0]*} + \lambda \mathcal{S}_k^{[2]} \mathcal{J} \mathcal{S}_k^{[0]*} - \bar\nu \mathcal{S}_k^{[0]} \mathcal{J} \mathcal{S}_k^{[2]*} + \lambda\bar\nu \mathcal{S}_k^{[2]} \mathcal{J} \mathcal{S}_k^{[1]*} = -\mathcal{T}_k(\lambda) \mathcal{W}_k \mathcal{T}_k^*(\nu).$$

Therefore by (5.13) we have

$$\Psi_k(\lambda) = -\mathcal{J}\mathcal{T}_k(\bar\lambda)\mathcal{W}_k\mathcal{T}_k^*(\bar\lambda)\mathcal{J} \quad \text{and} \quad \det \Psi_k(\lambda) = |\det \mathcal{A}_k|^2 \times \det \mathcal{W}_k, \tag{5.21}$$

i.e., the matrix $\Psi_k(\lambda)$ is no longer constant in λ and it is invertible if (and only if) the matrices $\mathcal{A}_k$, $\mathcal{W}_k^{[1]}$, and $\mathcal{W}_k^{[2]}$ are invertible, compare with $\Psi_k(\lambda) = \Psi_k$ in the case of general linear dependence on λ. However, the invertibility of the weight matrix $\Psi_k(\lambda)$ can occur only when system (5.17) with the coefficients specified in (5.18) corresponds to the linear Hamiltonian difference system from (2.6) with $A_k := I - \mathcal{A}_k^{-1}$, $B_k := \mathcal{A}_k^{-1}\mathcal{B}_k$, $C_k := \mathcal{C}_k\mathcal{A}_k^{-1}$, $E_k \equiv 0$, $F_k = \mathcal{W}_k^{[2]}$, and $G_k = -\mathcal{W}_k^{[1]}$, see Remark 1.2.1(iv) and the identities in (1.29).

In addition, formula (5.4) yields that the multiplication of $z_{k+1}(\lambda)$ by $\mathcal{T}_k^*(\bar\lambda)\mathcal{J}$ produces a backward shift in the second component. More precisely, if $z_k(\lambda) = (x_k^*(\lambda), u_k^*(\lambda))^*$ solves system (5.17) with (5.18), then by using the partially shifted notation

$$z_k^{[s]}(\lambda) := (x_{k+1}^*(\lambda), u_k^*(\lambda))^* \tag{5.22}$$

we obtain

$$\mathcal{T}_k^*(\bar\lambda)\mathcal{J} z_{k+1}(\lambda) = z_k^{[s]}(\lambda) \quad \text{and} \quad z_{k+1}^*(\lambda)\mathcal{J}\mathcal{T}_k(\bar\lambda) = -\big[\mathcal{T}_k^*(\bar\lambda)\mathcal{J} z_{k+1}(\lambda)\big]^* = -z_k^{[s]*}(\lambda). \tag{5.23}$$

Hence identity (5.9) can be written as

$$\Delta\big[z_k^*(\lambda)\mathcal{J} z_k(\nu)\big] = -(\bar\lambda-\nu) z_{k+1}^*(\lambda)\mathcal{J}\mathcal{T}_k(\bar\lambda)\mathcal{W}_k\mathcal{T}_k^*(\bar\nu)\mathcal{J} z_{k+1}(\nu) = (\bar\lambda-\nu) z_k^{[s]*}(\lambda)\mathcal{W}_k z_k^{[s]}(\nu). \tag{5.24}$$

Since the weight matrix $\mathcal{W}_k$ in (5.24) is independent of λ, we can associate with the system in hand a semi-inner product and a semi-norm, which are independent of λ, see (5.41) below. We note that the latter observation is crucial for the invariance of the limit circle case derived in Section 5.4. ▲

In the following example we investigate the connection between the linear Hamiltonian difference system from (2.6) and system ($\mathcal{S}_\lambda$).

Example 5.2.3. According to Remark 1.2.1(iv), see identity (1.28), system (2.6) can be written as system ($\mathcal{S}_\lambda$) with the coefficient matrix $\mathcal{S}_k(\lambda) = \left(\begin{smallmatrix} \mathcal{A}_k(\lambda) & \mathcal{B}_k(\lambda) \\ \mathcal{C}_k(\lambda) & \mathcal{D}_k(\lambda) \end{smallmatrix}\right)$, where

$$\left.\begin{aligned} \mathcal{A}_k(\lambda) &:= \widetilde{A}_k(\lambda) = (I - A_k - \lambda E_k)^{-1}, \quad \mathcal{B}_k(\lambda) := \widetilde{A}_k(\lambda)(B_k + \lambda F_k), \\ \mathcal{C}_k(\lambda) &:= (C_k + \lambda G_k)\widetilde{A}_k(\lambda), \quad \mathcal{D}_k(\lambda) := I - A_k^* - \lambda E_k^* + (C_k + \lambda G_k)\widetilde{A}_k(\lambda)(B_k + \lambda F_k). \end{aligned}\right\} \tag{5.25}$$

We claim that the matrix $\widetilde{A}_k(\lambda)$ is polynomial in λ, and consequently the corresponding system ($\mathcal{S}_\lambda$) is polynomial in λ. Let us fix $k \in [0,\infty)_{\mathbb{Z}}$. By the definition of the determinant, the function $d(\lambda) := \det(I - A_k - \lambda E_k)$ is a polynomial of degree at most n. The assumption on the existence of $\widetilde{A}_k(\lambda)$ for all $\lambda \in \mathbb{C}$ then implies that $d(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$. Thus,

$d(\lambda) \equiv d \neq 0$ on $\mathbb{C}$ and the matrix $I - A_k - \lambda E_k$ is unimodular. This yields that $\widetilde{A}_k(\lambda)$ is also a polynomial matrix in λ (of degree at most $n-1$) and hence by (5.25), the matrix $\mathbb{S}_k(\lambda)$ is in this case polynomial of degree at most $n+1$. Although we do not calculate the matrix $\Lambda(\bar{\lambda}, \nu)$ explicitly, the corresponding Lagrange identity can be written as in (5.24) with $\mathcal{W}_k$ replaced by $-\mathcal{J} W_k$; cf. [36, Formula (2.55)]. Indeed, one easily observes that

$$\mathbb{S}_k(\lambda)\,\mathcal{J}\,\mathbb{S}_k^*(\nu) = \begin{pmatrix} \mathcal{A}_k(\lambda)\,\mathcal{B}_k^*(\nu) - \mathcal{B}_k(\lambda)\,\mathcal{A}_k^*(\nu) & \mathcal{A}_k(\lambda)\,\mathcal{D}_k^*(\nu) - \mathcal{B}_k(\lambda)\,\mathcal{C}_k^*(\nu) \\ \mathcal{C}_k(\lambda)\,\mathcal{B}_k^*(\nu) - \mathcal{D}_k(\lambda)\,\mathcal{A}_k^*(\nu) & \mathcal{C}_k(\lambda)\,\mathcal{D}_k^*(\nu) - \mathcal{D}_k(\lambda)\,\mathcal{C}_k^*(\nu) \end{pmatrix},$$

and similarly as in (5.23) we get

$$z_{k+1}(\lambda) = -\mathcal{J}\,\mathcal{T}_k^{*-1}(\bar{\lambda})\,z_k^{[s]}(\lambda), \quad \text{where} \quad \mathcal{T}_k(\lambda) := \begin{pmatrix} 0 & \mathcal{A}_k(\lambda) \\ -I & \mathcal{C}_k(\lambda) \end{pmatrix}.$$

Therefore, following the calculation in the proof of Theorem 5.1.6 we obtain

$$\begin{aligned} \Delta\Big[z_k^*(\lambda)\,\mathcal{J}\,z_k(\nu)\Big] &= z_k^{[s]*}(\lambda)\,\mathcal{T}_k^{-1}(\bar{\lambda})\,[\mathcal{J} - \mathbb{S}_k(\bar{\lambda})\,\mathcal{J}\,\mathbb{S}_k^*(\bar{\nu})]\,\mathcal{T}_k^{*-1}(\bar{\nu})\,z_k^{[s]}(\nu) \\ &= -(\bar{\lambda} - \nu)\,z_k^{[s]*}(\lambda)\,\mathcal{J}\,W_k\,z_k^{[s]}(\nu). \end{aligned} \tag{5.26}$$

Especially, if $E_k \equiv 0$, then the matrix $\widetilde{A}_k(\lambda) \equiv \widetilde{A}_k$ does not depend on λ and in this case system (2.6) can be written as system (5.17) with the special quadratic dependence on λ specified in (5.18) with $\mathcal{A}_k$ being invertible, see also [142, Formula (2.3) and Lemma 2.2]. ▲

In the last example of this section we consider system (S_λ) with the truly analytic (i.e., nonpolynomial) dependence on λ, which was studied in [48, 49].

Example 5.2.4. Let $\mathbb{S}_k^{[j]} := (1/j!)\,E_k^j$ for $j = 0, 1, \dots,$ where $E_k \in \mathbb{C}^{2n\times 2n}$ is Hamiltonian for all $k \in [0,\infty)_{\mathbb{Z}}$, i.e., $E_k^*\mathcal{J} + \mathcal{J}E_k = 0$. Then $\mathbb{C}_{\mathbb{S}} = \mathbb{C}$ and the coefficient matrix $\mathbb{S}_k(\lambda)$ is of the exponential type, i.e.,

$$\mathbb{S}_k(\lambda) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}\,E_k^j = \exp(\lambda E_k). \tag{5.27}$$

Then by (5.8), (1.9)–(1.10), and the Hamiltonian property of E_k we obtain

$$\Lambda(\lambda, \bar{\nu}) = \sum_{j=1}^{\infty} \frac{(-1)^j\,(\lambda - \bar{\nu})^{j-1}}{j!}\,\mathcal{J}\,(E_k^*)^j, \tag{5.28}$$

compare with [49, pg. 6] or [48, Section 2]. The Lagrange identity has the same form as in (5.9) with the corresponding $\Lambda_k(\bar{\lambda}, \nu)$. Especially, let $n = 1$ and consider the matrix $\mathbb{S}_k(\lambda)$ as in (5.27) with

$$E_k \equiv E := \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}.$$

Then by (1.9) we have $\det \mathbb{S}_k(\lambda) = \mathrm{e}^{\lambda \operatorname{tr} E} = \mathrm{e}^{2i\lambda} = \mathrm{e}^{-2\operatorname{im}(\lambda)}\,\mathrm{e}^{2i\operatorname{re}(\lambda)}$ for all $k \in [0,\infty)_{\mathbb{Z}}$ and any $\lambda \in \mathbb{C}$. Thus $|\det \mathbb{S}_k(\lambda)| = \mathrm{e}^{-2\operatorname{im}(\lambda)}$ and it is equal to one if only if $\lambda \in \mathbb{R}$, which agrees with the symplecticity of the matrix $\mathbb{S}_k(\lambda)$ on the real line; compare with Theorem 5.1.3 and see also Example 5.3.9. ▲

5.3 Weyl–Titchmarsh theory

In this section we focus on an eigenvalue problem and the Weyl–Titchmarsh theory for system ($\mathcal{S}_\lambda$) with analytic or polynomial dependence on λ. We show that the main results of Chapter 2 remain valid also for the latter system, when we modify the corresponding Atkinson-type conditions to this more general setting. The solutions are weighted with respect to the Hermitian matrix $\Psi_k(\lambda)$ defined in (5.13), that is,

$$\|z\|_{\Psi(\lambda)} := \sqrt{\langle z, z\rangle_{\Psi(\lambda)}} \quad \text{and} \quad \langle z, \tilde{z}\rangle_{\Psi(\lambda)} := \sum_{k=0}^{\infty} z_{k+1}^* \Psi_k(\lambda)\tilde{z}_{k+1}. \tag{5.29}$$

The expression of the bilinear form in (5.29) justifies the restriction of $\lambda \in \mathbb{C}_{\mathbb{S}}$ to those values for which $\Psi_k(\lambda) \geq 0$. Since this condition can be violated for some $\lambda \in \mathbb{C}_{\mathbb{S}}$, we denote by $\mathbb{C}_\Psi$ and $\mathbb{C}_{\Psi,N}$ the subsets of $\mathbb{C}_{\mathbb{S}}$ such that

$$\mathbb{C}_{\Psi,N} := \left\{\lambda \in \mathbb{C}_{\mathbb{S}} \mid \Psi_k(\lambda) \geq 0 \text{ for all } k \in [0,N]_{\mathbb{Z}}\right\},$$
$$\mathbb{C}_{\Psi} := \left\{\lambda \in \mathbb{C}_{\mathbb{S}} \mid \Psi_k(\lambda) \geq 0 \text{ for all } k \in [0,\infty)_{\mathbb{Z}}\right\},$$

where $N \in [0,\infty)_{\mathbb{Z}}$. An example with $\mathbb{C}_\Psi \subsetneq \mathbb{C}_{\mathbb{S}}$ can be found in [A20, Example 5.7] for a continuous analogue of system ($\mathcal{S}_\lambda$). Our treatment is based on the Lagrange identity derived in Theorem 5.1.6 and a construction of the Weyl disks. Nevertheless, the proofs are basically the same as for the linear dependence on λ in Chapter 2 and hence they are omitted. For brevity, we do not keep the precise identification of the minimal assumptions as in Chapter 2 and slightly simplify some formulations.

Throughout this section, let $\alpha \in \Gamma$ be given, see (2.19), and $\mathcal{Z}(\lambda), \widetilde{\mathcal{Z}}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ be the two components of the fundamental matrix $\Phi_k(\lambda) = \big(\mathcal{Z}_k(\lambda), \widetilde{\mathcal{Z}}_k(\lambda)\big)$ of system ($\mathcal{S}_\lambda$) satisfying $\Phi_0(\lambda) = (\alpha^*, -\mathcal{J}\alpha^*)$, i.e., the solutions $\mathcal{Z}(\lambda)$ and $\widetilde{\mathcal{Z}}(\lambda)$ are determined by the initial conditions $\mathcal{Z}_0(\lambda) = \alpha^*$ and $\widetilde{\mathcal{Z}}_0(\lambda) = -\mathcal{J}\alpha^*$, compare with (2.22). The fundamental matrix $\Phi(\lambda)$ then satisfies the identities

$$\Phi_k^*(\bar{\lambda})\mathcal{J}\Phi_k(\lambda) = \mathcal{J} \quad \text{and} \quad \Phi_k(\lambda)\mathcal{J}\Phi_k^*(\bar{\lambda}) = \mathcal{J} \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}},$$

see Theorem 5.1.6 and compare with Lemma 2.1.6.

Now, let us fix $N \in [0,\infty)_{\mathbb{Z}}$ and $\beta \in \Gamma$. If we associate with system ($\mathcal{S}_\lambda$) the following *eigenvalue problem*

$$(\mathcal{S}_\lambda), \quad k \in [0,N]_{\mathbb{Z}}, \quad \lambda \in \mathbb{C}_{\mathbb{S}}, \quad \alpha z_0(\lambda) = 0, \quad \beta z_{N+1}(\lambda) = 0, \tag{5.30}$$

then it follows as in Theorem 2.2.3 that the eigenvalues of problem (5.30) are characterized by $\det \beta\widetilde{\mathcal{Z}}_{N+1}(\lambda) = 0$ and the corresponding eigenfunctions are of the form $\widetilde{\mathcal{Z}}(\lambda)d$ with a nonzero $d \in \operatorname{Ker} \beta\widetilde{\mathcal{Z}}_{N+1}(\lambda)$. Moreover, we introduce the following hypothesis, compare with Hypothesis 2.2.2.

Hypothesis 5.3.1 (Weak Atkinson condition – finite). For any $\lambda \in \mathbb{C}_{\Psi,N}\setminus\mathbb{R}$ every column $z(\lambda)$ of the solution $\widetilde{\mathcal{Z}}(\lambda)$ satisfies

$$\sum_{k=0}^{N} z_{k+1}^*(\lambda)\Psi_k(\lambda)z_{k+1}(\lambda) > 0.$$

Then, under Hypothesis 5.3.1, all eigenvalues of problem (5.30) restricted to the set $\mathbb{C}_{\Psi,N}$ are real and eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the semi-inner product $\langle\cdot,\cdot\rangle_{\Psi(\lambda),N}$ defined similarly as in (5.29) with the sum over the finite discrete interval $[0,N]_{\mathbb{Z}}$.

The $M(\lambda)$-function for system $(\mathcal{S}_\lambda)$ is defined in the same way as in Definition 2.2.4, i.e., $M_k(\lambda) := -[\beta\,\widetilde{\mathcal{Z}}_k(\lambda)]^{-1}\beta\,\mathcal{Z}_k(\lambda)$, and it satisfies the properties established in Lemma 2.2.5 and Theorem 2.2.7. In particular, $M_k^*(\lambda) = M_k(\bar\lambda)$ and $M_k(\lambda)$ is analytic in λ.

For the rest of this section we consider system $(\mathcal{S}_\lambda)$ on $[0,\infty)_{\mathbb{Z}}$ and study its square summable solutions. For this purpose we will need the following condition.

Hypothesis 5.3.2 (Weak Atkinson condition – infinite). For $\lambda \in \mathbb{C}_{\mathcal{S}}\setminus\mathbb{R}$ such that $\lambda, \bar\lambda \in \mathbb{C}_\Psi$ there exists $N_4 \in [0,\infty)_{\mathbb{Z}}$ such that for $\nu \in \{\lambda, \bar\lambda\}$ we have

$$\sum_{k=0}^{N_4} z_{k+1}^*(\nu)\,\Psi_k(\nu)\,z_{k+1}(\nu) > 0 \tag{5.31}$$

for every column $z(\nu)$ of the solution $\widetilde{\mathcal{Z}}(\nu)$ of system $(\mathcal{S}_\nu)$.

We note that the number N_4 in Hypothesis 5.3.2 depends in general on the chosen λ (and $\bar\lambda$). This is a weaker condition than in Hypothesis 2.3.7, where it was considered for all $\lambda \in \mathbb{C}$. The results of this section are phrased in terms of the following set

$$\mathbb{C}_A := \Big\{\lambda \in \mathbb{C}_{\mathcal{S}}\setminus\mathbb{R} \mid \text{Hypothesis 5.3.2 holds at } \lambda\Big\},$$

which is associated with the above Atkinson-type condition. Then, by definition, we have $\lambda \in \mathbb{C}_A$ if and only if $\bar\lambda \in \mathbb{C}_A$, i.e., the set $\mathbb{C}_A$ is symmetric with respect to the real axis. This observation is also very important for the development of the present theory.

For $M \in \mathbb{C}^{n\times n}$ we define the *Weyl solution* $\mathcal{X}(\lambda, M) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ of system $(\mathcal{S}_\lambda)$ by

$$\mathcal{X}_k(\lambda, M) := \mathbf{\Phi}_k(\lambda)\,(I,\ M^*)^*,$$

where $\mathbf{\Phi}(\lambda)$ is the fundamental matrix of system $(\mathcal{S}_\lambda)$ specified above, cf. (2.23). Moreover, we utilize the Hermitian matrix-valued function $\mathcal{E} : [0,\infty)_{\mathbb{Z}} \times \mathbb{C}_A \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ given by

$$\mathcal{E}_k(\lambda, M) := i\delta(\lambda)\,\mathcal{X}_k^*(\lambda, M)\,\mathcal{J}\,\mathcal{X}_k(\lambda, M),$$

compare with (2.32). This function is used for the definition of the Weyl disk $\mathcal{D}_k(\lambda)$ and the Weyl circle $\mathcal{C}_k(\lambda)$, i.e.,

$$\mathcal{D}_k(\lambda) := \Big\{M \in \mathbb{C}^{n\times n} \mid \mathcal{E}_k(\lambda, M) \le 0\Big\}, \quad \mathcal{C}_k(\lambda) := \Big\{M \in \mathbb{C}^{n\times n} \mid \mathcal{E}_k(\lambda, M) = 0\Big\}.$$

Since for $k = 0$ we have $\mathcal{E}_0(\lambda, M) = -2\delta(\lambda)\,\mathrm{im}(M)$, it follows from (5.12) that for $\lambda \in \mathbb{C}_A$ and $k \in [1,\infty)_{\mathbb{Z}}$ the elements of $\mathcal{D}_k(\lambda)$ are characterized by the inequality

$$\sum_{j=0}^{k-1} \mathcal{X}_{j+1}^*(\lambda, M)\,\Psi_j(\lambda)\,\mathcal{X}_{j+1}(\lambda, M) \le \frac{\mathrm{im}(M)}{\mathrm{im}(\lambda)}, \tag{5.32}$$

compare with Theorem 2.3.5. Similarly, the elements of $\mathcal{C}_k(\lambda)$ are characterized by the equality in (5.32). The following geometric description of the Weyl disk and the Weyl circle can be derived as in Section 2.3. If we set

$$\mathcal{G}_k(\lambda) := i\delta(\lambda)\,\widetilde{\mathcal{Z}}_k^*(\lambda)\,\mathcal{J}\,\mathcal{Z}_k(\lambda), \quad \mathcal{H}_k(\lambda) := i\delta(\lambda)\,\widetilde{\mathcal{Z}}_k^*(\lambda)\,\mathcal{J}\,\widetilde{\mathcal{Z}}_k(\lambda), \tag{5.33}$$

then $\mathcal{H}_k(\lambda)$ is Hermitian, $\mathcal{H}_0(\lambda) = 0$, and identity (5.12) yields

$$\mathcal{H}_k(\lambda) = 2\,|\mathrm{im}(\lambda)| \sum_{j=0}^{k-1} \widetilde{\mathcal{Z}}_{j+1}^*(\lambda)\,\Psi_j(\lambda)\,\widetilde{\mathcal{Z}}_{j+1}(\lambda) \ge 0. \tag{5.34}$$

This shows that $\mathcal{H}_k(\lambda)$ is nondecreasing in $k \in [0,\infty)_{\mathbb{Z}}$. Moreover, the symmetry of the set $\mathbb{C}_A$ with respect to the real axis and Hypothesis 5.3.2 guarantee that the matrices $\mathcal{H}_k(\lambda)$ and $\mathcal{H}_k(\bar\lambda)$ are positive definite for $k \in [N_4+1,\infty)_{\mathbb{Z}}$. We summarize the main properties of the Weyl disks in the following theorem.

Theorem 5.3.3. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}_A$, and suppose that Hypothesis 5.3.2 holds. Then for all $k \in [N_4+1,\infty)_{\mathbb{Z}}$ the Weyl disk and the Weyl circle admit the representations*

$$\mathcal{D}_k(\lambda) = \Big\{\mathcal{P}_k(\lambda) + \mathcal{R}_k(\lambda)\, V\, \mathcal{R}_k(\bar{\lambda}) \mid V \in \mathbb{V}\Big\}, \quad \mathcal{C}_k(\lambda) = \Big\{\mathcal{P}_k(\lambda) + \mathcal{R}_k(\lambda)\, U\, \mathcal{R}_k(\bar{\lambda}) \mid U \in \mathbb{U}\Big\}, \tag{5.35}$$

where the center $\mathcal{P}_k(\lambda)$ and the matrix radii $\mathcal{R}_k(\lambda)$, $\mathcal{R}_k(\bar{\lambda})$ are defined by

$$\mathcal{P}_k(\lambda) := -\mathcal{H}_k^{-1}(\lambda)\,\mathcal{G}_k(\lambda), \quad \mathcal{R}_k(\lambda) := \mathcal{H}_k^{-1/2}(\lambda), \quad \mathcal{R}_k(\bar{\lambda}) := \mathcal{H}_k^{-1/2}(\bar{\lambda}), \tag{5.36}$$

and $\mathbb{U}$, $\mathbb{V}$ are the sets defined in (2.39). Moreover, the Weyl disk $\mathcal{D}_k(\lambda)$ is closed, convex, and $\mathcal{D}_k(\lambda) \subseteq \mathcal{D}_j(\lambda)$ for all $k, j \in [N_4+1,\infty)_{\mathbb{Z}}$ with $k \geq j$.

Proof. The proof follows the same arguments as in Theorem 2.3.8. In particular, for the representations given in (5.35) we utilize the identity

$$\mathcal{E}_k(\lambda, M) = [\mathcal{H}_k^{-1}(\lambda)\,\mathcal{G}_k(\lambda) + M]^*\, \mathcal{H}_k(\lambda)\, [\mathcal{H}_k^{-1}(\lambda)\,\mathcal{G}_k(\lambda) + M] - \mathcal{H}_k^{-1}(\bar{\lambda}) \tag{5.37}$$

for $k \in [N_4+1,\infty)_{\mathbb{Z}}$, which is obtained by completing $\mathcal{E}_k(\lambda, M)$ to a square, see formulas (2.38) and (2.43). Expression (5.37) uses the invertibility of $\mathcal{H}_k(\lambda)$ and $\mathcal{H}_k(\bar{\lambda})$ for $k \in [N_4+1,\infty)_{\mathbb{Z}}$, which is guaranteed by the assumption $\lambda \in \mathbb{C}_A$. In fact, the motivation for the complicated form of Hypothesis 5.3.2 comes from the above symmetry argument with respect to λ and $\bar{\lambda}$. ∎

The latter properties of the Weyl disks imply that the intersection of all $\mathcal{D}_k(\lambda)$ for $k \in [N_4+1,\infty)_{\mathbb{Z}}$ is nonempty, closed, and convex. Thus we define the limiting Weyl disk

$$\mathcal{D}_+(\lambda) := \lim_{k\to\infty} \mathcal{D}_k(\lambda) = \bigcap\nolimits_{k\in[N_4+1,\infty)_{\mathbb{Z}}} \mathcal{D}_k(\lambda).$$

From Theorem 5.3.3 and inequality (5.32) we obtain the following result. It extends Corollaries 2.3.11 and 2.3.12 to the case of the analytic dependence on λ.

Theorem 5.3.4. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}_A$, and suppose that Hypothesis 5.3.2 holds. Then*

$$\mathcal{D}_+(\lambda) = \Big\{\mathcal{P}_+(\lambda) + \mathcal{R}_+(\lambda)\, V\, \mathcal{R}_+(\bar{\lambda}) \mid V \in \mathbb{V}\Big\}, \tag{5.38}$$

where the limiting center $\mathcal{P}_+(\lambda)$ and the limiting matrix radii $\mathcal{R}_+(\lambda)$, $\mathcal{R}_+(\bar{\lambda})$ are given by

$$\mathcal{P}_+(\lambda) := \lim_{k\to\infty} \mathcal{P}_k(\lambda), \quad \mathcal{R}_+(\lambda) := \lim_{k\to\infty} \mathcal{R}_k(\lambda) \geq 0, \quad \mathcal{R}_+(\bar{\lambda}) := \lim_{k\to\infty} \mathcal{R}_k(\bar{\lambda}) \geq 0. \tag{5.39}$$

In addition, a matrix $M \in \mathbb{C}^{n\times n}$ belongs to the limiting Weyl disk $D_+(\lambda)$ if and only if

$$\sum_{k=0}^{\infty} \mathcal{X}_{k+1}^*(\lambda, M)\, \Psi_k(\lambda)\, \mathcal{X}_{k+1}(\lambda, M) \leq \frac{\operatorname{im}(M)}{\operatorname{im}(\lambda)}. \tag{5.40}$$

We note that by (5.34) and (5.36) the limit of $\mathcal{R}_k(\lambda)$ as $k \to \infty$ exist and is positive semidefinite, while the proof of the existence of the limit of the matrices $\mathcal{P}_k(\lambda)$ is based on the fixed point argument as in Theorem 2.3.9. The statement of Theorem 5.3.4 follows from Theorem 5.3.3 and formula (5.32).

Let $\lambda \in \mathbb{C}_\Psi$ be fixed. We now turn our attention to the number of linearly independent square summable solutions of system (S_λ). By $\ell^2_{\Psi(\lambda)}$ we denote the space of all sequence on $[0,\infty)_{\mathbb{Z}}$, which are square summable with respect to the weight $\Psi(\lambda)$, i.e.,

$$\ell^2_{\Psi(\lambda)} := \Big\{z \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n} \mid \|z\|_{\Psi(\lambda)} < \infty\Big\},$$

where the semi-norm $\|\cdot\|_{\Psi(\lambda)}$ is defined in (5.29) and $\Psi_k(\lambda) \geq 0$ on $[0,\infty)_{\mathbb{Z}}$. The space $\ell^2_{\Psi(\lambda)}$ generally depends on the value of λ, but in some special cases it may be independent of λ, see Example 5.2.1 with $\Psi_k(\lambda) \equiv \Psi_k$. Furthermore, in view of (5.24) or (5.26) in Examples 5.2.2 and 5.2.3, i.e., we may consider for system ($\mathcal{S}_\lambda$) with the coefficients specified in (5.18) or (5.25) the space

$$\ell^2_{\mathcal{W}} := \Big\{ z \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n} \mid \sum_{k=0}^{\infty} z_k^{[s]*}\, \mathcal{W}_k\, z_k^{[s]} < \infty \Big\}, \tag{5.41}$$

which does not depend on λ, see also Section 5.4.

We are interested in the subspace $\mathcal{N}(\lambda) \subseteq \ell^2_{\Psi(\lambda)}$ consisting of all square summable solutions of system ($\mathcal{S}_\lambda$), i.e.,

$$\mathcal{N}(\lambda) := \Big\{ z \in \ell^2_{\Psi(\lambda)} \mid z \text{ solves system } (\mathcal{S}_\lambda) \Big\}. \tag{5.42}$$

If $\lambda \in \mathbb{C}_A$, then by inequality (5.40) the columns of the Weyl solution $\mathcal{X}(\lambda, M)$ corresponding to the matrices $M \in \mathcal{D}_+(\lambda)$ are linearly independent and all belong to $\mathcal{N}(\lambda)$. This means that $n \leq \dim \mathcal{N}(\lambda) \leq 2n$ for all $\lambda \in \mathbb{C}_A$, which justifies the classification of system ($\mathcal{S}_\lambda$) as being in the *limit point case* if $\dim \mathcal{N}(\lambda) = n$, and as being in the *limit circle case* if $\dim \mathcal{N}(\lambda) = 2n$. The remaining cases with $n+1 \leq \dim \mathcal{N}(\lambda) \leq 2n-1$ are called *intermediate*. Moreover, one can verify that the results of Theorem 2.4.1–Corollary 2.4.10 in Section 2.4 hold with exactly the same proofs also in the case of the analytic dependence on λ. Especially, the following extension of Theorem 2.4.8 to the analytic dependence on λ is true.

Theorem 5.3.5. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}_A$, and suppose that Hypothesis 5.3.2 holds. Then system ($\mathcal{S}_\lambda$) has exactly $n + \operatorname{rank} \mathcal{R}_+(\lambda)$ linearly independent square summable solutions, i.e.,*

$$\dim \mathcal{N}(\lambda) = n + \operatorname{rank} \mathcal{R}_+(\lambda),$$

where $\mathcal{R}_+(\lambda)$ is the matrix radius of the limiting Weyl disk $\mathcal{D}_+(\lambda)$ defined in (5.39).

By combining Theorem 5.3.5 and identity (5.38) we obtain the following limit point and limit circle classification of system ($\mathcal{S}_\lambda$) in terms of the rank of $\mathcal{R}_+(\lambda)$, compare with Theorem 2.4.3 and Corollary 2.4.10.

Corollary 5.3.6. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}_A$, and Hypothesis 5.3.2 hold. Then system ($\mathcal{S}_\lambda$) is*

(i) *in the limit point case if and only if $\mathcal{R}_+(\lambda) = 0$, in which case $\mathcal{D}_+(\lambda) = \{\mathcal{P}_+(\lambda)\}$ and $\mathcal{D}_+(\bar\lambda) = \{\mathcal{P}_+(\bar\lambda)\}$,*

(ii) *in the limit circle case if and only if $\mathcal{R}_+(\lambda)$ is invertible.*

Remark 5.3.7. We note that the results of Chapter 3 regarding the Weyl–Titchmarsh theory for discrete symplectic systems with jointly varying endpoints hold in the same way for system with the analytic dependence on λ under the appropriate strong Atkinson-type conditions including all nontrivial solutions, see Hypotheses 3.1.1 and Hypothesis 2.4.11.

Finally, we illustrate the results of the Weyl–Titchmarsh theory for system ($\mathcal{S}_\lambda$) by two interesting examples with the exponential dependence on λ discussed in Example 5.2.4.

Example 5.3.8. In this example we show that the discrete symplectic system

$$z_{k+1}(\lambda) = \exp(\lambda \mathcal{J})\, z_k(\lambda). \tag{5.43}$$

is in the limit point case for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and we calculate the unique $2n \times n$ solution (up to an invertible multiple) of system (5.43) whose columns lie in $\ell^2_{\Psi(\lambda)}$ and form a basis of $\mathcal{N}(\lambda)$, i.e., the Weyl solution $\mathcal{X}(\lambda, \mathcal{P}_+(\lambda))$. System (5.43) corresponds to the system from Example 5.2.4 with $E_k := E \equiv \mathcal{J}$, which satisfies the condition $E_k^* \mathcal{J} + \mathcal{J} E_k = 0$. Moreover, we have $\mathbb{S}_k(\lambda) = \exp(\lambda \mathcal{J}) = (\cos\lambda) I + (\sin\lambda)\mathcal{J}$ and $\mathbb{C}_{\mathbb{S}} = \mathbb{C}$, see also [16, Example 11.3.4].

For simplicity we perform the calculations below in the scalar case, i.e., for $n = 1$. The general case follows with the same arguments upon multiplication by the $n \times n$ or $2n \times 2n$ identity matrices at appropriate places. If we choose $\alpha = (1, 0)$, then the fundamental matrix $\Phi_k(\lambda)$ of system (5.43) with $\Phi_0(\lambda) = I$ is given by

$$\Phi_k(\lambda) = \exp(k\lambda\mathcal{J}) = (\cos k\lambda) I + (\sin k\lambda)\mathcal{J} = \begin{pmatrix} \cos k\lambda & \sin k\lambda \\ -\sin k\lambda & \cos k\lambda \end{pmatrix}, \quad k \in [0, \infty)_{\mathbb{Z}},$$

and so $\widetilde{\mathcal{Z}}_k(\lambda) = (\sin k\lambda,\ \cos k\lambda)^\top$. Since the powers of $\mathcal{J}$ repeat in a cycle of length four, we obtain for any $k \in [0, \infty)_{\mathbb{Z}}$ by (5.28) that $\Lambda_k(\bar{\lambda}, \lambda) = -I$ for all $\lambda \in \mathbb{R}$, while for $\lambda \in \mathbb{C}\setminus\mathbb{R}$ we calculate (with $p := \operatorname{im}(\lambda) \neq 0$)

$$\begin{aligned} \Lambda_k(\bar{\lambda}, \lambda) &= \sum_{j=1}^{\infty} \frac{(-2ip)^{j-1}}{j!} \mathcal{J}^{j+1} = \frac{1}{2ip} \sum_{j=1}^{\infty} \frac{(-1)^{j+1} (2ip)^{2j}}{(2j)!} \mathcal{J} + \frac{1}{2ip} \sum_{j=0}^{\infty} \frac{(-1)^{j+1} (2ip)^{2j+1}}{(2j+1)!} I \\ &= \frac{\cosh 2p - 1}{2p} i\mathcal{J} - \frac{\sinh 2p}{2p} I = \frac{\sinh p}{p} \Big[(\sinh p) i\mathcal{J} - (\cosh p) I \Big], \end{aligned}$$

where we used the well-known identities $\sinh 2x = 2 \sinh x \cosh x$, $\cosh 2x = 2\sinh^2 x + 1$, $i \sinh p = \sin(ix)$, and $\cosh p = \cos(ip)$. Hence by (5.13) we have

$$\Psi_k(\lambda) \equiv \Psi(\lambda) = \begin{cases} \dfrac{\sinh p}{p} \begin{pmatrix} \cosh p & -i\sinh p \\ i\sinh p & \cosh p \end{pmatrix} > 0 & \text{for all } \lambda \in \mathbb{C}\setminus\mathbb{R}, \\ I \quad \text{for all } \lambda \in \mathbb{R}, \end{cases} \tag{5.44}$$

see also (5.14). Thus $\mathbb{C}_\Psi = \mathbb{C}$. By the definitions of $\mathcal{H}_k(\lambda)$ and $\mathcal{G}_k(\lambda)$ in (5.33) we get

$$\begin{aligned} \mathcal{H}_k(\lambda) &= i\delta(\lambda)(\sin k\bar{\lambda} \cos k\lambda - \cos k\bar{\lambda} \sin k\lambda) = \delta(\lambda) \sinh(2k \operatorname{im}(\lambda)), \\ \mathcal{G}_k(\lambda) &= -i\delta(\lambda)(\sin k\bar{\lambda} \sin k\lambda + \cos k\bar{\lambda} \cos k\lambda) = -i\delta(\lambda) \cosh(2k \operatorname{im}(\lambda)). \end{aligned}$$

Note that the same value for $\mathcal{H}_k(\lambda)$ is of course obtained from formula (5.34) after some calculations. This shows that Hypothesis 5.3.2 is satisfied for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and any $N_4 \in [0, \infty)_{\mathbb{Z}}$, i.e., $\mathbb{C}_A = \mathbb{C}\setminus\mathbb{R}$. The relations in (5.36) yield

$$\mathcal{P}_k(\lambda) = i \coth(2k \operatorname{im}(\lambda)) \quad \text{and} \quad \mathcal{R}_k(\lambda) = 1/\sqrt{\sinh(2k|\operatorname{im}(\lambda)|)} \quad \text{for all } k \in [1, \infty)_{\mathbb{Z}}.$$

The center and radius of the limiting disk $\mathcal{D}_+(\lambda)$ are then

$$\mathcal{P}_+(\lambda) = \lim_{k\to\infty} \mathcal{P}_k(\lambda) = i\delta(\lambda) \quad \text{and} \quad \mathcal{R}_+(\lambda) = \lim_{k\to\infty} \mathcal{R}_k(\lambda) = 0,$$

which shows that system (5.43) is in the limit point case for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ by Corollary 5.3.6(i). Moreover, the space $\mathcal{N}(\lambda)$ of square summable solutions of system (5.43) with $\lambda \in \mathbb{C}\setminus\mathbb{R}$ is generated by the Weyl solution

$$\mathcal{X}_k(\lambda, \mathcal{P}_+(\lambda)) = \Phi_k(\lambda) \begin{pmatrix} 1 \\ \mathcal{P}_+(\lambda) \end{pmatrix} = \begin{pmatrix} \cos k\lambda + i\delta(\lambda) \sin k\lambda \\ -\sin k\lambda + i\delta(\lambda) \cos k\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ i\delta(\lambda) \end{pmatrix} \mathrm{e}^{i\delta(\lambda) k\lambda},$$

for which (we again substitute $p := \operatorname{im}(\lambda)$) we have

$$\begin{aligned}\big\| \mathcal{X}(\lambda, \mathcal{P}_+(\lambda)) \big\|^2_{\Psi(\lambda)} &= \sum_{k=0}^{\infty} \mathcal{X}^*_{k+1}(\lambda, \mathcal{P}_+(\lambda))\, \Psi_k(\lambda)\, \mathcal{X}_{k+1}(\lambda, \mathcal{P}_+(\lambda)) \\ &= \frac{2 \sinh p}{p} \times [\cosh p + \delta(\lambda) \sinh p] \times \sum_{k=0}^{\infty} \mathrm{e}^{-2|p|(k+1)} \\ &= \frac{2 \sinh p}{p} \times [\cosh p + \delta(\lambda) \sinh p] \times \frac{\mathrm{e}^{-2|p|}}{1 - e^{-2|p|}} = \frac{1}{|p|}.\end{aligned}$$

This shows that $\|\mathcal{X}(\lambda, \mathcal{P}_+(\lambda))\|_{\Psi(\lambda)} = 1/\sqrt{|\operatorname{im}(\lambda)|} < \infty$, and so indeed $\mathcal{X}(\lambda, \mathcal{P}_+(\lambda)) \in \ell^2_{\Psi(\lambda)}$ for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$. On the other hand, we also have

$$\begin{aligned}\big\| \widetilde{\mathcal{Z}}(\lambda) \big\|^2_{\Psi(\lambda)} &= \sum_{k=0}^{\infty} \widetilde{\mathcal{Z}}^*_{k+1}(\lambda)\, \Psi_k(\lambda)\, \widetilde{\mathcal{Z}}_{k+1}(\lambda) \overset{(5.34)}{=} \frac{1}{2|\operatorname{im}(\lambda)|} \lim_{k\to\infty} \mathcal{H}_k(\lambda) \\ &= \frac{1}{2|\operatorname{im}(\lambda)|} \lim_{k\to\infty} \sinh(2k|\operatorname{im}(\lambda)|) = \infty,\end{aligned}$$

i.e., $\widetilde{\mathcal{Z}}(\lambda) \notin \ell^2_{\Psi(\lambda)}$. Thus, again we get that $\dim \mathcal{N}(\lambda) = 1$ for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$, see also the proof of Theorem 2.4.3. Similarly, in arbitrary dimension n we get that the n columns of the Weyl solution $\mathcal{X}(\lambda, \mathcal{P}_+(\lambda))$ are linearly independent and they belong to $\ell^2_{\Psi(\lambda)}$, while the n columns of $\widetilde{\mathcal{Z}}(\lambda)$ are linearly independent and they do not belong to $\ell^2_{\Psi(\lambda)}$. Hence, $\dim \mathcal{N}(\lambda) = n$ and system (5.43) is in the limit point case for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. ▲

Now we give a counterexample for the invariance of the limit circle case when the dependence on λ is analytic and nonpolynomial.

Example 5.3.9. Let us consider system ($\mathcal{S}_\lambda$) with the coefficient matrix $\mathbb{S}_k(\lambda)$ from Example 5.2.4 with $E_k := E \equiv iI + \mathcal{J}$, i.e., the system

$$z_{k+1}(\lambda) = \exp(\lambda E) z_k(\lambda), \quad \exp(\lambda E) = \mathrm{e}^{i\lambda} [(\cos\lambda) I + (\sin\lambda) \mathcal{J}], \tag{5.45}$$

see again [16, Example 11.3.4]. Then $\mathbb{C}_{\mathbb{S}} = \mathbb{C}$ and the fundamental matrix $\Phi_k(\lambda)$ of system (5.45) corresponding to the choice $\alpha = (1, 0)$ is given by

$$\Phi_k(\lambda) = \exp(k\lambda E) = \mathrm{e}^{ik\lambda} [(\cos k\lambda) I + (\sin k\lambda) \mathcal{J}], \quad k \in [0, \infty)_{\mathbb{Z}}.$$

Since $(E^*)^j = -(-2i)^{j-1} E$ for $j \geq 1$, it follows by (5.28) that for any $k \in [0, \infty)_{\mathbb{Z}}$ we have $\Lambda_k(\bar{\lambda}, \lambda) = iE$ if $\lambda \in \mathbb{R}$ and $\Lambda_k(\bar{\lambda}, \lambda) = \frac{i}{4p} (\mathrm{e}^{4p} - 1) E$ if $\lambda \in \mathbb{C}\setminus\mathbb{R}$ with $p := \operatorname{im}(\lambda)$. Hence identity (5.13) yields

$$\Psi_k(\lambda) \equiv \Psi(\lambda) = \begin{cases} -\dfrac{i(\mathrm{e}^{4\operatorname{im}(\lambda)} - 1)}{4\operatorname{im}(\lambda)} E & \text{for all } \lambda \in \mathbb{C}\setminus\mathbb{R}, \\ -iE \quad \text{for all } \lambda \in \mathbb{R}. \end{cases} \tag{5.46}$$

Moreover, $iE \leq 0$, which yields through (5.46) that $\Psi(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}$, i.e., $\mathbb{C}_\Psi = \mathbb{C}$. Note that in this case $\Psi(\lambda)$ is singular for all $\lambda \in \mathbb{C}$, compare with (5.44).

The left-hand side of (5.31) with $z_k(\lambda) = \widetilde{z}_k(\lambda) = \mathrm{e}^{ik\lambda} (\sin k\lambda, \cos k\lambda)^\top$ has the form

$$\sum_{k=0}^{N_4} z^*_{k+1}(\lambda)\, \Psi(\lambda)\, z_{k+1}(\lambda) = \begin{cases} \dfrac{1 - \mathrm{e}^{-4(N_4+1)\operatorname{im}(\lambda)}}{4\operatorname{im}(\lambda)}, & \lambda \in \mathbb{C}\setminus\mathbb{R}, \\ N_4 + 1, & \lambda \in \mathbb{R}. \end{cases}$$

Therefore the inequality in (5.31) is satisfied for all $\lambda \in \mathbb{C}$ and any $N_4 \in [0,\infty)_{\mathbb{Z}}$, which implies $\mathbb{C}_A = \mathbb{C}\setminus\mathbb{R}$. By (5.33), (5.34), and (5.36) we get

$$\mathcal{H}_k(\lambda) = \delta(\lambda)\big(1 - \mathrm{e}^{-4k\,\mathrm{im}(\lambda)}\big)/2, \quad \mathcal{G}_k(\lambda) = -i\,\delta(\lambda)\mathrm{e}^{-2k\,\mathrm{im}(\lambda)}\cosh(2k\,\mathrm{im}(\lambda)),$$
$$\mathcal{P}_k(\lambda) = i\coth(2k\,\mathrm{im}(\lambda)), \quad \mathcal{R}_k^2(\lambda) = \frac{2}{\delta(\lambda)\big(1 - \mathrm{e}^{-4k\,\mathrm{im}(\lambda)}\big)}.$$

Thus by (5.39) we have $\mathcal{P}_+(\lambda) = i\delta(\lambda)$ for every $\lambda \in \mathbb{C}_A$ and $\mathcal{R}_+(\lambda) = \sqrt{2}$ for $\lambda \in \mathbb{C}_+$, while $\mathcal{R}_+(\lambda) = 0$ for $\lambda \in \mathbb{C}_-$. This shows by Corollary 5.3.6 that system (5.45) is in the limit circle case for $\lambda \in \mathbb{C}_+$ and in the limit point case for $\lambda \in \mathbb{C}_-$. The space $\mathcal{N}(\lambda)$ is then generated by the columns of the fundamental matrix $\Phi(\lambda)$ when $\lambda \in \mathbb{C}_+$, and by the Weyl solution $\mathcal{X}_k(\lambda, \mathcal{P}_+(\lambda)) \equiv (1,-i)^\top$ with $\|\mathcal{X}(\lambda,-i)\|_{\Psi(\lambda)} = 0$ when $\lambda \in \mathbb{C}_-$. For completeness we note that system (5.45) is in the limit point case also for all $\lambda \in \mathbb{R}$ with $\mathcal{X}(\lambda,-i)$ being the unique square summable solution (up to a constant multiple). ▲

5.4 Special quadratic dependence and limit circle case

In Example 5.3.9 it was shown that the dimension of $\mathcal{N}(\lambda)$ may vary with respect to λ, even when Hypothesis 5.3.2 is satisfied. In particular, system (S_λ) can be in the limit circle case for some value λ and in the limit point case for another one. This situation is not possible when the dependence on λ is only linear as we derived in Chapter 4 and stated in Theorem 2.4.17. In this section we prove a similar invariance of the limit circle case for system (S_λ) with the special quadratic dependence on λ from Example 5.2.2. In this case we can choose the associated space of square summable solutions to be independent of λ, which is a key ingredient for this result. Note that this property was trivially satisfied also in the previous chapters.

5.4.1 Results for one system

Let us consider system (5.17) with the coefficients specified in (5.18) or equivalently (suppressing the argument λ)

$$x_{k+1} = \mathcal{A}_k x_k + \big(\mathcal{B}_k + \lambda\mathcal{A}\mathcal{W}_k^{[2]}\big)u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \big(\mathcal{D}_k + \lambda\,\mathcal{C}_k\mathcal{W}_k^{[2]}\big)u_k - \lambda\,\mathcal{W}_k^{[1]}x_{k+1}, \qquad (\mathcal{Q}_\lambda)$$

where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{W}^{[1]},\mathcal{W}^{[2]} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{n\times n}$ are such that the matrix $\mathcal{S}_k^{[0]}$ in (5.18) satisfies the first equality in (5.3), $\mathcal{W}_k^{[1]}$ and $\mathcal{W}_k^{[2]}$ are Hermitian, and

$$\mathcal{W}_k := \operatorname{diag}\big\{\mathcal{W}_k^{[1]}, \mathcal{W}_k^{[2]}\big\} \geq 0 \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}. \tag{5.47}$$

Recalling the notation from (5.22), we associate with system ($\mathcal{Q}_\lambda$) the space of all square summable sequences with respect to the weight matrix $\mathcal{W}_k$ defined in (5.41), i.e., $\ell^2_{\mathcal{W}}$. Since from (5.21) and (5.24) one infers that $z^*_{k+1}(\lambda)\,\Psi_k(\lambda)\,z_{k+1}(\lambda) = z_k^{[s]*}(\lambda)\,\mathcal{W}_k\,z_k^{[s]}(\lambda)$ for all solutions of system ($\mathcal{Q}_\lambda$), the space of all solutions being in $\ell^2_{\mathcal{W}}$ is the same as the corresponding space $\mathcal{N}(\lambda)$ defined in (5.42). This means that we have

$$\mathcal{N}(\lambda) = \big\{z \in \ell^2_{\mathcal{W}} \mid z \text{ solves system } (\mathcal{Q}_\lambda)\big\}$$

and system ($\mathcal{Q}_\lambda$) is in the limit point case when $\dim\mathcal{N}(\lambda) = n$ and in the limit circle case when $\dim\mathcal{N}(\lambda) = 2n$.

The following result concerning the invariance of the limit circle case for system ($\mathcal{Q}_\lambda$) generalizes [142, Theorem 5.5] for system (2.6) with $E_k \equiv 0$, which corresponds to system ($\mathcal{Q}_\lambda$) with $\mathcal{A}_k$ being invertible on $[0,\infty)_{\mathbb{Z}}$. The proof is given in Subsection 5.4.2 below, where we establish a more general statement (Theorem 5.4.5) for two systems of the form ($\mathcal{Q}_\lambda$) as in Chapter 4.

Theorem 5.4.1. *Let (5.47) hold. If there exists $\lambda_0 \in \mathbb{C}$ such that system ($\mathcal{Q}_{\lambda_0}$) is in the limit circle case, then system ($\mathcal{Q}_\lambda$) is in the limit circle case for every $\lambda \in \mathbb{C}$.*

From Theorem 5.4.1 we obtain the following simple criterion for the limit circle case in terms of the norms of the coefficients $\mathcal{S}_k^{[0]}$ and $\mathcal{W}_k$; cf. Corollary 2.4.19. It extends the statement in [134, Theorem 6.3] for system (2.6) with $E_k \equiv 0$.

Corollary 5.4.2. *Let (5.47) hold and assume that*

$$\sum_{k=0}^{\infty} \|\mathcal{S}_k^{[0]} - I\|_1 < \infty \quad \textit{and} \quad \sum_{k=0}^{\infty} \|\mathcal{W}_k\|_1 < \infty. \tag{5.48}$$

Then system ($\mathcal{Q}_\lambda$) is in the limit circle case for all $\lambda \in \mathbb{C}$.

Proof. The conditions in (5.48) imply that system ($\mathcal{Q}_0$) is in the limit circle case. Therefore the result follows from Theorem 5.4.1. Alternatively, this statement can be derived as a special case of Corollary 5.4.6 below regarding two systems. ∎

In the scalar case we get from Theorems 5.4.1 and 5.3.5 the following limit point criterion for system ($\mathcal{Q}_\lambda$).

Theorem 5.4.3. *Let $n = 1$ and assume that condition (5.47) and Hypothesis 5.3.2 hold. If there exists $\lambda_0 \in \mathbb{C}$ such that system ($\mathcal{Q}_{\lambda_0}$) is in the limit point case, then system ($\mathcal{Q}_\lambda$) is in the limit point case for every $\lambda \in \mathbb{C}_A$.*

Proof. Assume that for some $\lambda_1 \in \mathbb{C}_A$ system ($\mathcal{Q}_{\lambda_1}$) is not in the limit point case. Then by $n = 1$ and Theorem 5.3.5 we know that ($\mathcal{Q}_{\lambda_1}$) is in the limit circle case. Consequently, by Theorem 5.4.1, system ($\mathcal{Q}_\lambda$) is in the limit circle case for all $\lambda \in \mathbb{C}$, which contradicts the original assumption that system ($\mathcal{Q}_{\lambda_0}$) is in the limit point case. ∎

By combining Theorems 5.4.1 and 5.4.3 we obtain the following extension of the Weyl alternative, i.e., the dichotomy between the limit point and limit circle classifications of system ($\mathcal{Q}_\lambda$) for all suitable λ; cf. Corollary 2.4.23.

Corollary 5.4.4 (Weyl alternative). *Let $n = 1$ and assume that condition (5.47) and Hypothesis 5.3.2 hold. Then system ($\mathcal{Q}_\lambda$) is either in the limit circle case for all $\lambda \in \mathbb{C}$, or in the limit point case for all $\lambda \in \mathbb{C}_A$.*

5.4.2 Results for two systems

Motivated by the results in Chapter 4, we consider instead of system ($\mathcal{Q}_\lambda$) two systems of the same form (suppressing the argument λ)

$$\hat{x}_{k+1} = \widehat{\mathcal{A}}_k \hat{x}_k + \big(\widehat{\mathcal{B}}_k + \lambda \widehat{\mathcal{A}} \mathcal{W}_k^{[2]}\big)\hat{u}_k, \quad \hat{u}_{k+1} = \widehat{\mathcal{C}}_k \hat{x}_k + \big(\widehat{\mathcal{D}}_k + \lambda \widehat{\mathcal{C}}_k \mathcal{W}_k^{[2]}\big)\hat{u}_k - \lambda \mathcal{W}_k^{[1]} \hat{x}_{k+1}, \quad (\widehat{\mathcal{Q}}_\lambda)$$
$$\tilde{x}_{k+1} = \widetilde{\mathcal{A}}_k \tilde{x}_k + \big(\widetilde{\mathcal{B}}_k + \lambda \widetilde{\mathcal{A}} \mathcal{W}_k^{[2]}\big)\tilde{u}_k, \quad \tilde{u}_{k+1} = \widetilde{\mathcal{C}}_k \tilde{x}_k + \big(\widetilde{\mathcal{D}}_k + \lambda \widetilde{\mathcal{C}}_k \mathcal{W}_k^{[2]}\big)\tilde{u}_k - \lambda \mathcal{W}_k^{[1]} \tilde{x}_{k+1}, \quad (\widetilde{\mathcal{Q}}_\lambda)$$

where $\widehat{\mathcal{A}}, \widehat{\mathcal{B}}, \widehat{\mathcal{C}}, \widehat{\mathcal{D}}_k, \widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{D}}_k, \mathcal{W}^{[1]}, \mathcal{W}^{[2]} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{n\times n}$ with $\mathcal{W}_k^{[1]}$ and $\mathcal{W}_k^{[2]}$ being Hermitian and satisfying (5.47). Note that the weight matrices $\mathcal{W}_k^{[1]}$ and $\mathcal{W}_k^{[2]}$ are the same in both systems ($\widetilde{\mathcal{Q}}_\lambda$) and ($\widehat{\mathcal{Q}}_\lambda$). This is justified by the requirement of having the same space of square summable functions associated with these systems. The coefficient matrices $\widehat{\mathcal{S}}^{[0]}, \widehat{\mathcal{S}}^{[1]}, \widehat{\mathcal{S}}^{[2]} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ and $\widetilde{\mathcal{S}}^{[0]}, \widetilde{\mathcal{S}}^{[1]}, \widetilde{\mathcal{S}}^{[2]} \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ of systems ($\widehat{\mathcal{Q}}_\lambda$) and ($\widetilde{\mathcal{Q}}_\lambda$), respectively, are defined analogously to (5.18). We assume that systems ($\widehat{\mathcal{Q}}_\lambda$) and ($\widetilde{\mathcal{Q}}_\lambda$) are in general non-symplectic, i.e., we do not impose that the matrices $\widetilde{\mathcal{S}}_k^{[0]}$ and $\widehat{\mathcal{S}}_k^{[0]}$ are such that the first equality in (5.3) holds. Instead we assume on $[0,\infty)_{\mathbb{Z}}$ the combined identity

$$\widetilde{\mathcal{S}}_k^{[0]*} \mathcal{J} \widehat{\mathcal{S}}_k^{[0]} = \mathcal{J}. \tag{5.49}$$

By using the block structure of $\widetilde{\mathcal{S}}_k^{[0]}$ and $\widehat{\mathcal{S}}_k^{[0]}$ we obtain that equality (5.49) is equivalent to

$$\widetilde{\mathcal{A}}_k^* \widehat{\mathcal{C}}_k = \widetilde{\mathcal{C}}_k^* \widehat{\mathcal{A}}_k, \quad \widetilde{\mathcal{D}}_k^* \widehat{\mathcal{B}}_k = \widetilde{\mathcal{B}}_k^* \widehat{\mathcal{D}}_k, \quad \widetilde{\mathcal{A}}_k^* \widehat{\mathcal{D}}_k - \widetilde{\mathcal{C}}_k^* \widehat{\mathcal{B}}_k = I \quad \text{and} \quad \widetilde{\mathcal{D}}_k^* \widehat{\mathcal{A}}_k - \widetilde{\mathcal{B}}_k^* \widehat{\mathcal{C}}_k = I. \tag{5.50}$$

Consequently, the coefficient matrices also satisfy the following identities

$$\widetilde{\mathcal{S}}_k^{[0]*} \mathcal{J} \widehat{\mathcal{S}}_k^{[1]} + \widetilde{\mathcal{S}}_k^{[1]*} \mathcal{J} \widehat{\mathcal{S}}_k^{[0]} = 0, \tag{5.51}$$
$$\widetilde{\mathcal{S}}_k^{[0]*} \mathcal{J} \widehat{\mathcal{S}}_k^{[2]} + \widetilde{\mathcal{S}}_k^{[1]*} \mathcal{J} \widehat{\mathcal{S}}_k^{[1]} + \widetilde{\mathcal{S}}_k^{[2]*} \mathcal{J} \widehat{\mathcal{S}}_k^{[0]} = 0, \tag{5.52}$$
$$\widetilde{\mathcal{S}}_k^{[1]*} \mathcal{J} \widehat{\mathcal{S}}_k^{[2]} + \widetilde{\mathcal{S}}_k^{[2]*} \mathcal{J} \widehat{\mathcal{S}}_k^{[1]} = 0, \quad \widetilde{\mathcal{S}}_k^{[2]*} \mathcal{J} \widehat{\mathcal{S}}_k^{[2]} = 0. \tag{5.53}$$

In addition, identity (5.49) is equivalent to

$$\widetilde{\mathcal{S}}_k^{[0]} \mathcal{J} \widehat{\mathcal{S}}_k^{[0]*} = \mathcal{J}, \tag{5.54}$$

which can be written in terms of the $n \times n$ blocks as

$$\widetilde{\mathcal{A}}_k \widehat{\mathcal{B}}_k^* = \widetilde{\mathcal{B}}_k \widehat{\mathcal{A}}_k^*, \quad \widetilde{\mathcal{D}}_k \widehat{\mathcal{C}}_k^* = \widetilde{\mathcal{C}}_k \widehat{\mathcal{D}}_k^*, \quad \widetilde{\mathcal{A}}_k \widehat{\mathcal{D}}_k^* - \widetilde{\mathcal{B}}_k \widehat{\mathcal{C}}_k^* = I \quad \text{and} \quad \widetilde{\mathcal{D}}_k \widehat{\mathcal{A}}_k^* - \widetilde{\mathcal{C}}_k \widehat{\mathcal{B}}_k^* = I. \tag{5.55}$$

Since identity (5.49) is equivalent also with $\widehat{\mathcal{S}}_k^{[0]*} \mathcal{J} \widetilde{\mathcal{S}}_k^{[0]} = \mathcal{J}$ and $\widehat{\mathcal{S}}_k^{[0]} \mathcal{J} \widetilde{\mathcal{S}}_k^{[0]*} = \mathcal{J}$, the block matrices satisfy the relations

$$\widehat{\mathcal{A}}_k^* \widetilde{\mathcal{C}}_k = \widehat{\mathcal{C}}_k^* \widetilde{\mathcal{A}}_k, \quad \widehat{\mathcal{D}}_k^* \widetilde{\mathcal{B}}_k = \widehat{\mathcal{B}}_k^* \widetilde{\mathcal{D}}_k, \quad \widehat{\mathcal{A}}_k^* \widetilde{\mathcal{D}}_k - \widehat{\mathcal{C}}_k^* \widetilde{\mathcal{B}}_k = I, \quad \widehat{\mathcal{D}}_k^* \widetilde{\mathcal{A}}_k - \widehat{\mathcal{B}}_k^* \widetilde{\mathcal{C}}_k = I, \tag{5.56}$$
$$\widehat{\mathcal{A}}_k \widetilde{\mathcal{B}}_k^* = \widehat{\mathcal{B}}_k \widetilde{\mathcal{A}}_k^*, \quad \widehat{\mathcal{D}}_k \widetilde{\mathcal{C}}_k^* = \widehat{\mathcal{C}}_k \widetilde{\mathcal{D}}_k^*, \quad \widehat{\mathcal{A}}_k \widetilde{\mathcal{D}}_k^* - \widehat{\mathcal{B}}_k \widetilde{\mathcal{C}}_k^* = I, \quad \widehat{\mathcal{D}}_k \widetilde{\mathcal{A}}_k^* - \widehat{\mathcal{C}}_k \widetilde{\mathcal{B}}_k^* = I. \tag{5.57}$$

Note that (5.49) or (5.54) trivially implies that $\det \widetilde{\mathcal{S}}_k^{[0]*} \times \det \widehat{\mathcal{S}}_k^{[0]} = 1$.

Let $\widehat{\mathbb{S}}_k(\lambda) := \widehat{\mathcal{S}}_k^{[0]} + \lambda \widehat{\mathcal{S}}_k^{[1]} + \lambda^2 \widehat{\mathcal{S}}_k^{[2]}$ and $\widetilde{\mathbb{S}}_k(\lambda) := \widetilde{\mathcal{S}}_k^{[0]} + \lambda \widetilde{\mathcal{S}}_k^{[1]} + \lambda^2 \widetilde{\mathcal{S}}_k^{[2]}$ be the coefficient matrices of systems ($\widehat{\mathcal{Q}}_\lambda$) and ($\widetilde{\mathcal{Q}}_\lambda$), respectively. Then

$$\widetilde{\mathbb{S}}_k^*(\lambda) \mathcal{J} \widehat{\mathbb{S}}_k(\bar{\lambda}) = \mathcal{J} \quad \text{and} \quad \widetilde{\mathbb{S}}_k(\lambda) \mathcal{J} \widehat{\mathbb{S}}_k^*(\bar{\lambda}) = \mathcal{J} \quad \text{with} \quad \widehat{\mathbb{S}}_k^{-1}(\lambda) = -\mathcal{J} \widetilde{\mathbb{S}}_k^*(\bar{\lambda}) \mathcal{J}, \tag{5.58}$$

which means that $\widehat{\mathbb{S}}_k(\lambda)$ and $\widetilde{\mathbb{S}}_k(\lambda)$ are invertible for all $k \in [0,\infty)_{\mathbb{Z}}$ and any $\lambda \in \mathbb{C}$. Therefore any initial value problems associated with systems ($\widehat{\mathcal{Q}}_\lambda$) and ($\widetilde{\mathcal{Q}}_\lambda$) posses unique solutions on $[0,\infty)_{\mathbb{Z}}$ for any initial value given at any point in $[0,\infty)_{\mathbb{Z}}$. Moreover, the fundamental matrices of systems ($\widehat{\mathcal{Q}}_\lambda$) and ($\widetilde{\mathcal{Q}}_\lambda$) are in this case invertible on the discrete interval $[0,\infty)_{\mathbb{Z}}$. Since by (5.19) we have

$$\det \widehat{\mathbb{S}}_k(\lambda) = \det \widehat{\mathcal{S}}_k^{[0]} \quad \text{and} \quad \det \widetilde{\mathbb{S}}_k(\lambda) = \det \widetilde{\mathcal{S}}_k^{[0]}, \tag{5.59}$$

it follows that

$$\det \widetilde{\mathbb{S}}_k^*(\lambda) \times \det \widehat{\mathbb{S}}_k(\lambda) = 1 \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}} \text{ and any } \lambda \in \mathbb{C}. \tag{5.60}$$

Similarly as in Example 5.2.2, see (5.20), we define the $2n \times 2n$ matrices

$$\widehat{\mathcal{T}}_k(\lambda) := \begin{pmatrix} 0 & \widehat{\mathcal{A}}_k \\ -I & \widehat{\mathcal{C}}_k - \lambda \mathcal{W}_k^{[1]} \widehat{\mathcal{A}}_k \end{pmatrix} \quad \text{and} \quad \widetilde{\mathcal{T}}_k(\lambda) := \begin{pmatrix} 0 & \widetilde{\mathcal{A}}_k \\ -I & \widetilde{\mathcal{C}}_k - \lambda \mathcal{W}_k^{[1]} \widetilde{\mathcal{A}}_k \end{pmatrix}. \tag{5.61}$$

If $\hat{z}(\lambda), \tilde{z}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ solve systems $(\widehat{Q}_\lambda)$ and $(\widetilde{Q}_\lambda)$, respectively, then by using the inverse formula in (5.58) we get that the multiplication of $\hat{z}_{k+1}(\lambda)$ by $\widehat{\mathcal{T}}_k^*(\bar{\lambda})\mathcal{J}$ yields the backward shift in the second component of $\hat{z}_{k+1}(\lambda)$ and the multiplication of $\tilde{z}_{k+1}(\lambda)$ by $\widetilde{\mathcal{T}}_k^*(\bar{\lambda})\mathcal{J}$ yields the backward shift in the second component of $\tilde{z}_{k+1}(\lambda)$, i.e.,

$$\left.\begin{aligned} \hat{z}_k^{[s]}(\lambda) &:= (\hat{x}_{k+1}^*(\lambda), \hat{u}_k^*(\lambda))^* = \widehat{\mathcal{T}}_k^*(\bar{\lambda})\mathcal{J}\hat{z}_{k+1}(\lambda), \\ \tilde{z}_k^{[s]}(\lambda) &:= (\tilde{x}_{k+1}^*(\lambda), \tilde{u}_k^*(\lambda))^* = \widetilde{\mathcal{T}}_k^*(\bar{\lambda})\mathcal{J}\tilde{z}_{k+1}(\lambda), \end{aligned}\right\} \tag{5.62}$$

compare with (5.23). The same notation will be also used for matrix-valued solutions, in particular for the fundamental matrices of systems $(\widehat{Q}_\lambda)$ and $(\widetilde{Q}_\lambda)$. By similar calculations as in (5.24) we obtain for any solution $\widehat{\mathcal{Z}}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times m}$ of system $(\widehat{Q}_\lambda)$ and any solution $\widetilde{\mathcal{Z}}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times m}$ of system $(\widetilde{Q}_\lambda)$ the Lagrange-type identity

$$\Delta\Big[\widetilde{\mathcal{Z}}_k^*(\lambda)\mathcal{J}\widehat{\mathcal{Z}}_k(\nu)\Big] = (\bar{\lambda} - \nu)\,\widetilde{\mathcal{Z}}_k^{[s]*}(\lambda)\,\mathcal{W}_k\,\widehat{\mathcal{Z}}_k^{[s]}(\nu), \tag{5.63}$$

where we employed the identities in (5.57) and the notation from (5.62). In addition, by the summation of both sides of (5.63) we get

$$\widetilde{\mathcal{Z}}_{k+1}^*(\lambda)\mathcal{J}\widehat{\mathcal{Z}}_{k+1}(\nu) = \widetilde{\mathcal{Z}}_0^*(\lambda)\mathcal{J}\widehat{\mathcal{Z}}_0(\nu) + (\bar{\lambda} - \nu)\sum_{j=0}^{k} \widetilde{\mathcal{Z}}_j^{[s]*}(\lambda)\,\mathcal{W}_j\,\widehat{\mathcal{Z}}_j^{[s]}(\nu), \tag{5.64}$$

compare with (5.10). In the following result we use the space $\ell^2_{\mathcal{W}}$ defined in (5.41).

Theorem 5.4.5. *Let (5.47) hold. If there exists $\lambda_0 \in \mathbb{C}$ such that all solutions of systems $(\widehat{Q}_{\lambda_0})$ and $(\widetilde{Q}_{\lambda_0})$ belong to $\ell^2_{\mathcal{W}}$, then all solutions of systems $(\widehat{Q}_\lambda)$ and $(\widetilde{Q}_\lambda)$ belong to $\ell^2_{\mathcal{W}}$ for any $\lambda \in \mathbb{C}$.*

Proof. Let the assumptions be satisfied for $\lambda_0 \in \mathbb{C}$ and $\lambda \in \mathbb{C}\setminus\{\lambda_0\}$ be fixed. For $\nu \in \{\lambda, \lambda_0\}$ we denote by $\widehat{\Phi}(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ and $\widetilde{\Phi}(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ the fundamental matrices of systems $(\widehat{Q}_\nu)$ and $(\widetilde{Q}_\nu)$, respectively, such that $\widehat{\Phi}_0(\nu) = I = \widetilde{\Phi}_0(\nu)$. First we prove that all solutions of system $(\widehat{Q}_\lambda)$ belong to $\ell^2_{\mathcal{W}}$. Since $\widehat{\Phi}_k(\lambda)$ and $\widehat{\Phi}_k(\lambda_0)$ are invertible on $[0,\infty)_{\mathbb{Z}}$, there exists $\widehat{\Omega} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ such that

$$\widehat{\Phi}_k(\lambda) = \widehat{\Phi}_k(\lambda_0)\,\widehat{\Omega}_k \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}, \tag{5.65}$$

Then by a direct calculation we obtain that $\widehat{\Omega}_k$ satisfies the recurrence relation

$$\widehat{\Omega}_{k+1} = \Big[I + (\lambda - \lambda_0)\,\Upsilon_k\Big]\widehat{\Omega}_k, \quad \text{where} \quad \Upsilon_k := \widehat{\Phi}_{k+1}^{-1}(\lambda_0)\Big[\widehat{\mathbb{S}}_k^{[1]} + (\lambda + \lambda_0)\widehat{\mathbb{S}}_k^{[2]}\Big]\widehat{\Phi}_k(\lambda_0). \tag{5.66}$$

Since $\widehat{\Phi}_{k+1}^{-1}(\lambda_0) = \widehat{\Phi}_k^{-1}(\lambda_0)\,\widehat{\mathbb{S}}_k^{-1}(\lambda_0)$, we have

$$\det\Big[I + (\lambda - \lambda_0)\,\Upsilon_k\Big] = \det\Big[\widehat{\Phi}_k^{-1}(\lambda_0)\widehat{\mathbb{S}}_k^{-1}(\lambda_0)\widehat{\mathbb{S}}_k(\lambda)\widehat{\Phi}_k(\lambda_0)\Big] \overset{(5.59)}{=} \det\Big(\widehat{\mathbb{S}}_k^{[0]}\Big)^{-1} \times \det\widehat{\mathbb{S}}_k^{[0]} = 1,$$

i.e., the matrix $I + (\lambda - \lambda_0)\,\Upsilon_k$ is invertible on $[0,\infty)_{\mathbb{Z}}$. By using (5.50) and the $n \times n$ block structure of the matrices $\widehat{\mathbb{S}}_k(\lambda_0)$, $\widehat{\mathbb{S}}_k^{[1]}$, $\widehat{\mathbb{S}}_k^{[2]}$, $\widehat{\mathcal{T}}_k(\lambda_0)$, $\widetilde{\mathcal{T}}_k(\lambda_0)$, $\mathcal{W}_k$, we get the identity

$$\Big[\widehat{\mathbb{S}}_k^{[1]} + (\lambda_0 + \bar{\lambda}_0)\widehat{\mathbb{S}}_k^{[2]}\Big]\mathcal{J}\,\widetilde{\mathbb{S}}_k^*(\bar{\lambda}_0) = -\widehat{\mathcal{T}}_k(\lambda_0)\,\mathcal{W}_k\,\widetilde{\mathcal{T}}_k^*(\bar{\lambda}_0). \tag{5.67}$$

Moreover, a simple calculation yields

$$\widehat{\mathbb{S}}_k^{[1]} + (\lambda + \lambda_0)\widehat{\mathbb{S}}_k^{[2]} = T_k\Big[\widehat{\mathbb{S}}_k^{[1]} + (\lambda_0 + \bar{\lambda}_0)\widehat{\mathbb{S}}_k^{[2]}\Big], \quad \text{where} \quad T_k := \begin{pmatrix} I & 0 \\ (\bar{\lambda}_0 - \lambda)\,\mathcal{W}_k^{[1]} & I \end{pmatrix}. \tag{5.68}$$

Hence by using (5.67) and (5.68) in the definition of Υ_k in (5.66) and then applying (5.62) we can equivalently expressed Υ_k as

$$\Upsilon_k = Q_k^{-1}\,\widetilde{\mathbf{\Phi}}_{k+1}^*(\lambda_0)\,\mathcal{J}^*\widetilde{\mathcal{T}}_k(\bar{\lambda}_0)\,\mathcal{W}_k\,\widetilde{\mathcal{T}}_k^*(\bar{\lambda}_0)\,\mathcal{J}\,\widehat{\mathbf{\Phi}}_{k+1}(\lambda_0) \overset{(5.62)}{=} Q_k^{-1}\,\widetilde{\mathbf{\Phi}}_k^{[s]*}(\lambda_0)\,\mathcal{W}_k\,\widehat{\mathbf{\Phi}}_k^{[s]}(\lambda_0), \tag{5.69}$$

where we put

$$Q_k := -\widetilde{\mathbf{\Phi}}_{k+1}^*(\lambda_0)\,\mathcal{J}\,T_k^{-1}\,\widehat{\mathbf{\Phi}}_{k+1}(\lambda_0). \tag{5.70}$$

Since $\det T_k \equiv 1$, we get for any $k \in [0,\infty)_{\mathbb{Z}}$ the equality

$$\begin{aligned} \det Q_k &= \det \widetilde{\mathbf{\Phi}}_{k+1}^*(\lambda_0) \times \det \widehat{\mathbf{\Phi}}_{k+1}(\lambda_0) = \det \widetilde{\mathbf{\Phi}}_k^*(\lambda_0) \times \det \widehat{\mathbf{\Phi}}_k(\lambda_0) \times \det \widetilde{\mathbb{S}}_k^*(\lambda_0) \times \det \widehat{\mathbb{S}}_k(\lambda_0) \\ &\overset{(5.60)}{=} \det \widetilde{\mathbf{\Phi}}_k^*(\lambda_0) \times \det \widehat{\mathbf{\Phi}}_k(\lambda_0) = \cdots = \det \widetilde{\mathbf{\Phi}}_0^*(\lambda_0) \times \det \widehat{\mathbf{\Phi}}_0(\lambda_0) = 1. \end{aligned}$$

Now we show that there exists $\kappa > 0$ such that

$$\|Q_k^{-1}\|_\sigma \le \kappa < \infty \quad \text{on } [0,\infty)_{\mathbb{Z}}. \tag{5.71}$$

Since $\mathcal{W}_k \ge 0$, the Cauchy–Schwarz and arithmetic-geometric mean inequalities yield

$$|\xi^*\mathcal{W}_k\zeta| \le (\xi^*\mathcal{W}_k\xi)^{1/2}(\zeta^*\mathcal{W}_k\zeta)^{1/2} \le \tfrac{1}{2}(\xi^*\mathcal{W}_k\xi + \zeta^*\mathcal{W}_k\zeta)$$

for any $\xi,\zeta \in \mathbb{C}^{2n}$ and $k \in [0,\infty)_{\mathbb{Z}}$. Hence, for any sequences $\hat{z},\tilde{z} \in \ell^2_{\mathcal{W}}$ we have

$$\Big|\sum_{k=0}^{\infty} \tilde{z}_k^{[s]*}\mathcal{W}_k\hat{z}_k^{[s]}\Big| \le \sum_{k=0}^{\infty} \big|\tilde{z}_k^{[s]*}\mathcal{W}_k\hat{z}_k^{[s]}\big| \le \tfrac{1}{2}\sum_{k=0}^{\infty}(\tilde{z}_k^{[s]*}\mathcal{W}_k\tilde{z}_k^{[s]} + \hat{z}^{[s]*}\mathcal{W}_k\hat{z}_k^{[s]}) < \infty.$$

The last inequality with $\hat{z}$ and $\tilde{z}$ being the columns of $\widehat{\mathbf{\Phi}}(\lambda_0)$ and $\widetilde{\mathbf{\Phi}}(\lambda_0)$, respectively, and the assumption that all solutions of systems ($\widehat{\mathcal{Q}}_{\lambda_0}$) and ($\widetilde{\mathcal{Q}}_{\lambda_0}$) belong to $\ell^2_{\mathcal{W}}$ imply that there exists $\varepsilon > 0$ such that

$$\sum_{k=0}^{\infty}\Big\|\widetilde{\mathbf{\Phi}}_k^{[s]*}(\lambda_0)\,\mathcal{W}_k\,\widehat{\mathbf{\Phi}}_k^{[s]}(\lambda_0)\Big\|_\sigma \overset{(1.7)}{\le} \sum_{k=0}^{\infty}\Big\|\widetilde{\mathbf{\Phi}}_k^{[s]*}(\lambda_0)\,\mathcal{W}_k\,\widehat{\mathbf{\Phi}}_k^{[s]}(\lambda_0)\Big\|_1 \le \varepsilon < \infty. \tag{5.72}$$

Since $\mathcal{J}T_k^{-1} = \mathcal{J} - (\bar{\lambda}_0 - \lambda)\operatorname{diag}\big\{\mathcal{W}_k^{[1]},0\big\}$, the sequence Q_k in (5.70) can be written as

$$\begin{aligned} Q_k &= \widetilde{\mathbf{\Phi}}_{k+1}^*(\lambda_0)\Big[(\bar{\lambda}_0 - \lambda)\operatorname{diag}\big\{\mathcal{W}_k^{[1]},0\big\} - \mathcal{J}\Big]\widehat{\mathbf{\Phi}}_{k+1}(\lambda_0) \\ &\overset{(5.64)}{=} -\mathcal{J} + 2i\,\operatorname{im}(\lambda_0)\sum_{j=0}^{k}\widetilde{\mathbf{\Phi}}_j^{[s]*}(\lambda_0)\,\mathcal{W}_j\,\widehat{\mathbf{\Phi}}_j^{[s]}(\lambda_0) \\ &\qquad\qquad + (\bar{\lambda}_0 - \lambda)\,\widetilde{\mathbf{\Phi}}_k^{[s]*}(\lambda_0)\operatorname{diag}\big\{\mathcal{W}_k^{[1]},0\big\}\,\widehat{\mathbf{\Phi}}_k^{[s]}(\lambda_0). \end{aligned} \tag{5.73}$$

From the unitary invariance of the spectral norm, assumption (5.47), and the estimate in (5.72) we conclude that there exists $\tau > 0$ such that

$$\Big\|\widetilde{\mathbf{\Phi}}_k^{[s]*}(\lambda_0)\operatorname{diag}\big\{\mathcal{W}_k^{[1]},0\big\}\widehat{\mathbf{\Phi}}_k^{[s]}(\lambda_0)\Big\|_\sigma \le \Big\|\widetilde{\mathbf{\Phi}}_k^{[s]*}(\lambda_0)\,\mathcal{W}_k\,\widehat{\mathbf{\Phi}}_k^{[s]}(\lambda_0)\Big\|_\sigma \le \tau < \infty \tag{5.74}$$

for all $k \in [0,\infty)_{\mathbb{Z}}$. Upon taking the matrix norm in (5.73) we obtain for any $k \in [0,\infty)_{\mathbb{Z}}$ that

$$\begin{aligned} \|Q_k\|_\sigma \leq \|\mathcal{J}\|_\sigma + 2\,|\operatorname{im}(\lambda_0)| \sum_{j=0}^{k} \Big\| \tilde{\widehat{\Phi}}{}_j^{[s]*}(\lambda_0)\, \mathcal{W}_j\, \widehat{\Phi}_j^{[s]}(\lambda_0) \Big\|_\sigma \\ + |\bar{\lambda}_0 - \lambda|\, \Big\| \tilde{\widehat{\Phi}}{}_k^{[s]*}(\lambda_0)\, \operatorname{diag}\big\{\mathcal{W}_k^{[1]}, 0\big\}\, \widehat{\Phi}_k^{[s]}(\lambda_0) \Big\|_\sigma \overset{(5.72),(5.74)}{\leq} \omega < \infty, \end{aligned}$$

where $\omega := \|\mathcal{J}\|_\sigma + 2\,|\operatorname{im}(\lambda_0)|\,\varepsilon + |\bar{\lambda}_0 - \lambda|\,\tau$. Therefore the matrix Q_k is bounded on $[0,\infty)_{\mathbb{Z}}$ with $\det Q_k \equiv 1$. This implies that Q_k^{-1} is also bounded on $[0,\infty)_{\mathbb{Z}}$, i.e., inequality (5.71) holds. By combining the submultiplicative property of the spectral norm and (5.69), (5.71), (5.72) we then obtain

$$\sum_{k=0}^{\infty} \|\Upsilon_k\|_\sigma \leq \sum_{k=0}^{\infty} \|Q_k^{-1}\|_\sigma \times \Big\| \tilde{\widehat{\Phi}}{}_k^{[s]*}(\lambda_0)\, \mathcal{W}_k\, \widehat{\Phi}_k^{[s]}(\lambda_0) \Big\|_\sigma \leq \kappa\varepsilon < \infty.$$

Hence the same calculation as in (4.21), see also Proposition 1.1.4, implies that the fundamental matrix $\widehat{\Omega}_k$ of system (5.66) satisfies

$$\|\widehat{\Omega}_k\|_\sigma \leq \rho \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}} \text{ and some } \rho > 0. \tag{5.75}$$

With $\mathcal{K}_k(\lambda) := \tilde{\widehat{\Phi}}{}_k^{[s]*}(\lambda)\, \mathcal{W}_k\, \widehat{\Phi}_k^{[s]}(\lambda)$ we obtain from (5.65) and (5.47) that

$$\begin{aligned} \mathcal{K}_k(\lambda) &= \widehat{\Omega}_{k+1}^*\, \tilde{\widehat{\Phi}}{}_k^{[s]*}(\lambda_0)\, \operatorname{diag}\big\{\mathcal{W}_k^{[1]}, 0\big\}\, \widehat{\Phi}_k^{[s]}(\lambda_0)\, \widehat{\Omega}_{k+1} + \widehat{\Omega}_k^*\, \tilde{\widehat{\Phi}}{}_k^{[s]*}(\lambda_0)\, \operatorname{diag}\big\{0, \mathcal{W}_k^{[2]}\big\}\, \widehat{\Phi}_k^{[s]}(\lambda_0)\, \widehat{\Omega}_k \\ &\leq \widehat{\Omega}_{k+1}^*\, \mathcal{K}_k(\lambda_0)\, \widehat{\Omega}_{k+1} + \widehat{\Omega}_k^*\, \mathcal{K}_k(\lambda_0)\, \widehat{\Omega}_k. \end{aligned} \tag{5.76}$$

This implies through (5.75) and the submultiplicativity, self-adjointness, and unitary invariance of the spectral norm that

$$\begin{aligned} \sum_{k=0}^{\infty} \|\mathcal{K}_k(\lambda)\|_\sigma &\overset{(5.76)}{\leq} \sum_{k=0}^{\infty} \Big(\big\| \widehat{\Omega}_{k+1}^*\, \mathcal{K}_k(\lambda_0)\, \widehat{\Omega}_{k+1} \big\|_\sigma + \big\| \widehat{\Omega}_k^*\, \mathcal{K}_k(\lambda_0)\, \widehat{\Omega}_k \big\|_\sigma \Big) \\ &\leq \sum_{k=0}^{\infty} \Big(\big\|\widehat{\Omega}_{k+1}\big\|_\sigma^2 + \big\|\widehat{\Omega}_k\big\|_\sigma^2 \Big) \times \|\mathcal{K}_k(\lambda_0)\|_\sigma \overset{(5.75)}{\leq} 2\rho^2 \sum_{k=0}^{\infty} \|\mathcal{K}_k(\lambda_0)\|_\sigma < \infty, \end{aligned}$$

because all solutions of system $(\widehat{\mathcal{Q}}_{\lambda_0})$ belong to $\ell^2_{\mathcal{W}}$. This shows that all solutions of system $(\widehat{\mathcal{Q}}_\lambda)$ belong to $\ell^2_{\mathcal{W}}$ as well. Analogously, by switching the roles of systems $(\widehat{\mathcal{Q}}_\lambda)$ and $(\tilde{\mathcal{Q}}_\lambda)$, we prove that all solutions of system $(\tilde{\mathcal{Q}}_\lambda)$ belong to $\ell^2_{\mathcal{W}}$. Since $\lambda \in \mathbb{C}\setminus\{\lambda_0\}$ was chosen arbitrarily, the proof is complete. ∎

Proof of Theorem 5.4.1. The statement of Theorem 5.4.1 follows immediately from Theorem 5.4.5, when it is applied in the case $\widehat{\mathbb{S}}_k^{[0]} \equiv \tilde{\mathbb{S}}_k^{[0]} := \mathbb{S}_k^{[0]}$, i.e., when all systems $(\mathcal{Q}_\lambda)$, $(\widehat{\mathcal{Q}}_\lambda)$, and $(\tilde{\mathcal{Q}}_\lambda)$ are equal. ∎

Now we give sufficient conditions in terms of the coefficients, which guarantee that all solutions of systems $(\widehat{\mathcal{Q}}_\lambda)$ and $(\tilde{\mathcal{Q}}_\lambda)$ belong to the space $\ell^2_{\mathcal{W}}$. Similarly as in Corollary 4.2.3, the matrix norm $\|\cdot\|_1$ utilized in (5.77) below can be replaced by any other matrix norm because of their equivalence.

Corollary 5.4.6. *Let (5.47) hold and assume that*

$$\sum_{k=0}^{\infty} \big\| \widehat{\mathbb{S}}_k^{[0]} - I \big\|_1 < \infty, \quad \sum_{k=0}^{\infty} \big\| \widetilde{\mathbb{S}}_k^{[0]} - I \big\|_1 < \infty, \quad \textit{and} \quad \sum_{k=0}^{\infty} \| \mathcal{W}_k \|_1 < \infty. \tag{5.77}$$

Then all solutions of systems $(\widehat{\mathcal{Q}}_\lambda)$ *and* $(\widetilde{\mathcal{Q}}_\lambda)$ *belong to* $\ell^2_{\mathcal{W}}$ *for any* $\lambda \in \mathbb{C}$.

Proof. By Theorem 5.4.5 it suffices to show that there exists $\lambda_0 \in \mathbb{C}$ such that

$$\sum_{k=0}^{\infty} \big\| \widehat{\boldsymbol{\Phi}}_k^{[s]*}(\lambda_0)\, \mathcal{W}_k\, \widehat{\boldsymbol{\Phi}}_k^{[s]}(\lambda_0) \big\|_1 < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \big\| \widetilde{\boldsymbol{\Phi}}_k^{[s]*}(\lambda_0)\, \mathcal{W}_k\, \widetilde{\boldsymbol{\Phi}}_k^{[s]}(\lambda_0) \big\|_1 < \infty. \tag{5.78}$$

We show that these inequalities are satisfied for $\lambda_0 = 0$. Since $\widehat{\mathbb{S}}_k(0) = \widehat{\mathbb{S}}_k^{[0]}$, we obtain from the first condition in (5.77) and Proposition 1.1.4 that there exist $\varepsilon > 0$ such that $\big\| \widehat{\boldsymbol{\Phi}}_k(0) \big\|_1 \leq \varepsilon < \infty$ for all $k \in [0,\infty)_{\mathbb{Z}}$, where $\widehat{\boldsymbol{\Phi}}(0) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ represents a fundamental matrix of system $(\widehat{\mathcal{Q}}_0)$. Moreover, the second condition in (5.77) implies that there exists $\rho > 0$ such that $\big\| \widetilde{\mathcal{A}}_k - I \big\|_1 + \big\| \widetilde{\mathcal{C}}_k \big\|_1 \leq \rho < \infty$ for all $k \in [0,\infty)_{\mathbb{Z}}$. Hence from (5.61) we get

$$\begin{aligned} \big\| \widetilde{\mathcal{T}}_k^*(0)\, \mathcal{J} \big\|_1 &= \left\| \begin{pmatrix} 0 & 0 \\ -\widetilde{\mathcal{C}}_k^* & \widetilde{\mathcal{A}}_k^* - I \end{pmatrix} + I_{2n} \right\|_1 \leq \left\| \begin{pmatrix} 0 & 0 \\ -\widetilde{\mathcal{C}}_k^* & \widetilde{\mathcal{A}}_k^* - I \end{pmatrix} \right\|_1 + \| I_{2n} \|_1 \\ &\leq \big\| \widetilde{\mathcal{A}}_k - I \big\|_1 + \big\| \widetilde{\mathcal{C}}_k \big\|_1 + 2n \leq \kappa < \infty \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}, \end{aligned}$$

where we used the self-adjointness of the norm $\|\cdot\|_1$ and put $\kappa := 2n + \rho$. Since we have $\widehat{\boldsymbol{\Phi}}_k^{[s]}(0) = \widetilde{\mathcal{T}}_k^*(0)\, \mathcal{J}\, \widehat{\boldsymbol{\Phi}}_{k+1}(0)$ by (5.62), the last condition in (5.77) yields the estimate

$$\begin{aligned} \sum_{k=0}^{\infty} \big\| \widehat{\boldsymbol{\Phi}}_k^{[s]*}(0)\, \mathcal{W}_k\, \widehat{\boldsymbol{\Phi}}_k^{[s]}(0) \big\|_1 &\leq \sum_{k=0}^{\infty} \big\| \widehat{\boldsymbol{\Phi}}_k^{[s]}(0) \big\|_1^2 \times \| \mathcal{W}_k \|_1 \leq \sum_{k=0}^{\infty} \big\| \widetilde{\mathcal{T}}_k^*(0)\, \mathcal{J} \big\|_1^2 \times \big\| \widehat{\boldsymbol{\Phi}}_{k+1}(0) \big\|_1^2 \times \| \mathcal{W}_k \|_1 \\ &\leq \kappa^2 \varepsilon^2 \sum_{k=0}^{\infty} \| \mathcal{W}_k \|_1 < \infty. \end{aligned}$$

In a similar way we prove also the second inequality in (5.78). Therefore all solutions of systems $(\widehat{\mathcal{Q}}_0)$ and $(\widetilde{\mathcal{Q}}_0)$ belong to $\ell^2_{\mathcal{W}}$, and so the statement follows from Theorem 5.4.5. ∎

If the matrices $\widehat{\mathcal{A}}_k$ and $\widetilde{\mathcal{A}}_k$ are invertible on $[0,\infty)_{\mathbb{Z}}$, then systems $(\widehat{\mathcal{Q}}_\lambda)$ and $(\widetilde{\mathcal{Q}}_\lambda)$ are equivalent with the pair of the first order difference systems

$$\Delta \hat{z}_k(\lambda) = \big[\widehat{H}_k + \lambda\, \mathcal{J}\, \mathcal{W}_k \big]\, \hat{z}_k^{[s]}(\lambda) \quad \text{and} \quad \Delta \tilde{z}_k(\lambda) = \big[\widetilde{H}_k + \lambda\, \mathcal{J}\, \mathcal{W}_k \big]\, \tilde{z}_k^{[s]}(\lambda),$$

where $\mathcal{W}$ is from (5.47), the coefficient matrix $\widehat{H} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ has the form

$$\widehat{H}_k := \begin{pmatrix} I - \widehat{\mathcal{A}}_k^{-1} & \widehat{\mathcal{A}}_k^{-1} \widehat{\mathcal{B}}_k \\ \widehat{\mathcal{C}}_k \widehat{\mathcal{A}}_k^{-1} & \widehat{\mathcal{D}}_k - \widehat{\mathcal{C}}_k \widehat{\mathcal{A}}_k^{-1} \widehat{\mathcal{B}}_k - I \end{pmatrix} \overset{(5.50)}{=} \begin{pmatrix} I - \widehat{\mathcal{A}}_k^{-1} & \widehat{\mathcal{A}}_k^{-1} \widehat{\mathcal{B}}_k \\ \widehat{\mathcal{C}}_k \widehat{\mathcal{A}}_k^{-1} & \widehat{\mathcal{A}}_k^{*-1} - I \end{pmatrix},$$

and $\widetilde{H} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ is given analogously, cf. Examples 5.2.2 and 5.2.3. Especially, if we take $\widehat{\mathcal{A}}_k \equiv \widetilde{\mathcal{A}}_k$, then the first identities in (5.50) and (5.55) imply that $\widetilde{H}_k = \mathcal{J} H_k^* \mathcal{J}$ for all $k \in [0,\infty)_{\mathbb{Z}}$. In this case we obtain from Theorem 5.4.5 the following generalization of [142, Theorem 5.5] for two non-Hermitian linear Hamiltonian difference systems

$$\Delta \hat{z}_k(\lambda) = \big[\widehat{H}_k + \lambda\, \widehat{W}_k \big]\, \hat{z}_k^{[s]}(\lambda), \tag{$\widehat{\mathrm{H}}_\lambda$}$$

$$\Delta \tilde{z}_k(\lambda) = \big[\mathcal{J} \widehat{H}_k^* \mathcal{J} + \lambda\, \widehat{W}_k \big]\, \tilde{z}_k^{[s]}(\lambda), \tag{$\widetilde{\mathrm{H}}_\lambda$}$$

where we put

$$\widehat{H}_k := \begin{pmatrix} \widehat{A}_k & \widehat{B}_k \\ \widehat{C}_k & -\widehat{A}_k^* \end{pmatrix} \quad \text{and} \quad \widehat{W}_k := \begin{pmatrix} 0 & \widehat{W}_k^{[2]} \\ -\widehat{W}_k^{[1]} & 0 \end{pmatrix} \tag{5.79}$$

with $\widehat{A}, \widehat{B}, \widehat{C}, \widehat{W}^{[1]}, \widehat{W}^{[2]} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{n\times n}$ being such that $I - \widehat{A}_k$ is invertible, $\widehat{W}_k^{[1]}, \widehat{W}_k^{[2]}$ are Hermitian, and $-\mathcal{J}\widehat{W}_k \geq 0$ on $[0,\infty)_{\mathbb{Z}}$. Note that we have ($\widehat{\mathrm{H}}_\lambda$)=($\widetilde{\mathrm{H}}_\lambda$)=(2.6) if $\widehat{B}_k = \widehat{B}_k^* = B_k$ and $\widehat{C}_k = \widehat{C}_k^* = C_k$.

Corollary 5.4.7. *Let $\widehat{H}, \widehat{W} \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ be as in (5.79). If there exists $\lambda_0 \in \mathbb{C}$ such that all solutions of systems ($\widehat{\mathrm{H}}_{\lambda_0}$) and ($\widetilde{\mathrm{H}}_{\lambda_0}$) belong to $\ell^2_{\mathcal{W}}$ with $\mathcal{W} := -\mathcal{J}\widehat{W}_k$, then all solutions of systems ($\widehat{\mathrm{H}}_\lambda$) and ($\widetilde{\mathrm{H}}_\lambda$) belong to $\ell^2_{\mathcal{W}}$ for any $\lambda \in \mathbb{C}$.*

Finally, we give an example illustrating the result of Theorem 5.4.5.

Example 5.4.8. Let $\kappa \geq 1$ be fixed and $q, r \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^1$ be nonnegative sequences such that $F(\kappa) := \sum_{k=0}^{\infty} F_k(\kappa) < \infty$ and $G(\kappa) := \sum_{k=0}^{\infty} G_k(\kappa) < \infty$, where

$$\left.\begin{aligned} F_k(\kappa) &:= \kappa^{8k+2} q_{4k} + \kappa^{8k+4} r_{4k+2} + \kappa^{8k+6} r_{4k+3} + \kappa^{8k+8} q_{4k+3}, \\ G_k(\kappa) &:= \kappa^{8k} r_{4k} + \kappa^{8k+2} r_{4k+1} + \kappa^{8k+4} q_{4k+1} + \kappa^{8k+6} q_{4k+2}. \end{aligned}\right\} \tag{5.80}$$

For example, we may choose

$$q_k = r_k := 1/(c^k \kappa^{3k}) \quad \text{with } c > 1/\kappa,$$

because in this case the numbers $F(\kappa)$ and $G(\kappa)$ are multiples of the convergent series $\sum_{k=0}^{\infty} 1/(c\kappa)^{4k}$. By substituting $1/\kappa$ instead of κ in (5.80) we can see that the series $F(1/\kappa)$ and $G(1/\kappa)$ are convergent as well.

Let us consider systems ($\widehat{\mathcal{Q}}_\lambda$) and ($\widetilde{\mathcal{Q}}_\lambda$) with $n = 1$ and with the following coefficients: (i) for all $k \in [0,\infty)_{\mathbb{Z}}$ we put $\mathcal{W}_k^{[1]} := q_k$ and $\mathcal{W}_k^{[2]} := r_k$, (ii) for $k \in [0,\infty)_{\mathbb{Z}}$ even ($k = 2j$) we set $\widehat{\mathcal{A}}_k = \widehat{\mathcal{D}}_k := (-1)^j/\kappa$, $\widehat{\mathcal{B}}_k = \widehat{\mathcal{C}}_k := 0$ and $\widetilde{\mathcal{A}}_k = \widetilde{\mathcal{D}}_k := (-1)^j\kappa$, $\widetilde{\mathcal{B}}_k = \widetilde{\mathcal{C}}_k := 0$, while for $k \in [0,\infty)_{\mathbb{Z}}$ odd ($k = 2j+1$) we define $\widehat{\mathcal{A}}_k = \widehat{\mathcal{D}}_k := 0$, $\widehat{\mathcal{B}}_k = -\widehat{\mathcal{C}}_k := (-1)^j/\kappa$ and $\widetilde{\mathcal{A}}_k = \widetilde{\mathcal{D}}_k := 0$, $\widetilde{\mathcal{B}}_k = -\widetilde{\mathcal{C}}_k := (-1)^j\kappa$. This means that

$$\widehat{\mathcal{S}}_k^{[0]} = (1/\kappa)\mathcal{J}^k, \quad \widetilde{\mathcal{S}}_k^{[0]} = \kappa\mathcal{J}^k, \quad \text{and} \quad \mathcal{W}_k = \operatorname{diag}\left\{\mathcal{W}_k^{[1]}, \mathcal{W}_k^{[2]}\right\} = \operatorname{diag}\{q_k, r_k\} \geq 0. \tag{5.81}$$

Then conditions (5.47) and (5.49) are satisfied. We show that all solutions of systems ($\widehat{\mathcal{Q}}_0$) and ($\widetilde{\mathcal{Q}}_0$) with the coefficients specified above belong to the corresponding space $\ell^2_{\mathcal{W}}$. The fundamental matrices $\widehat{\mathbf{\Phi}}_k(0)$ and $\widetilde{\mathbf{\Phi}}_k(0)$ of systems ($\widehat{\mathcal{Q}}_0$) and ($\widetilde{\mathcal{Q}}_0$) determined by the initial conditions $\widehat{\mathbf{\Phi}}_0(0) = I = \widetilde{\mathbf{\Phi}}_0(0)$ are equal to

$$\widehat{\mathbf{\Phi}}_k(0) = (1/\kappa^k)\mathcal{J}^{(k^2-k)/2} \quad \text{and} \quad \widetilde{\mathbf{\Phi}}_k(0) = \kappa^k\mathcal{J}^{(k^2-k)/2} \quad \text{for all } k \in [0,\infty)_{\mathbb{Z}}.$$

From (5.61) we obtain $\widehat{\mathcal{T}}_k(0) = \left(\begin{smallmatrix} 0 & \widehat{\mathcal{A}}_k \\ -I & \widehat{\mathcal{C}}_k \end{smallmatrix}\right)$ and $\widetilde{\mathcal{T}}_k(0) = \left(\begin{smallmatrix} 0 & \widetilde{\mathcal{A}}_k \\ -I & \widetilde{\mathcal{C}}_k \end{smallmatrix}\right)$, and thus for $k \in [0,\infty)_{\mathbb{Z}}$ we get

$$\widehat{\mathbf{\Phi}}_k^{[s]}(0) = \widetilde{\mathcal{T}}_k^*(0)\mathcal{J}\widehat{\mathbf{\Phi}}_{k+1}(0) = \left(\left(\hat{z}_k^{[1]}\right)^{[s]}, \left(\hat{z}_k^{[2]}\right)^{[s]}\right) = \begin{cases} \dfrac{(-1)^j}{\kappa^{4j+1}}\begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix}, & k = 4j, \\[2ex] \dfrac{(-1)^j}{\kappa^{4j+2}}\begin{pmatrix} 0 & 1 \\ 0 & \kappa \end{pmatrix}, & k = 4j+1, \\[2ex] \dfrac{(-1)^{j+1}}{\kappa^{4j+3}}\begin{pmatrix} 0 & 1 \\ \kappa & 0 \end{pmatrix}, & k = 4j+2, \\[2ex] \dfrac{(-1)^{j+1}}{\kappa^{4j+4}}\begin{pmatrix} 1 & 0 \\ -\kappa & 0 \end{pmatrix}, & k = 4j+3, \end{cases}$$

where $\big(\hat{z}_k^{[1]}\big)^{[s]}$ and $\big(\hat{z}_k^{[2]}\big)^{[s]}$ mean the partial shift applied to $\hat{z}_k^{[1]}$ and $\hat{z}_k^{[2]}$, respectively, i.e., $\big(\hat{z}_k^{[1]}\big)^{[s]} = \big(\hat{x}_{k+1}^{[1]}, \hat{u}_k^{[1]}\big)^{\top}$ and $\big(\hat{z}_k^{[2]}\big)^{[s]} = \big(\hat{x}_{k+1}^{[2]}, \hat{u}_k^{[2]}\big)^{\top}$. Similarly, the matrix

$$\widetilde{\Phi}_k^{[s]}(0) = \widetilde{\mathcal{T}}_k^*(0)\mathcal{J}\widetilde{\Phi}_{k+1}(0) = \Big(\big(\tilde{z}_k^{[1]}\big)^{[s]}, \big(\tilde{z}_k^{[2]}\big)^{[s]}\Big)$$

has the same form as $\widehat{\Phi}_k^{[s]}(0)$ above but with κ replaced by $1/\kappa$. By direct calculations we then have

$$\sum_{k=0}^{\infty} \big(\hat{z}_k^{[1]}\big)^{[s]*} \mathcal{W}_k \big(\hat{z}_k^{[1]}\big)^{[s]} = F(1/\kappa) < \infty, \quad \sum_{k=0}^{\infty} \big(\hat{z}_k^{[2]}\big)^{[s]*} \mathcal{W}_k \big(\hat{z}_k^{[2]}\big)^{[s]} = G(1/\kappa) < \infty,$$
$$\sum_{k=0}^{\infty} \big(\tilde{z}_k^{[1]}\big)^{[s]*} \mathcal{W}_k \big(\tilde{z}_k^{[1]}\big)^{[s]} = F(\kappa) < \infty, \quad \sum_{k=0}^{\infty} \big(\tilde{z}_k^{[2]}\big)^{[s]*} \mathcal{W}_k \big(\tilde{z}_k^{[2]}\big)^{[s]} = G(\kappa) < \infty.$$

This shows that $\hat{z}^{[1]}, \hat{z}^{[2]}, \tilde{z}^{[1]}, \tilde{z}^{[2]} \in \ell^2_{\mathcal{W}}$, i.e., the assumptions of Theorem 5.4.5 are satisfied with $\lambda_0 = 0$. Therefore, by this theorem, all solutions of systems $(\widehat{\mathcal{Q}}_\lambda)$ and $(\widetilde{\mathcal{Q}}_\lambda)$ with (5.81) belong to $\ell^2_{\mathcal{W}}$ for any $\lambda \in \mathbb{C}$. Observe that the statement of Corollary 5.4.6 cannot be applied, because $\sum_{k=0}^{\infty} \big\|\widehat{\mathbb{S}}_k^{[0]} - I\big\|_1 = \infty = \sum_{k=0}^{\infty} \big\|\widetilde{\mathbb{S}}_k^{[0]} - I\big\|_1$, i.e., the first two conditions in (5.77) are now violated. Note also that systems $(\widehat{\mathcal{Q}}_\lambda)$ and $(\widetilde{\mathcal{Q}}_\lambda)$ in this example cannot be written as systems $(\widehat{\mathrm{H}}_\lambda)$ and $(\widetilde{\mathrm{H}}_\lambda)$, respectively, because the coefficient matrices $\widehat{\mathcal{A}}_k = 0 = \widetilde{\mathcal{A}}_k$ for $k \in [0, \infty)_{\mathbb{Z}}$ odd are singular. ▲

Remark 5.4.9. We note that Example 5.4.8 with $\kappa = 1$ illustrates the application of Theorem 5.4.1, since in this case the systems $(\widehat{\mathcal{Q}}_\lambda)$ and $(\widetilde{\mathcal{Q}}_\lambda)$ coincide.

5.5 Bibliographical notes

The results of this chapter were published in [A17] and their generalization to symplectic systems on time scales was established in [A20]. More precisely, Theorem 5.1.3, Example 5.2.3, and Section 5.4 were published only for systems on time scales and they are explicitly presented for the first time in the discrete case, while Examples 5.3.9 and 5.4.8 are taken almost verbatim from [A20]. However it is worth noticing that the results of [A17, A20] were stated without the shift in the definition of the space $\ell^2_{\Psi(\lambda)}$. A generalization of the invariance of the limit circle case for system $(\mathcal{S}_\lambda)$ with the special polynomial dependence on λ described in Example 5.2.3 will be a subject of our future research.

To compare the discrete with the continuous, to search for analogies between them, and ultimately to effect their unification, are patterns of mathematical development that did not begin with Zeno, and certainly did not end with Leibnitz and Newton, nor even with Riemann and Stieltjes. Such a pattern of investigation is especially appropriate to the theory of boundary problems, for which the discrete and the continuous pervade both physical origins and mathematical methods.

Frederick Valentine Atkinson, see [9, pg. v]

Chapter 6

Nohomogeneous problem and maximal and minimal linear relations

In the last two chapters we return to the case of linear dependence on the spectral parameter and focus on the discrete symplectic systems from the "operator-theoretic" point of view. To the best of the author's knowledge, this direction in the theory of discrete symplectic systems was completely untouched before the publications [A18,A21], see also the bibliographical notes in Sections 6.5 and 7.4. However, instead of system ($\mathcal{S}_\lambda$) with the matrices $\mathbb{S}_k, \mathcal{S}_k, \mathcal{V}_k, \Psi_k$ we now consider the underlying discrete symplectic system in the so-called *time-reversed* form with the matrices $\mathbb{S}_k, \mathscr{S}_k, \mathscr{V}_k, \psi_k$ (to avoid any confusion we use two different fonts), i.e.,

$$z_k(\lambda) = \mathbb{S}_k(\lambda) z_{k+1}(\lambda) \quad \text{with} \quad \mathbb{S}_k(\lambda) := \mathscr{S}_k + \lambda \mathscr{V}_k, \tag{$\mathscr{S}_\lambda$}$$

where $\lambda \in \mathbb{C}$ is the same as in the previous chapters and $\mathscr{S}_k, \mathscr{V}_k \in \mathbb{C}^{2n\times 2n}$ are such that

$$\mathscr{S}_k^* \mathscr{J} \mathscr{S}_k = \mathscr{J}, \quad \mathscr{S}_k^* \mathscr{J} \mathscr{V}_k \ \text{ is Hermitian}, \quad \mathscr{V}_k^* \mathscr{J} \mathscr{V}_k = 0, \quad \text{and} \quad \psi_k := \mathscr{J} \mathscr{S}_k \mathscr{J} \mathscr{V}_k^* \mathscr{J} \geq 0, \tag{6.1}$$

where the matrix $\mathscr{J}$ is (again) as in (1.12). This change is mainly motivated by the absence of the shift in the definition of the associated semi-inner product and semi-norm, see (6.8) and compare with (2.54) and (2.26). Consequently, it produces a more natural form of the Green function associated with nonhomogeneous discrete symplectic systems, see Lemma 6.3.2, and allows us to associate with system ($\mathscr{S}_\lambda$) a densely defined operator, see Theorem 6.4.5 and compare with [132].

Regardless the transition from system ($\mathcal{S}_\lambda$) to ($\mathscr{S}_\lambda$), the results of the previous chapters remain valid also for system ($\mathscr{S}_\lambda$) with the changes given for the definition of the semi-inner product and semi-norm. More precisely, one easily observes that the first three identities in (6.1) are the same as in (2.1). Therefore (again) the matrix $\mathscr{S}_k$ is symplectic, for any $\lambda \in \mathbb{C}$ we have

$$\mathbb{S}_k^*(\bar{\lambda})\mathcal{J}\mathbb{S}_k(\lambda) = \mathcal{J}, \quad \mathbb{S}_k^{-1}(\lambda) = -\mathcal{J}\mathbb{S}_k^*(\bar{\lambda})\mathcal{J}, \quad |\det \mathbb{S}_k(\lambda)| = |\det \mathcal{S}_k| = 1, \tag{6.2}$$

and the identities for the matrices $\mathcal{S}_k, \mathcal{V}_k, \mathbb{S}_k(\lambda)$ can be modified similarly as in Lemma 2.1.1, see Lemma 6.1.1. Consequently system ($\mathcal{S}_\lambda$) is equivalent to system ($\mathbb{S}_\lambda$), where we put $\mathbb{S}_k(\lambda) := \mathbb{S}_k^{-1}(\lambda)$, i.e., $\mathcal{S}_k := -\mathcal{J}\mathcal{S}_k^*\mathcal{J}$ and $\mathcal{V}_k := -\mathcal{J}\mathcal{V}_k^*\mathcal{J}$. But in that case we obtain

$$\begin{aligned} z_k^*(\lambda)\,\psi_k\, z_k(\lambda) &= z_{k+1}^*(\lambda)\,\mathbb{S}_k^*(\lambda)\,\psi_k\,\mathbb{S}_k(\lambda) z_{k+1}(\lambda) = -z_{k+1}^*(\lambda)\mathcal{V}_k^*\mathcal{J}\mathcal{S}_k z_{k+1}(\lambda) \\ &= z_{k+1}^*(\lambda)\,\mathcal{J}\mathcal{V}_k\mathcal{J}\mathcal{S}_k^*\mathcal{J} z_{k+1}(\lambda) = z_{k+1}^*(\lambda)\,\Psi_k\, z_{k+1}(\lambda), \end{aligned} \tag{6.3}$$

which among others justifies the replacement of Ψ_k by ψ_k. On the other hand, systems ($\mathbb{S}_\lambda$) and ($\mathcal{S}_\lambda$) lead to different spaces of square summable sequences, see (2.53) and (6.9). Finally, we remark that although the presence of the shift in the semi-inner product could be suppressed similarly as in [A17], the approach based on the time-reversed system seems to be more natural in the present situation and it is, in addition, traditionally used in connection with the second order Sturm–Liouville difference equations, see e.g. [96,147]. Moreover, while for the analyses of the square summable solutions developed in the previous chapters we it was justifiable to "ignore" finite discrete intervals, in this and the following chapters we concern with system ($\mathcal{S}_\lambda$) on a discrete interval $\mathcal{I}_\mathbb{Z}$, which is finite or unbounded above, i.e., $\mathcal{I}_\mathbb{Z} = [0, N+1)_\mathbb{Z}$ with $N \in \mathbb{N} \cup \{0, \infty\}$.

Given the inherent positive semidefiniteness of the sequence ψ defined in (6.1), it is reasonable to consider the construction of *operators* in connection with system ($\mathcal{S}_\lambda$), their extensions and their spectral theory. However we will see that the natural map associated with system ($\mathcal{S}_\lambda$) does give rise only to a multivalued or non-densely defined operator, see Section 6.4. Hence the approach dealing with *linear relations* instead of operators is utilized as it provides powerful tools for the investigation of *multivalued* linear operators in a Hilbert space, especially for non-densely defined linear operators. The beginning of the general theory of linear relations can be traced back to [8], where it was established as a generalization of the results carried out in [167], see also [42,45,46,82,143,144,146,175] and the references therein. Moreover, a short introduction to this theory is provided in the Appendix of this thesis and the reader should be acquainted with its content before reading Section 6.4. The study of linear relations in connection with the linear Hamiltonian differential system from (2.5) was initiated in [128] and further developed, e.g., in [13,83,104,116]. One of the first occurrences of this concept in the discrete theory can be found in [28,29] for the second order Sturm–Liouville difference equations, while their extension to the linear Hamiltonian difference systems was given recently in [79,134,135, 155,161], compare with [142,163]. Hence, in this chapter we aim to associate the minimal and maximal linear relations to the (time-reversed) discrete symplectic systems and to establish their fundamental properties in analogy with [116, Section 2] for system (2.5) and with [134, Section 5] for system (2.6).

The rest of this chapter is organized as follows. In the following section we give some basic properties of system ($\mathcal{S}_\lambda$). In Section 6.2 we focus on the definiteness (or the strong Atkinson) condition for system ($\mathcal{S}_\lambda$), which plays a crucial role in the present theory, and derive some equivalent characterizations. A nonhomogeneous discrete symplectic system is investigated in Section 6.3. Concluding with Section 6.4, the maximal and minimal linear relations associated with the (time-reversed) discrete symplectic systems are introduced and their fundamental properties, such as a relationship between the deficiency indices of the minimal relation in a suitable Hilbert space and the number of square summable solutions of system ($\mathcal{S}_\lambda$), are established. In this final section we also present a sufficient condition guaranteeing the existence of a densely defined operator associated with the time-reversed discrete symplectic system.

6.1 Preliminaries

The conditions for the coefficients $\mathcal{S}_k$ and $\mathcal{V}_k$ of system ($\mathcal{S}_\lambda$) given in (6.1) can be expressed in several equivalent forms, which are summarized in the following statement, compare with Lemma 2.1.1. Moreover, the same arguments as in (2.1.3) can be used for the calculation of $|\det \mathbb{S}_k(\lambda)|$.

Lemma 6.1.1. *Let $n \in \mathbb{N}$ be given. For any $k \in [0,\infty)_\mathbb{Z}$ the following conditions are equivalent.*

(i) *The matrices $\mathcal{S}_k$ and $\mathcal{V}_k$ satisfy the first three conditions in (6.1), i.e., $\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}$, $\mathcal{S}_k^* \mathcal{J} \mathcal{V}_k$ is Hermitian, and $\mathcal{V}_k^* \mathcal{J} \mathcal{V}_k = 0$.*

(ii) *The matrix $\mathbb{S}_k(\lambda)$ defined in ($\mathcal{S}_\lambda$) satisfies the first equality in (6.2), i.e., $\mathbb{S}_k^*(\bar{\lambda}) \mathcal{J} \mathbb{S}_k(\lambda) = \mathcal{J}$, for all $\lambda \in \mathbb{C}$.*

(iii) *The matrices $\mathcal{S}_k$ and $\mathcal{V}_k$ satisfy $\mathcal{S}_k \mathcal{J} \mathcal{S}_k^* = \mathcal{J}$ and $\mathcal{V}_k \mathcal{J} \mathcal{V}_k^* = 0$, and $\mathcal{V}_k \mathcal{J} \mathcal{S}_k^*$ is Hermitian.*

(iv) *The matrix $\mathbb{S}_k(\lambda)$ in ($\mathcal{S}_\lambda$) satisfies $\mathbb{S}_k(\lambda) \mathcal{J} \mathbb{S}_k^*(\bar{\lambda}) = \mathcal{J}$ for all $\lambda \in \mathbb{C}$.*

Moreover, if any of conditions (i), (ii), (iii) or (iv) holds, then

$$\mathbb{S}_k(\lambda) = (I - \lambda \mathcal{J} \Psi_k) \mathcal{S}_k, \quad \Psi_k^* = \Psi_k, \quad \Psi_k \mathcal{J} \Psi_k = 0, \quad \text{and} \quad (I - \lambda \mathcal{J} \Psi_k)^{-1} = (I + \lambda \mathcal{J} \Psi_k), \tag{6.4}$$

where Ψ_k is defined (without the requirement of the positive semidefiniteness) in (6.1). Consequently, $|\det \mathbb{S}_k(\lambda)| = |\det \mathcal{S}_k(\lambda)| = 1$ for all $\lambda \in \mathbb{C}$.

The invertibility of all matrices $\mathbb{S}_k(\lambda)$ guarantees the (global) existence and uniqueness of a solution of any initial value problem associated with system ($\mathcal{S}_\lambda$). Especially, if $z(\lambda) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$ solves system ($\mathcal{S}_\lambda$) and satisfies $z_s(\lambda) = 0$ for some $s \in \mathcal{I}_\mathbb{Z}^+$, then $z(\lambda)$ is only a trivial solution, i.e., $z_k(\lambda) = 0$ for all $k \in \mathcal{I}_\mathbb{Z}^+$.

The latter lemma also establishes the correspondence between the matrix pairs $\{\mathcal{S}, \mathcal{V}\}$ and $\{\mathcal{S}, \Psi\}$. More precisely, if $\mathcal{S}_k, \mathcal{V}_k \in \mathbb{C}^{2n\times 2n}$ satisfies the first three conditions in (6.1), then the equalities in (6.4) hold. On the other hand, if $\mathcal{S}_k, \Psi_k \in \mathbb{C}^{2n\times 2n}$ are such that $\mathcal{S}_k$ satisfies the first relation in (6.1), $\Psi_k^* = \Psi_k$, and $\Psi_k \mathcal{J} \Psi_k = 0$, then the second and third conditions in (6.1) hold with $\mathcal{V}_k := \mathcal{J}^* \Psi_k \mathcal{S}_k = -\mathcal{J} \Psi_k \mathcal{S}_k$. Simultaneously, from (6.4) one can easily conclude that system ($\mathcal{S}_\lambda$) can be equivalently written as

$$\mathcal{J}\big(z_k(\lambda) - \mathcal{S}_k z_{k+1}(\lambda)\big) = \lambda \Psi_k z_k(\lambda),$$

which gives rise to a natural linear map $\mathscr{L} : \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n\times m} \to \mathbb{C}(\mathcal{I}_\mathbb{Z})^{2n\times m}$ associated with system ($\mathcal{S}_\lambda$), namely

$$\mathscr{L}(z)_k := \mathcal{J}(z_k - \mathcal{S}_k z_{k+1}), \tag{6.5}$$

where $m \in \mathbb{N}$ corresponds to the dimension of z and typically we consider $m = 1$. Hence system ($\mathcal{S}_\lambda$) is equivalent to

$$\mathscr{L}(z(\lambda))_k = \lambda \Psi_k z_k(\lambda), \quad k \in \mathcal{I}_\mathbb{Z}. \tag{6.6}$$

From this point of view, the second approach seems to be more natural in the present situation, i.e., to "fix" the sequences of matrices $\mathcal{S}, \Psi \in \mathbb{C}(\mathcal{I}_\mathbb{Z})^{2n\times 2n}$ such that

$$\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \Psi_k^* = \Psi_k, \quad \Psi_k \mathcal{J} \Psi_k = 0, \quad \text{and} \quad \Psi_k \geq 0 \quad \text{for all } k \in \mathcal{I}_\mathbb{Z}. \tag{6.7}$$

In any case, throughout Chapters 6 and 7 we employ the following notation.

Notation 6.1.2. The number $n \in \mathbb{N}$ is fixed, the discrete interval $\mathcal{I}_\mathbb{Z}$ is given either finite or infinite, i.e., $\mathcal{I}_\mathbb{Z} = [0, N+1)_\mathbb{Z}$ with $N \in \mathbb{N} \cup \{0, \infty\}$, and the sequences $\mathcal{S}, \mathcal{V}, \Psi \in \mathbb{C}(\mathcal{I}_\mathbb{Z})^{2n\times 2n}$ are such that the conditions in (6.1) are satisfied for all $k \in \mathcal{I}_\mathbb{Z}$.

With respect to the fact that $\Psi \in \mathbb{C}(\mathcal{I}_\mathbb{Z})^{2n\times 2n}$ is a sequence with positive semidefinite terms, we define a semi-inner product on $\mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$ by

$$\langle z, v\rangle_\Psi := \sum_{k\in\mathcal{I}} z_k^* \Psi_k v_k, \tag{6.8}$$

where $z, v \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$. Here we are intentionally sloppy when suppressing the subscript $\mathbb{Z}$ in the notation of the discrete interval $\mathcal{I}_\mathbb{Z}$ in the sum on the right-hand side of the latter identity. The same convention is applied throughout the last two chapters. The associated linear space of all square summable sequences is denoted by

$$\ell_\Psi^2(\mathcal{I}_\mathbb{Z}) = \ell_\Psi^2 := \Big\{z \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n} \mid \|z\|_\Psi < \infty\Big\}, \tag{6.9}$$

where $\|\cdot\|_\Psi := \sqrt{\langle\cdot,\cdot\rangle_\Psi}$ denotes the associated semi-norm.

The equivalent expression of system $(\mathcal{S}_\lambda)$ given in (6.6) also justifies the following form of the associated nonhomogeneous system

$$z_k(\lambda) = \mathbb{S}_k(\lambda) z_{k+1}(\lambda) - \mathcal{J}\Psi_k f_k \quad \text{or equivalently} \quad \mathcal{L}(z(\lambda))_k = \lambda\Psi_k z_k(\lambda) + \Psi_k f_k, \quad k \in \mathcal{I}_\mathbb{Z}, \quad (\mathcal{S}_\lambda^f)$$

where $z(\lambda), f \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n\times m}$ with $m \in \mathbb{N}$ being the same as discussed when defining the linear map $\mathcal{L}$ in (6.5). For simplicity, $(\mathcal{S}_\nu^g)$ refers to the nonhomogeneous system of the form given in $(\mathcal{S}_\lambda^f)$ with λ replaced by ν and f replaced by g. In addition, instead of $(\mathcal{S}_\lambda^0)$ we write only $(\mathcal{S}_\lambda)$ and for this system we apply the same convention, i.e., by $(\mathcal{S}_\nu)$ we denote the system of the form given in $(\mathcal{S}_\lambda)$ with the parameter λ replaced by ν. Let us point out that the equivalent expression given in $(\mathcal{S}_\lambda^f)$ will play a key role in Section 6.4, where we sometimes write only $\mathcal{L}(z(\lambda)) = \lambda\Psi z(\lambda) + \Psi f$ and $\mathcal{L}(z) = \Psi f$ with the meaning $\mathcal{L}(z(\lambda))_k = \lambda\Psi_k z_k(\lambda) + \Psi_k f_k$ and $\mathcal{L}(z)_k = \Psi_k f_k$ for all $k \in \mathcal{I}_\mathbb{Z}$, respectively. For convenience, we also put $\mathcal{L}^*(z)_k := [\mathcal{L}(z)_k]^*$.

The next result presents an absolutely essential tool used throughout this chapter, compare with Lemma 2.1.5 and Theorem 2.1.7.

Theorem 6.1.3 (Extended Lagrange identity). *Let $\lambda, \nu \in \mathbb{C}$, $m \in \mathbb{N}$, and $f, g \in \mathbb{C}(\mathcal{I}_\mathbb{Z})^{2n\times m}$ be given. If the sequences $\mathcal{Z}(\lambda), \mathcal{Z}(\nu) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n\times m}$ solve systems $(\mathcal{S}_\lambda^f)$ and $(\mathcal{S}_\nu^g)$ on $\mathcal{I}_\mathbb{Z}$, respectively, then for any $s, t \in \mathcal{I}_\mathbb{Z}$ with $s \le t$ we have*

$$\Delta\Big[\mathcal{Z}_k^*(\lambda)\mathcal{J}\mathcal{Z}_k(\nu)\Big] = (\bar\lambda - \nu)\mathcal{Z}_k^*(\lambda)\Psi_k\mathcal{Z}_k(\nu) + f_k^*\Psi_k\mathcal{Z}_k(\nu) - \mathcal{Z}_k^*(\lambda)\Psi_k g_k, \tag{6.10}$$

$$\mathcal{Z}_k^*(\lambda)\mathcal{J}\mathcal{Z}_k(\nu)\Big|_s^{t+1} = \sum_{k=s}^{t}\Big\{(\bar\lambda - \nu)\mathcal{Z}_j^*(\lambda)\Psi_j\mathcal{Z}_j(\nu) + f_j^*\Psi_j\mathcal{Z}_j(\nu) - \mathcal{Z}_j^*(\lambda)\Psi_j g_j\Big\}. \tag{6.11}$$

Especially, if $\nu = \bar\lambda$ and $f_k \equiv g_k \equiv 0$ on $\mathcal{I}_\mathbb{Z}$, we have the Wronskian-type identity

$$\mathcal{Z}_k^*(\lambda)\mathcal{J}\mathcal{Z}_k(\bar\lambda) \equiv \mathcal{Z}_0^*(\lambda)\mathcal{J}\mathcal{Z}_0(\bar\lambda) \quad \text{on } \mathcal{I}_\mathbb{Z}^+. \tag{6.12}$$

Proof. Let $\mathcal{Z}(\lambda), \mathcal{Z}(\nu) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n\times m}$ solve systems $(\mathcal{S}_\lambda^f)$ and $(\mathcal{S}_\nu^g)$. Since by (6.4) we have $\mathbb{S}_k^{*-1}(\lambda)\mathcal{J}\mathbb{S}_k^{-1}(\nu) = \mathcal{J} + (\bar\lambda - \nu)\Psi_k$, we see that

$$\begin{aligned}
\Delta\Big[\mathcal{Z}_k^*(\lambda)\mathcal{J}\mathcal{Z}_k(\nu)\Big] &= \Big[\mathcal{Z}_k(\lambda) + \mathcal{J}\Psi_k f_k\Big]^*\mathbb{S}_k^{*-1}(\lambda)\mathcal{J}\mathbb{S}_k^{-1}(\nu)\Big[\mathcal{Z}_k(\nu) + \mathcal{J}\Psi_k g_k\Big] - \mathcal{Z}_k^*(\lambda)\mathcal{J}\mathcal{Z}_k(\nu) \\
&= \mathcal{Z}_k^*(\lambda)\Big[\mathbb{S}_k^{*-1}(\lambda)\mathcal{J}\mathbb{S}_k^{-1}(\nu) - \mathcal{J}\Big]\mathcal{Z}_k(\nu) + f_k^*\Psi_k\mathcal{Z}_k(\nu) - \mathcal{Z}_k^*(\lambda)\Psi_k g_k \\
&= (\bar\lambda - \nu)\mathcal{Z}_k^*(\lambda)\Psi_k\mathcal{Z}_k(\nu) + f_k^*\Psi_k\mathcal{Z}_k(\nu) - \mathcal{Z}_k^*(\lambda)\Psi_k g_k.
\end{aligned}$$

Identities (6.11) and (6.12) are only simple consequences of the latter equality. ■

Let us note that, by using the equivalent expression of system ($\mathcal{S}_\lambda^f$), identity (6.11) can be also written as

$$\big(\mathcal{Z}(\lambda),\mathcal{Z}(\nu)\big)_k\Big|_s^{t+1}=\sum_{k=s}^{t}\Big\{\mathcal{L}^*(\mathcal{Z}(\lambda))_k\,\mathcal{Z}_k(\nu)-\mathcal{Z}_k^*(\lambda)\,\mathcal{L}(\mathcal{Z}(\nu))_k\Big\},\tag{6.13}$$

where we use for any $Z,\mathcal{Z}\in\mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n\times m}$ and $k\in\mathcal{I}_\mathbb{Z}^+$ the notation

$$(Z,\mathcal{Z})_k:=Z_k^*\mathcal{J}\mathcal{Z}_k.$$

Moreover, under the assumptions of Theorem 6.1.3 with $m=1$, $\mathcal{Z}(\lambda)=z$, $\mathcal{Z}(\nu)=v$, $\lambda=0=\nu$, $s=0$, and $t=N$ we get from (6.11) and (6.8) that

$$(z,v)_k\Big|_0^{N+1}=\langle f,v\rangle_\Psi-\langle z,g\rangle_\Psi,\tag{6.14}$$

where the left-hand side of (6.14) means $\lim_{k\to\infty}(z,v)_k-(z,v)_0$ if $\mathcal{I}_\mathbb{Z}=[0,\infty)_\mathbb{Z}$. Nevertheless identity (6.14) and the Cauchy–Schwarz inequality imply that the latter limit exists finite whenever $z,v,f,g\in\ell_\Psi^2$.

Throughout this and the following chapters we denote by $\Theta(\lambda)\in\mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n\times 2n}$ a fundamental matrix of system ($\mathcal{S}_\lambda$). If, similarly as in Chapter 2, it is such that

$$\Theta_s^*(\bar\lambda)\mathcal{J}\Theta_s(\lambda)=\mathcal{J},\tag{6.15}$$

for some $s\in\mathcal{I}_\mathbb{Z}^+$, then as an immediate consequence of (6.12) we have for any $k\in\mathcal{I}_\mathbb{Z}^+$ that

$$\Theta_k^*(\bar\lambda)\mathcal{J}\Theta_k(\lambda)=\mathcal{J},\quad \Theta_k^{-1}(\lambda)=-\mathcal{J}\Theta_k^*(\bar\lambda)\mathcal{J},\quad\text{and}\quad \Theta_k(\lambda)\mathcal{J}\Theta_k^*(\bar\lambda)=\mathcal{J}.\tag{6.16}$$

Remark 6.1.4. Similarly as for linear Hamiltonian differential systems, there exists a unitary map $\mathcal{Q}:\mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}\to\mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$ preserving the square summability with respect to Ψ and such that system ($\mathcal{S}_0^f$) can be written in the *canonical* form, i.e., with $\mathcal{S}\equiv I$. Indeed, let $\Theta=\Theta(0)$ denote the fundamental matrix of system ($\mathcal{S}_0$) satisfying $\Theta_0=I$. Then it is invertible for all $k\in\mathcal{I}_\mathbb{Z}^+$ with $\Theta_k^{-1}=-\mathcal{J}\Theta_k^*\mathcal{J}$ and this inverse provides the canonical transformation, i.e., $\mathcal{Q}_k=\Theta_k^{-1}$ with $\mathcal{Q}(z)_k:=\Theta_k^{-1}z_k$. Hence system ($\mathcal{S}_0^f$) is equivalent with

$$-\mathcal{J}\Delta v_k=\widehat{\Psi}_k g_k,\quad k\in\mathcal{I}_\mathbb{Z},\tag{6.17}$$

where we put $v_k:=\mathcal{Q}(z)_k$, $g_k:=\mathcal{Q}(f)_k$, and $\widehat{\Psi}_k:=\Theta_k^*\Psi_k\Theta_k$, see also [79]. Moreover, one can easily verify that $v\in\ell_{\widehat{\Psi}}^2$ if and only if $z\in\ell_\Psi^2$. System (6.17) can be seen as a discrete counterpart of the canonical linear Hamiltonian differential system, i.e., nonhomogeneous system associated with (2.5), where $H(t)\equiv 0$, see e.g. [128] or [116, Subsection 2.2] and the references therein.

6.2 Definiteness condition

In this section we focus on the *definiteness condition* for system ($\mathcal{S}_\lambda$), which is closely related to the strong Atkinson condition applied to all $\lambda\in\mathbb{C}$, see Hypothesis 2.4.11 and Remark 6.2.7. Although it was shown in Chapter 2 that only the weak form of the Atkinson condition is sufficient for the development of the Weyl–Titchmarsh theory, we will need the strong version for (at least) two reasons:

(i) in the section devoted to the nonhomogeneous problem we utilize Theorem 2.4.12 in the setting of system ($\mathcal{S}_\lambda$), see Section 6.3;

(ii) it guarantees a certain (extremely important) uniqueness property for system ($\mathcal{S}_0^f$), see Lemmas 6.2.8 and 6.4.3, Theorem 6.4.2, Corollary 6.4.14, and Chapter 7.

Similar treatment in connection with the linear Hamiltonian differential and difference systems can be found in [13, 116] and [134], respectively.

Definition 6.2.1. System ($\mathcal{S}_\lambda$) is said to be *definite on a discrete interval* $\mathcal{I}_\mathbb{Z}^\mathrm{D} \subseteq \mathcal{I}_\mathbb{Z}$ provided the interval $\mathcal{I}_\mathbb{Z}^\mathrm{D}$ is nonempty and for each $\lambda \in \mathbb{C}$ any nontrivial solution $z(\lambda) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$ of system ($\mathcal{S}_\lambda$), i.e, $\mathscr{L}(z(\lambda))_k = \lambda \Psi_k z_k(\lambda)$ for all $k \in \mathcal{I}_\mathbb{Z}$, satisfies

$$\sum_{k\in\mathcal{I}^\mathrm{D}} z_k^*(\lambda)\Psi_k z_k(\lambda) > 0. \tag{6.18}$$

Remark 6.2.2. Alternatively, the definiteness condition for ($\mathcal{S}_\lambda$) can be stated in the following way: system ($\mathcal{S}_\lambda$) is definite on a (nonempty) discrete interval $\mathcal{I}_\mathbb{Z}^\mathrm{D} \subseteq \mathcal{I}_\mathbb{Z}$ if, for each $\lambda \in \mathbb{C}$, every solution $z(\lambda) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$ of system ($\mathcal{S}_\lambda$) for which

$$\sum_{k\in\mathcal{I}^\mathrm{D}} z_k^*(\lambda)\Psi_k z_k(\lambda) = 0, \tag{6.19}$$

is trivial on $\mathcal{I}_\mathbb{Z}^\mathrm{D}$, i.e., $z_k(\lambda) = 0$ for all $k \in \mathcal{I}_\mathbb{Z}^\mathrm{D}$, and consequently it is trivial on the whole interval $\mathcal{I}_\mathbb{Z}^+$. Furthermore, from the assumption of $\Psi_k \geq 0$ on $\mathcal{I}_\mathbb{Z}$ we get immediately that for any discrete interval $\widehat{\mathcal{I}}_\mathbb{Z} \subseteq \mathcal{I}_\mathbb{Z}$ such that $\mathcal{I}_\mathbb{Z}^\mathrm{D} \subseteq \widehat{\mathcal{I}}_\mathbb{Z}$ we have

$$\sum_{k\in\mathcal{I}^\mathrm{D}} z_k^*(\lambda)\Psi_k z_k(\lambda) \leq \sum_{k\in\widehat{\mathcal{I}}} z_k^*(\lambda)\Psi_k z_k(\lambda). \tag{6.20}$$

Therefore one easily concludes that the definiteness of system ($\mathcal{S}_\lambda$) on $\mathcal{I}_\mathbb{Z}^\mathrm{D}$ guarantees the definiteness of ($\mathcal{S}_\lambda$) on every discrete "interval superset" $\widehat{\mathcal{I}}_\mathbb{Z}$, especially on $\mathcal{I}_\mathbb{Z}$. Hence, definiteness of system ($\mathcal{S}_\lambda$) on some finite discrete subinterval $\mathcal{I}_\mathbb{Z}^\mathrm{D}$ implies, for every $\lambda \in \mathbb{C}$, that the semi-norm $\|\cdot\|_\Psi$ of any nontrivial solution of system ($\mathcal{S}_\lambda$) is nonzero. The converse of this last statement will be established in Lemma 6.2.6 below.

In the next lemma we show that for the definiteness of system ($\mathcal{S}_\lambda$) on a discrete interval $\mathcal{I}_\mathbb{Z}^\mathrm{D}$ it is not necessary to verify inequality (6.18) for all nontrivial solutions and every $\lambda \in \mathbb{C}$, but it suffices to do it only for one particular choice of $\lambda \in \mathbb{C}$.

Lemma 6.2.3. *System ($\mathcal{S}_\lambda$) is definite on a discrete interval $\mathcal{I}_\mathbb{Z}^\mathrm{D} \subseteq \mathcal{I}_\mathbb{Z}$ if and only if, for some $\lambda_0 \in \mathbb{C}$, each solution $z(\lambda_0) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$ of system ($\mathcal{S}_{\lambda_0}$) satisfying*

$$\sum_{k\in\mathcal{I}^\mathrm{D}} z_k^*(\lambda_0)\Psi_k z_k(\lambda_0) = 0 \tag{6.21}$$

is trivial on $\mathcal{I}_\mathbb{Z}^\mathrm{D}$, i.e., $z_k(\lambda_0) = 0$ for all $k \in \mathcal{I}_\mathbb{Z}^\mathrm{D}$.

Proof. We begin by assuming that for some $\lambda_0 \in \mathbb{C}$ each solution $z(\lambda_0)$ of system ($\mathcal{S}_{\lambda_0}$) satisfying (6.21) is necessarily trivial on $\mathcal{I}_\mathbb{Z}^\mathrm{D}$, i.e., $z_k(\lambda_0) = 0$ for all $k \in \mathcal{I}_\mathbb{Z}^\mathrm{D}$. Let $\lambda \in \mathbb{C}$ be arbitrary and $z(\lambda)$ be a solution of system ($\mathcal{S}_\lambda$) such that (6.19) holds. Given Remark 6.2.2, it suffices to show that $z(\lambda)$ is also trivial on $\mathcal{I}_\mathbb{Z}^\mathrm{D}$. Since $\Psi_k z_k(\lambda) = 0$ for all $k \in \mathcal{I}_\mathbb{Z}^\mathrm{D}$, we see by the equivalent expression given in (6.6) that $z(\lambda)$ solves also system ($\mathcal{S}_{\lambda_0}$) on $\mathcal{I}_\mathbb{Z}^\mathrm{D}$. Then, by the assumed definiteness of system ($\mathcal{S}_{\lambda_0}$) on $\mathcal{I}_\mathbb{Z}^\mathrm{D}$, condition (6.19) indeed implies that $z_k(\lambda) = 0$ for all $k \in \mathcal{I}_\mathbb{Z}^\mathrm{D}$. The converse is trivial. ∎

Example 6.2.4. The discrete symplectic systems (or their time-reversed form) investigated in Examples 2.5.1–2.5.3 are not definite, because they possess nontrivial solutions with the semi-norm equal to zero. On the other hand, we can provide a simple example of system ($\mathcal{S}_\lambda$) being definite on some nontrivial discrete interval $\mathcal{I}_\mathbb{Z}^{\mathrm{D}} \subseteq \mathcal{I}_\mathbb{Z}$; cf. Theorem 6.2.5. Let us consider the following system

$$\begin{pmatrix} x_k(\lambda) \\ u_k(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & -1/p_{k+1} \\ -q_k + \lambda w_k & 1 + (q_k - \lambda w_k)/p_{k+1} \end{pmatrix} \begin{pmatrix} x_{k+1}(\lambda) \\ u_{k+1}(\lambda) \end{pmatrix}, \quad k \in \mathcal{I}_\mathbb{Z}, \tag{6.22}$$

where $p \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+ \setminus \{0\})$ and $q, w \in \mathbb{C}(\mathcal{I}_\mathbb{Z})$ are (only) real-valued sequences such that $w_k \geq 0$ and $p_{k+1} \neq 0$ for all $k \in \mathcal{I}_\mathbb{Z}$. Then the conditions in (6.1) are satisfied with

$$\mathcal{S}_k = \begin{pmatrix} 1 & -1/p_{k+1} \\ -q_k & 1 + q_k/p_{k+1} \end{pmatrix}, \quad \mathcal{V}_k = \begin{pmatrix} 0 & 0 \\ w_k & -w_k/p_{k+1} \end{pmatrix}, \quad \text{and} \quad \Psi_k = \begin{pmatrix} w_k & 0 \\ 0 & 0 \end{pmatrix},$$

see also (6.25). If $w_k > 0$ for at least two consecutive points of $\mathcal{I}_\mathbb{Z}$, i.e., for $k \in [a, b]_\mathbb{Z}$ with $a, b \in \mathcal{I}_\mathbb{Z}$ and $a < b$, then system (6.22) is definite on $[a, b]_\mathbb{Z}$, and thus, by Remark 6.2.2, also on any discrete interval $\mathcal{I}_\mathbb{Z}^{\mathrm{D}}$ such that $[a, b]_\mathbb{Z} \subseteq \mathcal{I}_\mathbb{Z}^{\mathrm{D}} \subseteq \mathcal{I}_\mathbb{Z}$. Indeed, let us denote by $z = (x, u)^\top \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^2$ a sequence satisfying system (6.22) with $\lambda = 0$ and assume that $\Psi_k z_k = 0$ on $[a, b]_\mathbb{Z}$, i.e., the pair x_k, u_k is such that

$$\Delta x_k = \frac{u_{k+1}}{p_{k+1}}, \qquad \Delta u_k = q_k x_{k+1} - \frac{q_k}{p_{k+1}} u_{k+1}, \quad \text{and} \quad w_k x_k = 0 \quad \text{for all } k \in [a, b]_\mathbb{Z}. \tag{6.23}$$

Then the positivity of w_k on $[a, b]_\mathbb{Z}$ and the third equality in (6.23) yield $x_k = 0$ for all $k \in [a, b]_\mathbb{Z}$, which implies that also $u_k = 0$ for all $k \in [a+1, b]_\mathbb{Z}$ by the first equality in (6.23). Hence $z_k = 0$ on $[a+1, b]_\mathbb{Z}$ and from the invertibility of the coefficient matrix in (6.22) we easily obtain $z \equiv 0$ on $[a, b]_\mathbb{Z}$. It means that z is only the trivial solution of system (6.22) with $\lambda = 0$ and the definiteness of system (6.22) on $[a, b]_\mathbb{Z}$ follows by Lemma 6.2.3.

In addition, we point out that system (6.22) corresponds to the second order Sturm–Liouville difference equation

$$-\Delta[p_k \Delta y_{k-1}(\lambda)] + q_k y_k(\lambda) = \lambda w_k y_k(\lambda), \quad k \in \mathcal{I}_\mathbb{Z}. \tag{6.24}$$

with $p \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)$ and $q, w \in \mathbb{C}(\mathcal{I}_\mathbb{Z})$. More precisely, let $y(\lambda) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+ \cup \{-1\})$ be a solution of equation (6.24) on the discrete interval $\mathcal{I}_\mathbb{Z}$. Then the pair $x_k(\lambda) := y_k(\lambda)$, $k \in \mathcal{I}_\mathbb{Z}^+ \cup \{-1\}$, and $u_k(\lambda) := p_k \Delta y_{k-1}(\lambda)$, $k \in \mathcal{I}_\mathbb{Z}^+$, satisfies system (6.22) for all $k \in \mathcal{I}_\mathbb{Z}$, compare with the system corresponding to equation (2.8) in the case $m = n = 1$ and see also [4, Example 3.8]. On the other hand, if $p_0 \neq 0$ and the pair $x(\lambda), u(\lambda) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)$ solves system (6.22), then $y(\lambda) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+ \cup \{-1\})$ defined as

$$y_k(\lambda) := \begin{cases} x_k(\lambda), & k \in \mathcal{I}_\mathbb{Z}^+, \\ x_0(\lambda) - u_0(\lambda)/p_0, & k = -1, \end{cases}$$

satisfies equation (6.24) for all $k \in \mathcal{I}_\mathbb{Z}$. ▲

System (6.22) is a particular case of system ($\mathcal{S}_\lambda$) with a special linear dependence on the parameter λ, which is analogous to system (S_λ) with (2.7). Namely, let

$$\mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \quad \mathcal{V}_k = \begin{pmatrix} 0 & 0 \\ \mathcal{W}_k \mathcal{A}_k & \mathcal{W}_k \mathcal{A}_k \end{pmatrix}, \quad \text{and} \quad \Psi_k = \begin{pmatrix} \mathcal{W}_k & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.25}$$

where $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D},\mathcal{W}\in\mathbb{C}(\mathcal{I}_{\mathbb{Z}})^{n\times n}$ are such that $\mathcal{S}_k^*\mathcal{J}\mathcal{S}_k=\mathcal{J}$ and $\mathcal{W}_k=\mathcal{W}_k^*\geq 0$ for all $k\in\mathcal{I}_{\mathbb{Z}}$. Then the conditions in (6.1) are satisfied on $\mathcal{I}_{\mathbb{Z}}$ and in the following theorem we derive a sufficient condition for the definiteness of system ($\mathcal{S}_\lambda$) with the coefficients as in (6.25). For completeness, we note that the latter system is equivalent with the pair of equations

$$x_k(\lambda)=\mathcal{A}_k\,x_{k+1}(\lambda)+\mathcal{B}_k\,u_{k+1}(\lambda),\quad u_k(\lambda)=\mathcal{C}_k\,x_{k+1}(\lambda)+\mathcal{D}_k\,u_{k+1}(\lambda)+\lambda\mathcal{W}_k\,x_k(\lambda).\tag{6.26}$$

Theorem 6.2.5. *Let us consider system ($\mathcal{S}_\lambda$) with the coefficients specified in (6.25). If there exists an index $m\in\mathcal{I}_{\mathbb{Z}}\setminus\{0\}$ such that the matrices $\mathcal{B}_{m-1}$, $\mathcal{W}_{m-1}$, and $\mathcal{W}_m$ are invertible (in fact, $\mathcal{W}_{m-1}$ and $\mathcal{W}_m$ are positive definite), then system ($\mathcal{S}_\lambda$) is definite on the discrete interval $[m-1,m]_{\mathbb{Z}}$, and thus also on $\mathcal{I}_{\mathbb{Z}}$.*

Proof. Let $\lambda\in\mathbb{C}$ be fixed and $z(\lambda)=(x^\top(\lambda),\ u^\top(\lambda))^\top\in\mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$ be a nontrivial solution of system ($\mathcal{S}_\lambda$) such that $\Psi_k\,z_k(\lambda)=0$ on $[m-1,m]_{\mathbb{Z}}$, i.e., $\mathcal{W}_{m-1}\,x_{m-1}(\lambda)=0$ and $\mathcal{W}_m\,x_m(\lambda)=0$. By Lemma 6.2.3 we have to show that $z_k(\lambda)=0$ on $[m-1,m]_{\mathbb{Z}}$. From the invertibility of $\mathcal{W}_{m-1}$ and $\mathcal{W}_m$ we obtain $x_{m-1}(\lambda)=x_m(\lambda)=0$. Hence

$$0=x_{m-1}(\lambda)\overset{(6.26)}{=}\mathcal{A}_{m-1}\,x_m(\lambda)+\mathcal{B}_{m-1}\,u_m(\lambda)=\mathcal{B}_{m-1}\,u_m(\lambda),$$

which yields also $u_m(\lambda)=0$ by the invertibility of $\mathcal{B}_{m-1}$. It means $z_m(\lambda)=0$ and consequently $z(\lambda)\equiv 0$ on $\mathcal{I}_{\mathbb{Z}}^+$. ∎

For the following result we fix $k_0\in\mathcal{I}_{\mathbb{Z}}^+$ and by $\Theta(\lambda)\in\mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n\times 2n}$ we denote the fundamental matrix of system ($\mathcal{S}_\lambda$) satisfying $\Theta_{k_0}(\lambda)=I$ for any $\lambda\in\mathbb{C}$. Hence $\Theta(\lambda)$ satisfies equality (6.15) with $s=k_0$ and thus also the relations in (6.16) for all $k\in\mathcal{I}_{\mathbb{Z}}^+$. The next result provides a characterization of the definiteness of system ($\mathcal{S}_\lambda$) which is analogous to that for system (2.5) established in [13, Proposition 2.11].

Lemma 6.2.6. *System ($\mathcal{S}_\lambda$) is definite on $\mathcal{I}_{\mathbb{Z}}$ if and only if there exists a finite discrete interval $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}}\subseteq\mathcal{I}_{\mathbb{Z}}$ over which the system is definite.*

Proof. If the discrete interval $\mathcal{I}_{\mathbb{Z}}$ is finite, the statement is trivial. Hence, let us consider the case $\mathcal{I}_{\mathbb{Z}}=[0,\infty)_{\mathbb{Z}}$. From Remark 6.2.2 we know that the definiteness of system ($\mathcal{S}_\lambda$) on a finite discrete interval $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}}$ implies definiteness on the discrete interval $[0,\infty)_{\mathbb{Z}}$. Thus, it remains to show the converse.

Assume that system ($\mathcal{S}_\lambda$) is definite on $[0,\infty)_{\mathbb{Z}}$. In light of Lemma 6.2.3, we need only to show the existence of a finite discrete interval $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}}$ over which system ($\mathcal{S}_\lambda$) is definite for one value $\lambda_0\in\mathbb{C}$. Thus, let $\lambda_0\in\mathbb{C}$ be fixed and for any finite discrete subinterval $\widetilde{\mathcal{I}}_{\mathbb{Z}}\subset[0,\infty)_{\mathbb{Z}}$ we define the set $s(\widetilde{\mathcal{I}}_{\mathbb{Z}})$ as

$$s(\widetilde{\mathcal{I}}_{\mathbb{Z}}):=\Big\{\xi\in\mathbb{C}^{2n}\ \Big|\ \|\xi\|_2=1\text{ and }\sum_{k\in\widetilde{\mathcal{I}}}\xi^*\,\Theta_k^*(\lambda_0)\,\Psi_k\,\Theta_k(\lambda_0)\,\xi=0\Big\},$$

where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{C}^{2n}$. Then $s(\widetilde{\mathcal{I}}_{\mathbb{Z}})$ is compact and it holds $s(\widehat{\mathcal{I}}_{\mathbb{Z}})\subseteq s(\widetilde{\mathcal{I}}_{\mathbb{Z}})$ whenever $\widetilde{\mathcal{I}}_{\mathbb{Z}}\subseteq\widehat{\mathcal{I}}_{\mathbb{Z}}$. Moreover, let $\big\{\mathcal{I}_{\mathbb{Z}}^{[m]}\big\}_{m\in\mathbb{N}}$ be a collection of nested finite discrete intervals $\mathcal{I}_{\mathbb{Z}}^{[1]}\subseteq\mathcal{I}_{\mathbb{Z}}^{[2]}\subseteq\cdots\subset[0,\infty)_{\mathbb{Z}}$ such that $\bigcup_{m\in\mathbb{N}}\mathcal{I}_{\mathbb{Z}}^{[m]}=[0,\infty)_{\mathbb{Z}}$. Suppose that there exists a vector $\xi\in\mathbb{C}^{2n}$ with $\|\xi\|_2=1$ such that for every $m\in\mathbb{N}$ we have $\sum_{k\in\mathcal{I}^{[m]}}\xi^*\,\Theta_k^*(\lambda_0)\,\Psi_k\,\Theta_k(\lambda_0)\,\xi=0$. Then also $\sum_{k=0}^{\infty}\xi^*\,\Theta_k^*(\lambda_0)\,\Psi_k\,\Theta_k(\lambda_0)\,\xi=0$, which implies that $\Theta_k(\lambda_0)\,\xi=0$ for all $k\in[0,\infty)_{\mathbb{Z}}$ by the definiteness of system ($\mathcal{S}_\lambda$) on $[0,\infty)_{\mathbb{Z}}$, see Remark 6.2.2. But it means that $\xi=0$, which contradicts the assumption $\|\xi\|_2=1$. Consequently

$$\bigcap_{m\in\mathbb{N}}s\big(\mathcal{I}_{\mathbb{Z}}^{[m]}\big)=\emptyset$$

for the collection of nested compact sets $s\big(\mathcal{I}_{\mathbb{Z}}^{[1]}\big) \supseteq s\big(\mathcal{I}_{\mathbb{Z}}^{[2]}\big) \supseteq \cdots$. Thus by the Cantor intersection theorem, q.v. [7, Theorem 3.25], there exists $m_0 \in \mathbb{N}$ such that $s\big(\mathcal{I}_{\mathbb{Z}}^{[m_0]}\big) = \emptyset$, which demonstrates the definiteness of system ($\mathcal{S}_\lambda$) on the interval $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}} = \mathcal{I}_{\mathbb{Z}}^{[m_0]} \subset [0,\infty)_{\mathbb{Z}}$. ■

Remark 6.2.7. From Lemmas 6.2.3 and 6.2.6 we conclude that the strong Atkinson condition stated similarly as in Hypothesis 2.4.11 for system ($\mathcal{S}_\lambda$) is equivalent to the definiteness of system ($\mathcal{S}_\lambda$) on $[0,\infty)_{\mathbb{Z}}$ and it suffices to verify this condition only for one $\lambda \in \mathbb{C}$. Note also that [26, Assumption 2.2] requires satisfaction of inequality (6.18) on every nonempty finite subinterval of $[0,\infty)_{\mathbb{Z}}$, which is significantly stronger than requiring the definiteness of system ($\mathcal{S}_\lambda$) on $[0,\infty)_{\mathbb{Z}}$ as seen, e.g., when ($\mathcal{S}_\lambda$) is definite on a finite discrete interval $[0,N]_{\mathbb{Z}} \subset [0,\infty)_{\mathbb{Z}}$ and $\Psi_k \equiv 0$ for $k \in [N+1,\infty)_{\mathbb{Z}}$.

Now we establish a basic result concerning the solvability of a boundary value problem associated with system ($\mathcal{S}_\lambda$), which will be utilized in the proof of Lemma 6.4.3. It provides the symplectic counterpart of the original Naimark's result known as the "Patching lemma", see [124, Lemma 2 in Section 17.3]. Analogous result for system (2.6) can be found in [135, Lemma 3.3].

Lemma 6.2.8. *Let system ($\mathcal{S}_\lambda$) be definite on a finite discrete interval $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}} \subseteq \mathcal{I}_{\mathbb{Z}}$ and a finite discrete interval $\widehat{\mathcal{I}}_{\mathbb{Z}} := [c,d]_{\mathbb{Z}}$ be given such that $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}} \subseteq \widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ with $c,d \in \mathcal{I}_{\mathbb{Z}}$. Then for any given $\xi,\eta \in \mathbb{C}^{2n}$ there exists $f \in \mathbb{C}(\widehat{\mathcal{I}}_{\mathbb{Z}})^{2n}$ such that the boundary value problem*

$$\mathcal{L}(z)_k = \Psi_k f_k, \quad z_c = \xi, \quad z_{d+1} = \eta, \quad k \in \widehat{\mathcal{I}}_{\mathbb{Z}}, \tag{6.27}$$

possesses a solution $z \in \mathbb{C}(\widehat{\mathcal{I}}_{\mathbb{Z}}^{+})^{2n}$.

Proof. Let $A = (a_{ij})_{i,j=1,\dots,2n}$ be the $2n \times 2n$ matrix with the elements $a_{ij} := \sum_{k=c}^{d} \varphi_k^{[i]*} \Psi_k \varphi_k^{[j]}$ for $i,j \in \{1,\dots,2n\}$, where $\varphi^{[1]},\dots,\varphi^{[2n]} \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^{+})^{2n}$ are linearly independent solutions of system ($\mathcal{S}_0$), i.e., $\mathcal{L}(\varphi^{[i]})_k = 0$ for all $k \in \mathcal{I}_{\mathbb{Z}}$ and $i \in \{1,\dots,2n\}$. Then the homogeneous system of algebraic equations $A\rho = 0$, where $\rho = (\rho_1,\dots,\rho_{2n})^{\top} \in \mathbb{C}^{2n}$, is equivalent to $\sum_{k=c}^{d} \varphi_k^{*} \Psi_k \varphi_k = 0$, where $\varphi := \sum_{i=1}^{2n} \rho_i \varphi^{[i]} \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^{+})^{2n}$. Since φ also solves system ($\mathcal{S}_0$), it follows from the assumption of the definiteness on $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}}$ and inequality (6.20) that φ is only the trivial solution of system ($\mathcal{S}_0$), i.e., $\sum_{i=1}^{2n} \rho_i \varphi_k^{[i]} \equiv 0$, which implies that $\rho_i = 0$ for all $i \in \{1,\dots,2n\}$. Thus, the matrix A is invertible.

Consequently, there exists a unique solution $\zeta = (\zeta_1,\dots,\zeta_{2n})^{\top} \in \mathbb{C}^{2n}$ of the nonhomogeneous system of algebraic equations

$$\zeta^{*} A = \eta^{*} \mathcal{J} \Theta_{d+1}, \tag{6.28}$$

where $\Theta := (\varphi^{[1]*},\dots,\varphi^{[2n]*})^{*}$ is a fundamental matrix of system ($\mathcal{S}_0$). If we put $h_k^{[1]} := \Theta_k \zeta$ for $k \in \widehat{\mathcal{I}}_{\mathbb{Z}}$, we get from (6.28) for all $i \in \{1,\dots,2n\}$ that

$$\sum_{k=c}^{d} h_k^{[1]*} \Psi_k \varphi_k^{[i]} = \eta^{*} \mathcal{J} \varphi_{d+1}^{[i]}. \tag{6.29}$$

Simultaneously the definiteness of system ($\mathcal{S}_\lambda$) guarantees the existence of a unique solution $z^{[1]} \in \mathbb{C}(\widehat{\mathcal{I}}_{\mathbb{Z}}^{+})^{2n}$ of the nonhomogeneous initial value problem

$$\mathcal{L}(z^{[1]})_k = \Psi_k h_k^{[1]}, \quad z_c^{[1]} = 0, \quad k \in \widehat{\mathcal{I}}_{\mathbb{Z}}.$$

Then, for all $i \in \{1,\dots,2n\}$, the fact $\mathcal{L}(\varphi^{[i]})_k \equiv 0$ and identity (6.13) yield

$$\sum_{k=c}^{d} h_k^{[1]*} \Psi_k \varphi_k^{[i]} = \sum_{k=c}^{d} \Big\{ \mathcal{L}^{*}(z^{[1]})_k \varphi_k^{[i]} - z_k^{[1]*} \mathcal{L}(\varphi^{[i]})_k \Big\} = (z^{[1]},\varphi^{[i]})_k \Big|_c^{d+1} = (z^{[1]},\varphi^{[i]})_{d+1}. \tag{6.30}$$

Upon combining (6.29) and (6.30) we obtain $z^{[1]}_{d+1} = \eta$, which means that $z^{[1]}$ solves the boundary value problem

$$\mathscr{L}(z^{[1]})_k = \Psi_k h^{[1]}_k, \quad z^{[1]}_c = 0, \quad z^{[1]}_{d+1} = \eta, \quad k \in \widehat{\mathcal{I}}_{\mathbb{Z}}.$$

Similarly, the nonhomogeneous system of algebraic equations $\omega^* A = \xi^* \mathcal{J}\, \Theta_c$ has a unique solution $\omega = (\omega_1, \dots, \omega_{2n})^\top \in \mathbb{C}^{2n}$. Then with $h^{[2]}_k := \Theta_k \omega$, $k \in \widehat{\mathcal{I}}_{\mathbb{Z}}$, we can calculate that $z^{[2]} \in \mathbb{C}(\widehat{\mathcal{I}}^+_{\mathbb{Z}})^{2n}$, being the unique solution of

$$\mathscr{L}(z^{[2]})_k = -\Psi_k h^{[2]}_k, \quad z^{[2]}_{d+1} = 0, \quad k \in \widehat{\mathcal{I}}_{\mathbb{Z}},$$

also satisfies $z^{[2]}_c = \xi$, i.e., it solves the boundary value problem

$$\mathscr{L}(z^{[2]})_k = -\Psi_k h^{[2]}_k, \quad z^{[2]}_c = \xi, \quad z^{[2]}_{d+1} = 0, \quad k \in \widehat{\mathcal{I}}_{\mathbb{Z}}.$$

Therefore, the sequence $z \in \mathbb{C}(\widehat{\mathcal{I}}^+_{\mathbb{Z}})^{2n}$ with the terms $z_k := z^{[1]}_k + z^{[2]}_k$ for all $k \in \widehat{\mathcal{I}}^+_{\mathbb{Z}}$ solves the boundary value problem (6.27) with $f_k := h^{[1]}_k - h^{[2]}_k$ for $k \in \widehat{\mathcal{I}}_{\mathbb{Z}}$, i.e., $f \in \mathbb{C}(\widehat{\mathcal{I}}_{\mathbb{Z}})^{2n}$. ∎

Finally, we derive yet another characterization of the definiteness of system ($\mathcal{S}_\lambda$), analogous to that given for linear Hamiltonian systems (2.5) and (2.6) with $E_k \equiv 0$ in [116, Sections 2.3 and 2.4] and [134, Sections 3 and 4], respectively.

For any $\lambda \in \mathbb{C}$ and nonempty finite discrete interval $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ with $k_0 \in \widehat{\mathcal{I}}^+_{\mathbb{Z}}$ we define the $2n \times 2n$ positive semidefinite matrix

$$\vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}}) := \sum_{k \in \widehat{\mathcal{I}}} \Theta^*_k(\lambda)\, \Psi_k\, \Theta_k(\lambda) \tag{6.31}$$

in terms of the fundamental matrix $\Theta(\lambda)$ of system ($\mathcal{S}_\lambda$) specified in the paragraph preceding Lemma 6.2.6. While the matrix $\vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ depends obviously on λ and $\widehat{\mathcal{I}}_{\mathbb{Z}}$, we next establish that its kernel and range do not, which implies also the independence of the rank of $\vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ on the value of λ.

Lemma 6.2.9. *For any nonempty finite discrete interval $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ such that $k_0 \in \widehat{\mathcal{I}}^+_{\mathbb{Z}}$, the subspaces* $\operatorname{Ker} \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ *and* $\operatorname{Ran} \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ *are independent of $\lambda \in \mathbb{C}$.*

Proof. Let $\lambda \in \mathbb{C}$ and $\xi \in \operatorname{Ker} \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ be fixed and put $z_k := \Theta_k(\lambda)\, \xi$ for all $k \in \widehat{\mathcal{I}}^+_{\mathbb{Z}}$. Then obviously $z \in \mathbb{C}(\widehat{\mathcal{I}}^+_{\mathbb{Z}})^{2n}$ solves system ($\mathcal{S}_\lambda$) on $\widehat{\mathcal{I}}_{\mathbb{Z}}$, i.e., $\mathscr{L}(z) = \lambda\, \Psi z$ on $\widehat{\mathcal{I}}_{\mathbb{Z}}$, while satisfying the initial condition $z_{k_0} = \xi$. Simultaneously, the sequence z solves also system ($\mathcal{S}_\nu$) on $\widehat{\mathcal{I}}_{\mathbb{Z}}$ for any $\nu \in \mathbb{C}$, i.e., $\mathscr{L}(z) = \nu\, \Psi z$ on $\widehat{\mathcal{I}}_{\mathbb{Z}}$, because the positive semidefiniteness of the elements in the sum on the right-hand side of (6.31) implies $\Psi_k z_k = \Psi_k\, \Theta_k(\lambda)\, \xi = 0$ for all $k \in \widehat{\mathcal{I}}_{\mathbb{Z}}$. Hence $z_k = \Theta_k(\nu)\, \zeta$ for all $k \in \widehat{\mathcal{I}}^+_{\mathbb{Z}}$ and some $\zeta \in \mathbb{C}^{2n}$. Since $\xi = z_{k_0} = \Theta_{k_0}(\nu)\, \zeta = \zeta$, it holds $z = \Theta(\lambda)\, \xi = \Theta(\nu)\, \xi$ on the discrete interval $\widehat{\mathcal{I}}^+_{\mathbb{Z}}$, which implies

$$0 = \xi^*\, \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})\, \xi = \sum_{k \in \widehat{\mathcal{I}}} z^*_k \Psi_k z_k = \xi^*\, \vartheta(\nu, \widehat{\mathcal{I}}_{\mathbb{Z}})\, \xi.$$

Therefore, $\operatorname{Ker} \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}}) \subseteq \operatorname{Ker} \vartheta(\nu, \widehat{\mathcal{I}}_{\mathbb{Z}})$ and by reversing the roles of the numbers λ and ν we obtain $\operatorname{Ker} \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}}) = \operatorname{Ker} \vartheta(\nu, \widehat{\mathcal{I}}_{\mathbb{Z}})$ for all $\lambda, \nu \in \mathbb{C}$. The independence of $\operatorname{Ran} \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ on $\lambda \in \mathbb{C}$ follows from the previous part and the fact that, as defined in (6.31), the matrix $\vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ is Hermitian, which yields $\operatorname{Ran} \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}}) = \operatorname{Ker} \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})^\perp$ by (1.6). ∎

The latter statement justifies the suppression of the parameter λ in the notation of the kernel, range, and rank of $\vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ for any nonempty finite discrete interval $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ with $k_0 \in \widehat{\mathcal{I}}_{\mathbb{Z}}^+$, i.e., henceforward we write only $\operatorname{Ker} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$, $\operatorname{Ran} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$, and $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$. In the following theorem we show that there exists a nonempty finite discrete interval $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$, which maximizes the value of $\operatorname{rank} \vartheta(\cdot)$.

Lemma 6.2.10. *There exists a nonempty finite discrete interval $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ with $k_0 \in \widehat{\mathcal{I}}_{\mathbb{Z}}^+$ such that for any finite discrete interval $\widetilde{\mathcal{I}}_{\mathbb{Z}}$ satisfying $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \widetilde{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ we have*

$$\operatorname{rank} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}}) = \operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}), \qquad \operatorname{Ran} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}}) = \operatorname{Ran} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}). \tag{6.32}$$

Proof. If the discrete interval $\mathcal{I}_{\mathbb{Z}}$ is finite, the statement is trivial. Hence, let us consider the case $\mathcal{I}_{\mathbb{Z}} = [0, \infty)_{\mathbb{Z}}$. For finite discrete intervals $\widehat{\mathcal{I}}_{\mathbb{Z}}$ and $\widetilde{\mathcal{I}}_{\mathbb{Z}}$ such that $k_0 \in \widehat{\mathcal{I}}_{\mathbb{Z}}^+ \subseteq \widetilde{\mathcal{I}}_{\mathbb{Z}}^+ \subset [0, \infty)_{\mathbb{Z}}$, we see that $\operatorname{Ker} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}}) \subseteq \operatorname{Ker} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$ by the definition of $\vartheta(\cdot)$ in (6.31). Then, given that the matrix $\vartheta(\cdot)$ is Hermitian for any finite discrete interval, we see that

$$\operatorname{Ran} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}) \overset{(1.6)}{=} [\operatorname{Ker} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})]^{\perp} \subseteq [\operatorname{Ker} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}})]^{\perp} \overset{(1.6)}{=} \operatorname{Ran} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}}),$$

and consequently $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}) \leq \operatorname{rank} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}})$ by (1.1), i.e., the value of $\operatorname{rank} \vartheta(\cdot)$ does not decrease when we extend the interval. Since at the same time $\operatorname{rank} \vartheta(\cdot) \leq 2n$, there must be a finite discrete interval $\widehat{\mathcal{I}}_{\mathbb{Z}}$ such that $\operatorname{rank} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}}) = \operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$, and thus also $\operatorname{Ran} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}}) = \operatorname{Ran} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$, for all finite discrete intervals $\widetilde{\mathcal{I}}_{\mathbb{Z}}$ containing $\widehat{\mathcal{I}}_{\mathbb{Z}}$. ∎

Finally, we describe a connection between the definiteness of system ($\mathcal{S}_\lambda$) and the matrix $\vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ for a finite discrete interval $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$.

Theorem 6.2.11. *For a nonempty finite discrete interval $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ with $k_0 \in \widehat{\mathcal{I}}_{\mathbb{Z}}^+$ and for $\vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ being defined as in (6.31) the following statements are equivalent.*

(i) *It holds* $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}) = 2n$.

(ii) *It holds* $\operatorname{Ker} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}) = \{0\}$.

(iii) *For some $\lambda \in \mathbb{C}$, every nontrivial solution $z(\lambda) \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$ of system ($\mathcal{S}_\lambda$) satisfies $\sum_{k \in \widehat{\mathcal{I}}} z_k^*(\lambda)\, \Psi_k\, z_k(\lambda) > 0$.*

(iv) *For some $\lambda \in \mathbb{C}$, a solution $z(\lambda) \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$ of system ($\mathcal{S}_\lambda$) is necessarily trivial, i.e. $z_k(\lambda) = 0$ for all $k \in \mathcal{I}_{\mathbb{Z}}^+$, when $\sum_{k \in \widehat{\mathcal{I}}} z_k^*(\lambda)\, \Psi_k\, z_k(\lambda) = 0$.*

Proof. The equivalence of (i) and (ii) is clear, while the equivalence of (iii) and (iv) follows from Remark 6.2.2 and Lemma 6.2.3. Hence it remains to show that the statements in (ii) and (iii) are equivalent.

Let us assume that (ii) is true and $z(\lambda) \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$ be an arbitrary nontrivial solution of system ($\mathcal{S}_\lambda$) for some $\lambda \in \mathbb{C}$, i.e., $z_k(\lambda) = \Theta_k(\lambda)\, \xi$ for some $\xi \in \mathbb{C}^{2n} \setminus \{0\}$ and all $k \in \mathcal{I}_{\mathbb{Z}}^+$. Since $\vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})$ is positive definite and $\xi \neq 0$, we obtain

$$0 < \xi^* \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})\, \xi = \sum_{k \in \widehat{\mathcal{I}}} \xi^*\, \Theta_k^*(\lambda)\, \Psi_k\, \Theta_k(\lambda)\, \xi = \sum_{k \in \widehat{\mathcal{I}}} z_k^*(\lambda)\, \Psi_k\, z_k(\lambda),$$

i.e., (iii) holds. Conversely, assume that the statement in (iii) is true. Let $\xi \in \mathbb{C}^{2n} \setminus \{0\}$ be fixed and put $z_k(\lambda) := \Theta_k(\lambda)\, \xi$ for all $k \in \mathcal{I}_{\mathbb{Z}}^+$. Then $z(\lambda) \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$ is a nontrivial solution of system ($\mathcal{S}_\lambda$) and, by (iii),

$$\xi^* \vartheta(\lambda, \widehat{\mathcal{I}}_{\mathbb{Z}})\, \xi = \sum_{k \in \widehat{\mathcal{I}}} z_k^*(\lambda)\, \Psi_k\, z_k(\lambda) > 0,$$

i.e., we have $\vartheta(\lambda,\widetilde{\mathcal{I}}_{\mathbb{Z}})\xi \neq 0$. Since the vector ξ was chosen arbitrarily, we conclude that $\operatorname{Ker}\vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}}) = \{0\}$, i.e., (ii) is satisfied, which completes the proof. ∎

As an immediate consequence of Lemma 6.2.6 and Theorem 6.2.11 we get the following corollary; cf. [116, Definition 2.14].

Corollary 6.2.12. *System ($\mathcal{S}_\lambda$) is definite on $\mathcal{I}_{\mathbb{Z}}$ if and only if, for some nonempty finite discrete interval $\widetilde{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$, one of the conditions listed in Theorem 6.2.11 is satisfied.*

6.3 Nonhomogeneous problem

Now we take the nonhomogeneous problem into consideration and extend the results of [A4, Section 5] to the case of general linear dependence on λ. For this purpose we naturally consider only the case $\mathcal{I}_{\mathbb{Z}} = [0,\infty)_{\mathbb{Z}}$ and we start this section by restating some fundamental results from the Weyl–Titchmarsh theory for system ($\mathcal{S}_\lambda$), which are related to the present study of system ($\mathcal{S}^f_\lambda$). As discussed at the beginning of this chapter, see (6.3), these results can be easily derived from Chapter 2 by appropriate changes in the definition of the semi-inner product and its weight matrix.

Throughout this section we assume that system ($\mathcal{S}_\lambda$) is definite on $[0,\infty)_{\mathbb{Z}}$ and "fix" the fundamental matrix $\Theta(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times 2n}$ by the initial condition $\Theta_0(\lambda) = (\alpha^*,\ -\mathcal{J}\alpha^*)$ for a given $\alpha \in \Gamma$, see (2.19) and (2.21). Since $\alpha \in \Gamma$, this fundamental matrix satisfies equality (6.15) with $s = 0$, and thus all the relations in (6.16) hold. We also denote by $\mathcal{Z}(\lambda), \widetilde{\mathcal{Z}}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ the two components of the fundamental matrix, i.e., we put $\Theta(\lambda) := (\mathcal{Z}(\lambda),\ \widetilde{\mathcal{Z}}(\lambda))$. If $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then the associated *Weyl disks* are defined in the same way as in Definition 2.3.1 with the $\mathcal{E}(M)$-function replaced by $\mathscr{E}(M) := i\delta(\lambda)\mathcal{X}^*(\lambda)\mathcal{J}\mathcal{X}(\lambda)$, where $M \in \mathbb{C}^{n\times n}$ and

$$\mathcal{X}_k(\lambda) := \Theta_k(\lambda,\alpha)(I,\ M^*)^*, \quad k \in [0,\infty)_{\mathbb{Z}}, \tag{6.33}$$

represents the *Weyl solution* of system ($\mathcal{S}_\lambda$); cf. identity (2.32) and Definition 2.2.1. Since system ($\mathcal{S}_\lambda$) is assumed to be definite on the discrete interval $[0,\infty)_{\mathbb{Z}}$, the limiting Weyl disk exists and it is a closed, convex, and nonempty subset of $\mathbb{C}^{n\times n}$; cf. Definition 2.3.10 and Remark 6.2.7. Hence the columns of the Weyl solution $\mathcal{X}(\lambda)$ defined through any matrix M from the limiting Weyl disk are linearly independent square summable solutions of system($\mathcal{S}_\lambda$), i.e., they belong to ℓ^2_Ψ; cf. Theorem 2.4.1. Consequently we adopt the terminology from Definition 2.4.2, i.e., system ($\mathcal{S}_\lambda$) is said to be in the *limit point case* and in the *limit circle case* if it possesses n and $2n$ linearly independent solutions in ℓ^2_Ψ, respectively. Finally, by $M_+(\lambda)$ we denote the *half-line Weyl–Titchmarsh $M(\lambda)$-function*, which is defined in accordance with Remark 2.3.17(i) and satisfies

$$M^*_+(\lambda) = M_+(\bar{\lambda}) \quad \text{for all } \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{6.34}$$

Moreover, if $\lambda,\nu \in \mathbb{C}\setminus\mathbb{R}$ and systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$), are both in the limit point or in the limit circle case, then

$$\lim_{k\to\infty} \mathcal{X}^{+*}_k(\lambda)\mathcal{J}\mathcal{X}^+_k(\nu) = 0, \tag{6.35}$$

where $\mathcal{X}^+(\lambda), \mathcal{X}^+(\nu) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n\times n}$ represent the Weyl solutions of systems ($\mathcal{S}_\lambda$) and ($\mathcal{S}_\nu$) defined by (6.33) through the matrices $M_+(\lambda)$ and $M_+(\nu)$, respectively; cf. Theorem 2.4.12. For simplicity, we put $\mathcal{X}^{+*}_k(\lambda) := [\mathcal{X}^+_k(\lambda)]^*$. The next result shows a useful relation between the Weyl solution $\mathcal{X}^+(\lambda)$ and $\widetilde{\mathcal{Z}}(\lambda)$.

Lemma 6.3.1. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and system ($\mathcal{S}_\lambda$) be definite on $[0,\infty)_\mathbb{Z}$. Then*

$$\mathcal{X}_k^+(\lambda)\,\widetilde{\mathcal{Z}}_k^*(\bar\lambda) - \widetilde{\mathcal{Z}}_k(\lambda)\mathcal{X}_k^{+*}(\bar\lambda) = \mathcal{J} \quad \text{for all } \ k \in [0,\infty)_\mathbb{Z}. \tag{6.36}$$

Proof. Identity (6.36) then follows by a direct calculation from the definition of $\mathcal{X}^+(\cdot)$, the third identity in (6.16), and equality (6.34). ∎

For $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and $k,s \in [0,\infty)_\mathbb{Z}$ we introduce the *Green function*

$$G_{k,s}(\lambda) := \begin{cases} \widetilde{\mathcal{Z}}_k(\lambda)\mathcal{X}_s^{+*}(\bar\lambda), & k \in [0,s]_\mathbb{Z}, \\ \mathcal{X}_k^+(\lambda)\,\widetilde{\mathcal{Z}}_s^*(\bar\lambda), & k \in [s+1,\infty)_\mathbb{Z}, \end{cases} \tag{6.37}$$

which can be equivalently expressed as

$$G_{k,s}(\lambda) = \begin{cases} \mathcal{X}_k^+(\lambda)\,\widetilde{\mathcal{Z}}_s^*(\bar\lambda), & s \in [0,k-1]_\mathbb{Z}, \\ \widetilde{\mathcal{Z}}_k(\lambda)\mathcal{X}_s^{+*}(\bar\lambda), & s \in [k,\infty)_\mathbb{Z}. \end{cases} \tag{6.38}$$

Let us note that in the literature we also find the terminology *resolvent kernel* for an analogous function in the continuous time case, see e.g. [107, page 15].

In the following lemma, we establish some fundamental properties of the Green function. We note that the given identities are presented in a more symmetric form, with respect to the variables k and s, than the corresponding identities for the Green function in the case of the system ($\mathbb{S}_\lambda$) with the special linear dependence on the spectral parameter given in [A4, Lemma 5.1].

Lemma 6.3.2. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and system ($\mathcal{S}_\lambda$) be definite on $[0,\infty)_\mathbb{Z}$. Then the Green function $G(\lambda) \in \mathbb{C}([0,\infty)_\mathbb{Z})^{2n\times 2n}$ possesses the following properties:*

(i) $G_{k,s}^*(\lambda) = G_{s,k}(\bar\lambda)$ *for all $k,s \in [0,\infty)_\mathbb{Z}$ such that $k \neq s$;*

(ii) $G_{k,k}^*(\lambda) = G_{k,k}(\bar\lambda) + \mathcal{J}$ *for all $k \in [0,\infty)_\mathbb{Z}$;*

(iii) *for any given $s \in [0,\infty)_\mathbb{Z}$ the function $G_{\cdot,s}(\lambda)$ satisfies the homogeneous system ($\mathcal{S}_\lambda$) for all $k \in [0,\infty)_\mathbb{Z}$ such that $k \neq s$, i.e., $G_{k,s}(\lambda) = \mathbb{S}_k(\lambda)\,G_{k+1,s}(\lambda)$ on the set*

$$\Big\{\{k,s\} \in [0,\infty)_\mathbb{Z} \times [0,\infty)_\mathbb{Z} \mid k \neq s\Big\};$$

(iv) $G_{k,k}(\lambda) = \mathbb{S}_k(\lambda)\,G_{k+1,k}(\lambda) - \mathcal{J}$ *for every $k \in [0,\infty)_\mathbb{Z}$;*

(v) *the columns of $G_{\cdot,s}(\lambda)$ belong to ℓ^2_Ψ for every $s \in [0,\infty)_\mathbb{Z}$ and the columns of $G_{k,\cdot}(\lambda)$ belong to ℓ^2_Ψ for every $k \in [0,\infty)_\mathbb{Z}$.*

Proof. The first property follows directly from the definition of $G_{k,s}(\lambda)$ given in (6.37). The second property can be obtained from (6.37) by means of identity (6.36). For the proof of the third property we distinguish two cases for the calculation of the value of $G_{k,s}(\lambda) - \mathbb{S}_k(\lambda)\,G_{k+1,s}(\lambda)$ and use the fact that $\mathcal{X}^+(\lambda)$ and $\widetilde{\mathcal{Z}}(\lambda)$ solve system ($\mathcal{S}_\lambda$), i.e.,

$$G_{k,s}(\lambda) - \mathbb{S}_k(\lambda)\,G_{k+1,s}(\lambda) = \begin{cases} [\mathcal{X}_k^+(\lambda) - \mathbb{S}_k(\lambda)\mathcal{X}_{k+1}^+(\lambda)]\,\widetilde{\mathcal{Z}}_s^*(\bar\lambda) = 0 & \text{when} \ \ s < k < k+1, \\ [\widetilde{\mathcal{Z}}_k(\lambda) - \mathbb{S}_k(\lambda)\,\widetilde{\mathcal{Z}}_{k+1}(\lambda)]\,\mathcal{X}_s^{+*}(\bar\lambda) = 0 & \text{when} \ \ s \geq k+1 > k. \end{cases}$$

Property (iv) can be proven by using the definition of $G(\lambda)$ in (6.37) together with identities (6.34) and (6.16). Finally, the columns of $G_{\cdot,s}(\lambda)$ belong to ℓ^2_Ψ for every $s \in [0,\infty)_\mathbb{Z}$

by the definition of the Green function and the square summability of the Weyl solution $\mathcal{X}^+(\lambda)$, because

$$\begin{aligned}\|G_{\cdot,s}(\lambda)e_j\|_\Psi^2 &= e_j^* \mathcal{X}_s^+(\bar\lambda)\Big(\sum_{k=0}^{s} \widetilde{\mathcal{Z}}_k^*(\lambda)\Psi_k \widetilde{\mathcal{Z}}_k(\lambda)\Big)\mathcal{X}_s^{+*}(\bar\lambda)e_j \\ &\quad + e_j^* \widetilde{\mathcal{Z}}_s(\bar\lambda)\Big(\sum_{k=s+1}^{\infty} \mathcal{X}_k^{*+}(\lambda)\Psi_k \mathcal{X}_k^+(\lambda)\Big)\widetilde{\mathcal{Z}}_s^*(\bar\lambda)e_j < \infty,\end{aligned}$$

while the columns of $G_{k,\cdot}^*(\lambda)$ are in ℓ_Ψ^2 for every $k \in [0,\infty)_\mathbb{Z}$ by the similar calculation and the equivalent expression given in (6.38). ∎

Let us associate with system ($\mathcal{S}_\lambda^f$) the sequence $\varkappa(\lambda) \in \mathbb{C}([0,\infty)_\mathbb{Z})^{2n}$, where

$$\varkappa_k(\lambda) := \sum_{s=0}^{\infty} G_{k,s}(\lambda)\Psi_s f_s, \quad k \in [0,\infty)_\mathbb{Z}. \tag{6.39}$$

By (6.38), it can be written as

$$\varkappa_k(\lambda) = \mathcal{X}_k^+(\lambda)\sum_{s=0}^{k-1} \widetilde{\mathcal{Z}}_s^*(\bar\lambda)\Psi_s f_s + \widetilde{\mathcal{Z}}_k(\lambda)\sum_{s=k}^{\infty} \mathcal{X}_s^{+*}(\bar\lambda)\Psi_s f_s, \tag{6.40}$$

which shows that $\varkappa(\lambda)$ is well defined for all $f \in \ell_\Psi^2$ by the Cauchy–Schwarz inequality, because the columns of $\mathcal{X}^+(\bar\lambda)$ are square summable, see also Lemma 6.3.2(v). Similarly as in [A4, Theorem 5.2] we show that the above defined function $\varkappa(\lambda)$ represents a square summable solution of system ($\mathcal{S}_\lambda^f$) with $f \in \ell_\Psi^2$.

Theorem 6.3.3. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, $f \in \ell_\Psi^2$, and system ($\mathcal{S}_\lambda$) be definite on $[0,\infty)_\mathbb{Z}$. The sequence $\varkappa(\lambda)$ defined in (6.39) solves system ($\mathcal{S}_\lambda^f$) on $[0,\infty)_\mathbb{Z}$, satisfies the initial condition $\alpha\varkappa_0(\lambda) = 0$, is square summable, i.e., $\varkappa(\lambda) \in \ell_\Psi^2$, and it holds*

$$\|\varkappa(\lambda)\|_\Psi \le \frac{1}{|\operatorname{im}(\lambda)|}\|f\|_\Psi. \tag{6.41}$$

In addition, if system ($\mathcal{S}_\lambda$) is in the limit point or limit circle case for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then

$$\lim_{k\to\infty} \mathcal{X}_k^{+*}(\nu)\mathcal{J}\varkappa_k(\lambda) = 0 \quad \textit{for every} \;\; \nu \in \mathbb{C}\setminus\mathbb{R}. \tag{6.42}$$

Proof. The form of $\varkappa_k(\lambda)$ given in (6.40) together with the similar expression of $\varkappa_{k+1}(\lambda)$, the fact that $\mathcal{X}^+(\lambda)$ and $\widetilde{\mathcal{Z}}(\lambda)$ solve system ($\mathcal{S}_\lambda$), and identity (6.36) yield

$$\begin{aligned}\varkappa_k(\lambda) - \mathbb{S}_k(\lambda)\varkappa_{k+1}(\lambda) &= \mathcal{X}_k^+(\lambda)\sum_{s=0}^{k-1} \widetilde{\mathcal{Z}}_s^*(\bar\lambda)\Psi_s f_s + \widetilde{\mathcal{Z}}_k(\lambda)\sum_{s=k}^{\infty} \mathcal{X}_s^{+*}(\bar\lambda)\Psi_s f_s \\ &\quad - \mathbb{S}_k(\lambda)\mathcal{X}_{k+1}^+(\lambda)\sum_{s=0}^{k} \widetilde{\mathcal{Z}}_s^*(\bar\lambda)\Psi_s f_s - \mathbb{S}_k(\lambda)\widetilde{\mathcal{Z}}_{k+1}(\lambda)\sum_{s=k+1}^{\infty} \mathcal{X}_s^{+*}(\bar\lambda)\Psi_s f_s \\ &= -\Big[\mathcal{X}_k^+(\lambda)\widetilde{\mathcal{Z}}_k^*(\bar\lambda) - \widetilde{\mathcal{Z}}_k(\lambda)\mathcal{X}_k^{+*}(\bar\lambda)\Big]\Psi_k f_k = -\mathcal{J}\Psi_k f_k,\end{aligned}$$

i.e., the sequence $\varkappa(\lambda)$ solves system ($\mathcal{S}_\lambda^f$).

The fulfillment of the boundary condition follows by the simple calculation

$$\alpha \varkappa_0(\lambda) = \alpha \tilde{\mathcal{Z}}_0(\lambda) \sum_{s=0}^{\infty} \mathcal{X}_s^{+*}(\bar{\lambda})\, \Psi_s f_s = -\alpha \mathcal{J} \alpha^* \sum_{s=0}^{\infty} \mathcal{X}_s^{+*}(\bar{\lambda})\, \Psi_s f_s = 0,$$

because $\tilde{\mathcal{Z}}_0(\lambda) = -\mathcal{J}\alpha^*$ and $\alpha \in \Gamma$.

Next, we prove the estimate in (6.41) which together with the assumption $f \in \ell^2_\Psi$ will imply that $\varkappa(\lambda) \in \ell^2_\Psi$. For every $r \in [0,\infty)_{\mathbb{Z}}$ we define the function

$$f_k^{[r]} := \begin{cases} f_k, & k \in [0,r]_{\mathbb{Z}}, \\ 0, & k \in [r+1,\infty)_{\mathbb{Z}}, \end{cases}$$

and the function

$$\varkappa_k^{[r]}(\lambda) := \sum_{s=0}^{\infty} G_{k,s}(\lambda)\, \Psi_s f_s^{[r]} = \sum_{s=0}^{r} G_{k,s}(\lambda)\, \Psi_s f_s. \tag{6.43}$$

Then $\varkappa^{[r]}(\lambda)$ solves system $(\mathcal{S}_\lambda^f)$ with f replaced by $f^{[r]}$. Applying the extended Lagrange identity from Theorem 6.1.3, we obtain

$$\begin{aligned} \lim_{k\to\infty} \varkappa_{k+1}^{[r]*}(\lambda)\, \mathcal{J} \varkappa_{k+1}^{[r]}(\lambda) = \varkappa_0^{[r]*}(\lambda)\, \mathcal{J} \varkappa_0^{[r]}(\lambda) + (\bar{\lambda} - \lambda) \sum_{k=0}^{\infty} \varkappa_k^{[r]*}(\lambda)\, \Psi_k \varkappa_k^{[r]}(\lambda) \\ + \sum_{k=0}^{\infty} f_k^{[r]*}\, \Psi_k \varkappa_k^{[r]}(\lambda) - \sum_{k=0}^{\infty} \varkappa_k^{[r]*}(\lambda)\, \Psi_k f_k^{[r]}. \end{aligned} \tag{6.44}$$

Since $\tilde{\mathcal{Z}}_0(\lambda) = -\mathcal{J}\alpha^*$ and $\alpha \in \Gamma$, we see that

$$\varkappa_0^{[r]*}(\lambda)\, \mathcal{J} \varkappa_0^{[r]}(\lambda) = \left(\sum_{s=0}^{r} \mathcal{X}_s^{+*}(\bar{\lambda})\, \Psi_s f_s \right)^* \tilde{\mathcal{Z}}_0^*(\lambda)\, \mathcal{J} \tilde{\mathcal{Z}}_0(\lambda) \left(\sum_{s=0}^{r} \mathcal{X}_s^{+*}(\bar{\lambda})\, \Psi_s f_s \right) = 0. \tag{6.45}$$

For every $k \in [r+1,\infty)_{\mathbb{Z}}$ we can also write

$$\varkappa_k^{[r]}(\lambda) = \mathcal{X}_k^{+}(\lambda)\, g_r(\lambda), \quad \text{where } g_r(\lambda) := \sum_{s=0}^{r} \tilde{\mathcal{Z}}_s^*(\bar{\lambda})\, \Psi_s f_s, \tag{6.46}$$

which, together with the fact that $M_+(\lambda)$ belongs to the limiting Weyl disk, yields

$$\begin{aligned} \frac{1}{\bar{\lambda} - \lambda} \lim_{k\to\infty} \varkappa_{k+1}^{[r]*}(\lambda)\, \mathcal{J} \varkappa_{k+1}^{[r]}(\lambda) &= \frac{i\,\delta(\lambda)}{2\,|\operatorname{im}(\lambda)|}\, g_r^*(\lambda) \Big(\lim_{k\to\infty} \mathcal{X}_{k+1}^{+*}(\lambda)\, \mathcal{J} \mathcal{X}_{k+1}^{+}(\lambda) \Big) g_r(\lambda) \\ &= \frac{1}{2\,|\operatorname{im}(\lambda)|}\, g_r^*(\lambda) \Big(\lim_{k\to\infty} \mathcal{E}_{k+1}(M_+(\lambda)) \Big) g_r(\lambda) \le 0. \end{aligned} \tag{6.47}$$

By using identities (6.44), (6.45), and (6.47), the assumption $\lambda \neq \bar{\lambda}$, the Hermitian property of Ψ_k, and the Cauchy–Schwarz inequality we get

$$\begin{aligned} \|\varkappa^{[r]}(\lambda)\|_\Psi^2 = \sum_{k=0}^{\infty} \varkappa_k^{[r]*}(\lambda)\, \Psi_k \varkappa_k^{[r]}(\lambda) &\le \frac{1}{2i\, \operatorname{im}(\lambda)} \left(\sum_{k=0}^{r} f_k^{[r]*}\, \Psi_k \varkappa_k^{[r]}(\lambda) - \sum_{k=0}^{r} \varkappa_k^{[r]*}(\lambda)\, \Psi_k f_k^{[r]} \right) \\ &\le \frac{1}{|\operatorname{im}(\lambda)|} \left| \sum_{k=0}^{r} \varkappa_k^{[r]*}(\lambda)\, \Psi_k f_k^{[r]} \right| \\ &\le \frac{1}{|\operatorname{im}(\lambda)|} \left(\sum_{k=0}^{r} \varkappa_k^{[r]*}(\lambda)\, \Psi_k \varkappa_k^{[r]}(\lambda) \right)^{1/2} \left(\sum_{k=0}^{r} f_k^{[r]*}\, \Psi_k f_k^{[r]} \right)^{1/2} \\ &\le \frac{1}{|\operatorname{im}(\lambda)|}\, \|\varkappa^{[r]}(\lambda)\|_\Psi \times \|f^{[r]}\|_\Psi, \end{aligned}$$

thereby yielding the inequality

$$\|\varkappa^{[r]}(\lambda)\|_{\Psi} \leq \frac{1}{|\operatorname{im}(\lambda)|}\,\|f^{[r]}\|_{\Psi} \leq \frac{1}{|\operatorname{im}(\lambda)|}\,\|f\|_{\Psi}, \tag{6.48}$$

because $\varkappa^{[r]}(\lambda) \in \ell^2_{\Psi}$. Upon combining identities (6.39) and (6.43) for any $k, r \in [0,\infty)_{\mathbb{Z}}$ we easily calculate

$$\varkappa_k(\lambda) - \varkappa_k^{[r]}(\lambda) = \sum_{s=r+1}^{\infty} G_{k,s}(\lambda)\,\Psi_s\, f_s.$$

Now, let $t \in [0,r]_{\mathbb{Z}}$. Then from the definition of $G(\lambda)$ given in (6.37) we obtain for every $k \in [0,t]_{\mathbb{Z}}$ that

$$\varkappa_k(\lambda) - \varkappa_k^{[r]}(\lambda) = \widetilde{\mathscr{Z}}_k(\lambda) \sum_{s=r+1}^{\infty} \mathscr{X}_s^{+*}(\bar{\lambda})\,\Psi_s\, f_s. \tag{6.49}$$

Since the columns of the Weyl solution $\mathscr{X}^+(\bar{\lambda})$ and the function f belong to ℓ^2_{Ψ}, it follows that $\sum_{s=0}^{\infty} \mathscr{X}_s^{+*}(\bar{\lambda})\,\Psi_s\, f_s < \infty$ by the Cauchy–Schwarz inequality. Hence the right-hand side of (6.49) tends to zero as $r \to \infty$ for every $k \in [0,t]_{\mathbb{Z}}$, which shows that $\varkappa^{[r]}$ converges uniformly to $\varkappa(\lambda)$ on the interval $[0,t]_{\mathbb{Z}}$. Moreover, by (6.48), we have

$$\sum_{k=0}^{t} \varkappa_k^{[r]*}(\lambda)\,\Psi_k\,\varkappa_k^{[r]}(\lambda) \leq \|\varkappa^{[r]}(\lambda)\|^2_{\Psi} \leq \frac{1}{|\operatorname{im}(\lambda)|^2}\,\|f\|^2_{\Psi},$$

from which, as a consequence of the uniform convergence of $\varkappa^{[r]}(\lambda)$ for $r \to \infty$ on $[0,t]_{\mathbb{Z}}$, we see that

$$\sum_{k=0}^{t} \varkappa_k^{*}(\lambda)\,\Psi_k\,\varkappa_k(\lambda) \leq \frac{1}{|\operatorname{im}(\lambda)|^2}\,\|f\|^2_{\Psi}. \tag{6.50}$$

As identity (6.50) is satisfied for any $t \in [0,\infty)_{\mathbb{Z}}$, the desired estimate in (6.41) follows.

Finally, to establish the existence of the limit in (6.42), assume that system ($\mathscr{S}_\lambda$) is in the limit point case for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $\nu \in \mathbb{C}\backslash\mathbb{R}$ is arbitrary. From the extended Lagrange identity in Theorem 6.1.3, for any $k, r \in [0,\infty)_{\mathbb{Z}}$, we obtain

$$\Big[\mathscr{X}_j^{+*}(\nu)\,\mathcal{J}\,\varkappa_j^{[r]}(\lambda)\Big]_0^{k+1} = (\bar{\nu} - \lambda) \sum_{j=0}^{k} \mathscr{X}_j^{+*}(\nu)\,\Psi_j\,\varkappa_j^{[r]}(\lambda) - \sum_{j=0}^{k} \mathscr{X}_j^{+*}(\nu)\,\Psi_j\, f_j^{[r]}(\lambda). \tag{6.51}$$

If $r \in [0,\infty)_{\mathbb{Z}}$ and $k \in [r+1,\infty)_{\mathbb{Z}}$, then identities (6.35) and (6.46) imply

$$\lim_{k\to\infty} \mathscr{X}_{k+1}^{+*}(\nu)\,\mathcal{J}\,\varkappa_{k+1}^{[r]}(\lambda) \overset{(6.46)}{=} \lim_{k\to\infty} \mathscr{X}_{k+1}^{+*}(\nu)\,\mathcal{J}\,\mathscr{X}_{k+1}^{+}(\lambda)\, g_r(\lambda) \overset{(6.35)}{=} 0.$$

Hence upon taking in (6.51) the limit for $k \to \infty$ we get

$$\mathscr{X}_0^{+*}(\nu)\,\mathcal{J}\,\varkappa_0^{[r]}(\lambda) = (\lambda - \bar{\nu}) \sum_{j=0}^{\infty} \mathscr{X}_j^{+*}(\nu)\,\Psi_j\,\varkappa_j^{[r]}(\lambda) + \sum_{j=0}^{\infty} \mathscr{X}_j^{+*}(\nu)\,\Psi_j\, f_j^{[r]}(\lambda). \tag{6.52}$$

Since, by the previous part, $\varkappa^{[r]}(\lambda)$ converges uniformly on finite subintervals of $[0,\infty)_{\mathbb{Z}}$ to $\varkappa(\lambda)$ as $r \to \infty$, identity (6.52) yields

$$\mathscr{X}_0^{+*}(\nu)\,\mathcal{J}\,\varkappa_0(\lambda) = (\lambda - \bar{\nu}) \sum_{j=0}^{\infty} \mathscr{X}_j^{+*}(\nu)\,\Psi_j\,\varkappa_j(\lambda) + \sum_{j=0}^{\infty} \mathscr{X}_j^{+*}(\nu)\,\Psi_j\, f_j(\lambda). \tag{6.53}$$

Simultaneously, as in (6.51), we obtain from (6.11) for every $k \in [0,\infty)_{\mathbb{Z}}$ that

$$\left[\mathcal{X}_j^{+*}(\nu)\,\mathcal{J}\,\varkappa_j(\lambda)\right]_0^{k+1} = (\bar{\nu}-\lambda)\sum_{j=0}^{k}\mathcal{X}_j^{+*}(\nu)\,\Psi_j\,\varkappa_j(\lambda) - \sum_{j=0}^{k}\mathcal{X}_j^{+*}(\nu)\,\Psi_j\,f_j(\lambda). \tag{6.54}$$

Hence by letting $k\to\infty$ in (6.54) and using equality (6.53) the limit in (6.42) is established.

An argument, similar to that given above, can be used also in the limit circle case to show the existence of the limit in (6.42), because all solutions of system ($\mathcal{S}_\lambda$) are square summable in that case. However, an alternative and more direct method of the proof is available. It utilizes the fact that $\widetilde{\mathcal{Z}}(\bar{\lambda}) \in \ell_\Psi^2$ in the limit circle case, which is not true in the limit point case, q.v. the proof of Theorem 2.4.3. More specifically, by (6.40) we have for every $k\in[0,\infty)_{\mathbb{Z}}$ that

$$\mathcal{X}_k^{+*}(\nu)\,\mathcal{J}\,\varkappa_k(\lambda) = \mathcal{X}_k^{+*}(\nu)\,\mathcal{J}\,\mathcal{X}_k^{+}(\lambda)\sum_{s=0}^{k-1}\widetilde{\mathcal{Z}}_s^{*}(\bar{\lambda})\,\Psi_s\,f_s + \mathcal{X}_k^{+*}(\nu)\,\mathcal{J}\,\widetilde{\mathcal{Z}}_k(\lambda)\sum_{s=k}^{\infty}\mathcal{X}_s^{+*}(\bar{\lambda})\,\Psi_s\,f_s. \tag{6.55}$$

The limit in (6.42) follows by the fact that both the terms on the right-hand side of (6.55) tend to zero as $k\to\infty$. Indeed, the zero limit of the first term is a consequence of the relation in (6.35) together with the convergence of the sum for $k\to\infty$, which we get by the Cauchy–Schwarz inequality. The second term tends to zero because $\mathcal{X}^{+*}(\nu)\,\mathcal{J}\,\widetilde{\mathcal{Z}}(\lambda)$ is bounded by equality (6.11) and the sum converges to zero as $k\to\infty$. ■

In the last result of this section, we extend [A4, Corollary 5.3] to the case of general linear dependence on the spectral parameter.

Corollary 6.3.4. *Let $\alpha\in\Gamma$, $\lambda\in\mathbb{C}\setminus\mathbb{R}$, $f\in\ell_\Psi^2$, and $\xi\in\mathbb{C}^n$. Assume that system ($\mathcal{S}_\lambda$) is definite on the discrete interval $[0,\infty)_{\mathbb{Z}}$, and define*

$$\hat{\varkappa}_k(\lambda) := \mathcal{X}_k^{+}(\lambda)\,\xi + \varkappa_k(\lambda), \quad k\in[0,\infty)_{\mathbb{Z}}, \tag{6.56}$$

where $\varkappa_k(\lambda)$ is given in (6.39). Then the sequence $\hat{\varkappa}(\lambda)\in\mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ represents a square summable solution of system ($\mathcal{S}_\lambda^f$) satisfying the initial condition $\alpha\,\hat{\varkappa}_0(\lambda)=\xi$ and

$$\|\hat{\varkappa}(\lambda)\|_\Psi \le \frac{1}{|\operatorname{im}(\lambda)|}\|f\|_\Psi + \|\mathcal{X}^{+}(\lambda)\,\xi\|_\Psi. \tag{6.57}$$

If system ($\mathcal{S}_\lambda$) is in the limit point or in the limit circle case for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$, we have

$$\lim_{k\to\infty}\mathcal{X}_k^{+*}(\nu)\,\mathcal{J}\,\hat{\varkappa}_k(\lambda) = 0 \quad \text{for every} \quad \nu\in\mathbb{C}\setminus\mathbb{R}. \tag{6.58}$$

Moreover, in the limit point case the sequence $\hat{\varkappa}(\lambda)$ is the unique square summable solution of system ($\mathcal{S}_\lambda^f$) satisfying $\alpha\,\hat{\varkappa}_0(\lambda)=\xi$, while in the limit circle case $\hat{\varkappa}(\lambda)$ is the unique solution of ($\mathcal{S}_\lambda^f$) being in ℓ_Ψ^2 such that $\alpha\,\hat{\varkappa}_0(\lambda)=\xi$ and

$$\lim_{k\to\infty}\mathcal{X}_k^{+*}(\bar{\lambda})\,\mathcal{J}\,\hat{\varkappa}_k(\lambda) = 0. \tag{6.59}$$

Proof. Since $\mathcal{X}^{+}(\lambda)\,\xi$ solves system ($\mathcal{S}_\lambda$) and $\Theta_0(\lambda,\alpha)=(\alpha^*,\ -\mathcal{J}\alpha^*)$, it follows from Theorem 6.3.3 that the sequence $\hat{\varkappa}(\lambda)$ solves the nonhomogeneous system ($\mathcal{S}_\lambda^f$) and satisfies $\alpha\,\hat{\varkappa}_0(\lambda)=\alpha\,\mathcal{X}_0^{+}(\lambda)\,\xi=\xi$. The estimate in (6.57) follows directly from (6.56) and (6.41) by the triangle inequality. The limit in (6.58) follows in the limit point or in the limit circle case from (6.35), (6.42), and from the calculation

$$\lim_{k\to\infty} \mathscr{X}_k^{+*}(\nu)\mathcal{J}\hat{z}_k(\lambda) = \lim_{k\to\infty} \left\{\mathscr{X}_k^{+*}(\nu)\mathcal{J}\mathscr{X}_k^{+}(\lambda)\,\xi + \mathscr{X}_k^{+*}(\nu)\mathcal{J}\varkappa_k(\lambda)\right\} = 0.$$

Finally, we prove the uniqueness of the solution in the limit point and limit circle cases. Assume that $z^{[1]}(\lambda), z^{[2]}(\lambda) \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ are two square summable solutions of system ($\mathcal{S}_\lambda^f$) satisfying $\alpha z_0^{[1]}(\lambda) = \xi = \alpha z_0^{[2]}(\lambda)$. Then $z_k(\lambda) := z_k^{[1]}(\lambda) - z_k^{[2]}(\lambda)$, $k \in [0,\infty)_{\mathbb{Z}}$, represents a square summable solution of the homogeneous system ($\mathcal{S}_\lambda$) and it satisfies $\alpha z_0(\lambda) = 0$. Since $z(\lambda) = \Theta(\lambda,\alpha)\zeta$ for some $\zeta \in \mathbb{C}^{2n}$, the initial condition $\alpha z_0(\lambda) = 0$ implies that $z_k(\lambda) = \widetilde{\mathscr{Z}}_k(\lambda)\eta$ for some $\eta \in \mathbb{C}^n$. If system ($\mathcal{S}_\lambda$) is in the limit point case, we have $z(\lambda) \notin \ell_\Psi^2$ for $\eta \neq 0$, because the columns of $\widetilde{\mathscr{Z}}(\lambda)$ do not belong to ℓ_Ψ^2 in this case, see Theorem 2.4.3. Therefore $\eta = 0$ and the uniqueness follows. On the other hand, if ($\mathcal{S}_\lambda$) is in the limit circle case and both $z^{[1]}(\lambda)$ and $z^{[2]}(\lambda)$ satisfy also the limit relation given in (6.59), then we obtain from the previous part and the first identity in (6.16) that

$$0 = \lim_{k\to\infty} \mathscr{X}_k^{+*}(\bar\lambda)\mathcal{J}z_k(\lambda) = \lim_{k\to\infty} \mathscr{X}_k^{+*}(\bar\lambda)\mathcal{J}\Theta_k(\lambda)\begin{pmatrix}0\\ \eta\end{pmatrix} = \lim_{k\to\infty} \begin{pmatrix} I & M_+^*(\bar\lambda)\end{pmatrix}\Theta_k^*(\bar\lambda)\mathcal{J}\Theta_k(\lambda)\begin{pmatrix}0\\ \eta\end{pmatrix} = \eta,$$

which implies the uniqueness of the solution $\hat{z}(\lambda)$ also in the latter case. ■

6.4 Maximal and minimal linear relations

Finally, in this section we come to the topic of linear relations. We focus on a pair of linear relations defined in terms of the linear map, $\mathscr{L}(\cdot)$, introduced in (6.5) in association with system ($\mathcal{S}_\lambda$). Let us mention that similar results for linear Hamiltonian differential and difference systems can be found in [13, 116, 134]. Moreover, we remind that a short introduction to the theory of linear relations is available in the Appendix and we recommend to read this passage now if it is not familiar to the reader.

For the present treatment we need to introduce some spaces in addition to the space of square summable sequences ℓ_Ψ^2 defined in (6.9). Namely, we denote by $\tilde{\ell}_\Psi^2$ the quotient space obtained by factoring out the kernel of the semi-norm $\|\cdot\|_\Psi$, i.e.,

$$\tilde{\ell}_\Psi^2 := \ell_\Psi^2 \Big/ \left\{ z \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n} \mid \|z\|_\Psi = 0 \right\}. \tag{6.60}$$

It is easy to see that $\tilde{\ell}_\Psi^2$ is a Banach space with respect to the norm generated by the quotient space map $\pi(z) := \tilde{z}$ and simultaneously it is a Hilbert space with respect to the associated inner product $\langle \tilde{z}, \tilde{v}\rangle_\Psi := \langle z, v\rangle_\Psi$, where z and f are elements of the equivalence classes $\tilde{z} \in \tilde{\ell}_\Psi^2$ and $\tilde{f} \in \tilde{\ell}_\Psi^2$, respectively; cf. [142, Lemma 2.5]. Henceforward, we denote by the superscript $\tilde{\cdot}$ the corresponding equivalence class, i.e., if $z \in \ell_\Psi^2$ then $\tilde{z} \in \tilde{\ell}_\Psi^2$ is such that $z \in \tilde{z}$. In addition, we point out that the value of $\|z\|_\Psi := \sqrt{\langle z, z\rangle_\Psi}$ for $z \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$ does not depend on z_{N+1} in the case of $\mathcal{I}_{\mathbb{Z}}$ being a finite discrete interval, which implies that the sequences $z, v \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$ such that $z_k \neq v_k$ only for $k = N+1$, belong to the same equivalence class. In addition, the product space $\tilde{\ell}_\Psi^{2\times 2} := \tilde{\ell}_\Psi^2 \times \tilde{\ell}_\Psi^2$ is also a pre-symplectic space with the associated function $[\cdot:\cdot] : \tilde{\ell}_\Psi^{2\times 2} \times \tilde{\ell}_\Psi^{2\times 2} \to \mathbb{C}$ given by

$$[\{\tilde{z}, \tilde{f}\} : \{\tilde{w}, \tilde{g}\}] := \langle \tilde{f}, \tilde{w}\rangle_\Psi - \langle \tilde{z}, \tilde{g}\rangle_\Psi, \tag{6.61}$$

see (6.14) and the second part of the Appendix. Moreover, we define the subspace

$$\ell^2_{\psi,0} := \begin{cases} \left\{z \in \mathbb{C}_0(\mathcal{I}^+_{\mathbb{Z}})^{2n} \cap \ell^2_\psi \mid z_0 = 0,\ z_{N+1} = 0\right\} & \text{if } \mathcal{I}_{\mathbb{Z}} = [0,N]_{\mathbb{Z}},\, N \in \mathbb{N} \cup \{0\}, \\ \left\{z \in \mathbb{C}_0(\mathcal{I}^+_{\mathbb{Z}})^{2n} \cap \ell^2_\psi \mid z_0 = 0\right\} & \text{if } \mathcal{I}_{\mathbb{Z}} = [0,\infty)_{\mathbb{Z}}, \end{cases}$$

and if $\mathcal{I}_{\mathbb{Z}} = [0,\infty)_{\mathbb{Z}}$ also

$$\ell^2_{\psi,1} := \left\{z \in \ell^2_\psi \mid \text{there exists } K \in [0,\infty)_{\mathbb{Z}} \text{ such that } \psi_k z_k = 0 \text{ for all } k \in [K,\infty)_{\mathbb{Z}}\right\}.$$

Finally, the space $\tilde{\ell}^2_{\psi,1}$ consists of equivalence classes similarly as $\tilde{\ell}^2_\psi$ and it is easy to see that $\tilde{\ell}^2_{\psi,1}$ is a dense subspace of $\tilde{\ell}^2_\psi$.

6.4.1 Linear relations and definiteness

Now we can introduce the *maximal linear relation* $T_{\max}$ as a subspace of $\tilde{\ell}^{2\times 2}_\psi$ given by

$$T_{\max} := \left\{\{\tilde{z}, \tilde{f}\} \in \tilde{\ell}^{2\times 2}_\psi \mid \text{there exists } z \in \tilde{z} \text{ such that } \mathscr{L}(z) = \psi f\right\}. \tag{6.62}$$

Note that when $\mathscr{L}(z) = \psi f$, then $\mathscr{L}(z) = \psi g$ for any $g \in \tilde{f}$, i.e., the definition of $T_{\max}$ does not depend on the choice of the representative $g \in \tilde{f}$. Similarly, we define the *pre-minimal linear relation*

$$T_0 := \left\{\{\tilde{z}, \tilde{f}\} \in \tilde{\ell}^{2\times 2}_\psi \mid \text{there exists } z \in \tilde{z} \cap \ell^2_{\psi,0} \text{ such that } \mathscr{L}(z) = \psi f\right\}, \tag{6.63}$$

which evidently satisfies $T_0 \subseteq T_{\max}$. In addition, by (A.6) we put

$$T_0 - \lambda I := \left\{\{\tilde{z}, \tilde{f}\} \in \tilde{\ell}^{2\times 2}_\psi \mid \text{ there exists } z \in \tilde{z} \cap \ell^2_{\psi,0} \text{ such that } \mathscr{L}(z) = \lambda\psi z + \psi f\right\}. \tag{6.64}$$

The consideration of linear relations (instead of operators) in our current context is natural given that the weight ψ, being present on the right-hand side of system ($\mathcal{S}^f_\lambda$) and in the definitions of the sequence spaces associated with $T_{\max}$ and $T_{\min}$, has terms none of which are positive definite, but all of which are only positive semidefinite, see (6.7). Moreover, this fact is affirmed by the following simple example, which is analogous to that found in [116, Section 2] for system (2.5).

Example 6.4.1. Let $n = 1$ and consider system ($\mathcal{S}^f_0$) with

$$\mathcal{S}_k \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \psi_k \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f_k = \begin{pmatrix} f^{[1]}_k \\ f^{[2]}_k \end{pmatrix}, \quad \text{and} \quad z_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix}. \tag{6.65}$$

Then $\mathscr{L}(z)$ and ($\mathcal{S}^f_0$), respectively, can be written as

$$\mathscr{L}(z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Delta z \quad \text{and} \quad \Delta z = \begin{pmatrix} \Delta x \\ \Delta u \end{pmatrix} = \begin{pmatrix} 0 \\ -f^{[1]} \end{pmatrix}. \tag{6.66}$$

Hence, for any $f \in \ell^2_\psi$, the sequence $z \in \mathbb{C}(\mathcal{I}^+_{\mathbb{Z}})^2$ with the terms

$$z_0 = 0 \quad \text{and} \quad z_k = \begin{pmatrix} 0 \\ -\sum_{j=0}^{k-1} f^{[1]}_j \end{pmatrix} \quad \text{for all } k \in \mathcal{I}^+_{\mathbb{Z}} \setminus \{0\}$$

solves the nonhomogeneous system in (6.66), i.e., system ($\mathcal{S}_0^f$) with the coefficients specified in (6.65). Since obviously $z \in \tilde{0}$, we have $\{\tilde{0}, \tilde{f}\} \in T_{\max}$ for any $\tilde{f} \in \tilde{\ell}^2_{\Psi}$. Thus, the multivalued part of the corresponding linear relation $T_{\max}$, i.e., $\operatorname{mul} T_{\max}$ defined in (A.3), is nontrivial, which means that $T_{\max}$ is not a graph of a linear operator. Let us also note that a solution of system ($\mathcal{S}_0^f$) in (6.66) which is an element of the space $\ell^2_{\Psi,0}$, of necessity is such that $x_k = 0$ for all $k \in \mathcal{I}_{\mathbb{Z}}^+$ with the consequence that $\operatorname{dom} T_0 = \{\tilde{0}\}$, i.e., the set $\operatorname{dom} T_0$ is not dense in ℓ^2_{Ψ}. ▲

The latter example illustrates yet another interesting situation: for any $c \in \mathbb{C}$ the sequence $z_k(\lambda) \equiv (0, c)^\top \in \tilde{0}$ represents a solution of the nonhomogeneous system in (6.66) with $f \in \tilde{0}$, i.e., $f_k^{[1]} \equiv 0$. It means that for the pair $\{\tilde{0}, \tilde{0}\}$ there exists *infinitely* many representatives $z \in \tilde{0}$ such that $\mathscr{L}(z) = \Psi f = 0$ and, at the same time, it implies that system ($\mathcal{S}_\lambda$) with the coefficients given in (6.65) is not definite on the discrete interval $\mathcal{I}_{\mathbb{Z}}$ by Lemma 6.2.3. In the next result we characterize the definiteness of system ($\mathcal{S}_\lambda$) in terms of the domain of $T_{\max}$.

Theorem 6.4.2. *System ($\mathcal{S}_\lambda$) is definite on the discrete interval $\mathcal{I}_{\mathbb{Z}}$ if and only if for any pair $\{\tilde{z}, \tilde{f}\} \in T_{\max}$ there exists a unique $z \in \tilde{z}$ such that $\mathscr{L}(z) = \Psi f$.*

Proof. Assume that system ($\mathcal{S}_\lambda$) is definite on $\mathcal{I}_{\mathbb{Z}}$. Let $\{\tilde{z}, \tilde{f}\} \in T_{\max}$ and $z^{[1]}, z^{[2]} \in \tilde{z}$ be two representatives of the equivalence class $\tilde{z}$ such that $\mathscr{L}(z^{[1]}) = \Psi f = \mathscr{L}(z^{[2]})$. If we put $v_k := z_k^{[1]} - z_k^{[2]}$ for all $k \in \mathcal{I}_{\mathbb{Z}}^+$, then $\mathscr{L}(v) = 0$ and $v \in \tilde{0} \in \tilde{\ell}^2_{\Psi}$, i.e., v solves system ($\mathcal{S}_0$) and satisfies $\sum_{k \in \mathcal{I}} v_k^* \Psi_k v_k = 0$. Therefore, by Remark 6.2.2, the definiteness of system ($\mathcal{S}_\lambda$) implies that $v_k \equiv 0$ on $\mathcal{I}_{\mathbb{Z}}^+$, i.e., $z^{[1]} \equiv z^{[2]}$ on $\mathcal{I}_{\mathbb{Z}}^+$ and so there exists only one representative $z \in \tilde{z}$ such that $\mathscr{L}(z) = \Psi f$.

To show the converse, assume that there is only one $z \in \tilde{z}$ for which $\mathscr{L}(z) = \Psi f$, whenever $\{\tilde{z}, \tilde{f}\} \in T_{\max}$. Let $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ be a finite discrete interval such that $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$ is maximal; cf. Lemma 6.2.10. If $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}) < 2n$, then there is a vector $\eta \in \mathbb{C}^{2n} \setminus \{0\}$ such that $\eta^* \vartheta(0, \widehat{\mathcal{I}}_{\mathbb{Z}}) \eta = \sum_{k \in \widehat{\mathcal{I}}} \eta^* \Theta_k^* \Psi_k \Theta_k \eta = 0$, where $\Theta_k := \Theta_k(0)$ is the fundamental matrix of system ($\mathcal{S}_\lambda$) specified in the paragraph preceding Lemma 6.2.6 with $k_0 \in \widehat{\mathcal{I}}_{\mathbb{Z}}^+$. If also $\sum_{k \in \mathcal{I}} \eta^* \Theta_k^* \Psi_k \Theta_k \eta = 0$, then $v := \Theta \eta \in \ell^2_{\Psi}$, $\mathscr{L}(v) = 0$, and $v \in \tilde{0}$. Given that the zero sequence is the unique representative of $\tilde{0}$ satisfying $\mathscr{L}(z) = 0$, it follows $v_k = \Theta_k \eta = 0$ for all $k \in \mathcal{I}_{\mathbb{Z}}^+$ and as a consequence $\eta = 0$, which contradicts the assumption $\eta \neq 0$. Hence, there is a finite discrete interval superset $\widetilde{\mathcal{I}}_{\mathbb{Z}} \supset \widehat{\mathcal{I}}_{\mathbb{Z}}$ such that $\vartheta(0, \widetilde{\mathcal{I}}_{\mathbb{Z}}) \eta \neq 0$. As a result, $\operatorname{Ker} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}}) \subset \operatorname{Ker} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$ and consequently $\operatorname{Ran} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}) \subset \operatorname{Ran} \vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}})$; thereby contradicting the maximality of $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$. Thus, $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}) = 2n$ and system ($\mathcal{S}_\lambda$) is definite on the discrete interval $\mathcal{I}_{\mathbb{Z}}$ by Theorem 6.2.11. ■

Naturally, the latter statement does not mean that the equivalence class $\tilde{z}$ contains only one representative as it can be easily seen, e.g., from the example discussed in Remark 6.4.4 below. Nevertheless, if system ($\mathcal{S}_\lambda$) is definite on the discrete interval $\mathcal{I}_{\mathbb{Z}}$ and $\{\tilde{z}, \tilde{f}\} \in T_{\max}$, then we denote the unique representative $z \in \tilde{z}$ satisfying $\mathscr{L}(z) = \Psi f$ by $\hat{z}$, i.e., $\hat{z} := z$. In this case, identities (6.14) and (6.61) yield

$$[\{\tilde{z}, \tilde{f}\} : \{\tilde{v}, \tilde{g}\}] = \langle f, \hat{v} \rangle_{\Psi} - \langle \hat{z}, g \rangle_{\Psi} = (\hat{z}, \hat{v})_k \Big|_0^{N+1} \tag{6.67}$$

for any $\{\tilde{z}, \tilde{f}\}, \{\tilde{v}, \tilde{g}\} \in T_{\max}$, where $f \in \tilde{f}$ and $g \in \tilde{g}$ are arbitrary representatives. Moreover, we obtain from Lemma 6.2.8 the following statement, compare with [135, Remark 3.2] and [147, Lemma 3.3].

Lemma 6.4.3. *Let system* ($\mathcal{S}_\lambda$) *be definite on a finite discrete interval* $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}} := [a,b]_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$*, where* $a, b \in \mathcal{I}_{\mathbb{Z}}$*. Then for any pairs* $\{\tilde{z}, \tilde{f}\}, \{\tilde{v}, \tilde{g}\} \in T_{\max}$ *there exists* $\{\tilde{r}, \tilde{h}\} \in T_{\max}$ *such that*

$$\hat{r}_k = \begin{cases} \hat{z}_k, & k \in [0,c]_{\mathbb{Z}}, \\ \hat{v}_k, & k \in [d+1,\infty)_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}^+, \end{cases}$$

where $c \in [0,a]_{\mathbb{Z}}$ *and* $d \in [b,\infty)_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}$.

In particular, there exists $\{\tilde{z}^{[i]}, \tilde{f}^{[i]}\} \in T_{\max}$ *such that* $\hat{z}_0^{[i]} = e_i$ *and* $\hat{z}_k^{[i]} = 0$ *for* $k \in [d+1,\infty)_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}^+$*, where* $i \in \{1,\dots,2n\}$ *is arbitrary and* $e_i = (0,\dots,1,\dots,0)^\top \in \mathbb{C}^{2n}$ *is the i-th canonical unit vector in* $\mathbb{C}^{2n}$*. If, in addition,* $\mathcal{I}_{\mathbb{Z}}$ *is a finite discrete interval, i.e.,* $\mathcal{I}_{\mathbb{Z}} = [0,N]_{\mathbb{Z}}$ *with* $N \in \mathbb{N} \cup \{0\}$*, then there exists* $\{\tilde{r}, \tilde{h}\} \in T_{\max}$ *such that* $\hat{r}_k = 0$ *for* $k \in [0,c]_{\mathbb{Z}}$ *and* $\hat{r}_{N+1} = e_i$*, where* $i \in \{1,\dots,2n\}$ *and* e_i *are the same as above.*

Proof. Let $\widehat{\mathcal{I}}_{\mathbb{Z}}$ be a finite discrete interval as in Lemma 6.2.8, the pairs $\{\tilde{z}, \tilde{f}\}, \{\tilde{v}, \tilde{g}\} \in T_{\max}$ be arbitrary, and define $\xi := \hat{z}_c$, $\eta := \hat{v}_{d+1}$. Then by the latter lemma there exist sequences $\ell \in \mathbb{C}(\widehat{\mathcal{I}}_{\mathbb{Z}})^{2n}$ and $s \in \mathbb{C}(\widehat{\mathcal{I}}_{\mathbb{Z}}^+)^{2n}$ such that

$$\mathscr{L}(s)_k = \psi_k \ell_k, \quad s_c = \xi, \quad s_{d+1} = \eta, \quad k \in \widehat{\mathcal{I}}_{\mathbb{Z}}.$$

If we put

$$r_k := \begin{cases} \hat{z}_k, & k \in [0,c]_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}, \\ s_k, & k \in [c+1,d]_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}, \\ \hat{v}_k, & k \in [d+1,\infty)_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}^+ \end{cases} \qquad h_k := \begin{cases} f_k, & k \in [0,c-1]_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}, \\ \ell_k, & k \in [c,d]_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}, \\ g_k, & k \in [d+1,\infty)_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}, \end{cases}$$

then obviously $r, h \in \ell_\psi^2$ and it can be verified by a direct calculation that $\mathscr{L}(r)_k = \psi_k h_k$ for all $k \in \mathcal{I}_{\mathbb{Z}}$, i.e., $\{\tilde{r}, \tilde{h}\} \in T_{\max}$ with $\hat{r}_k \equiv r_k$. The second part of the statement follows directly from Lemma 6.2.8. ∎

Remark 6.4.4. In Example 6.4.1 we constructed the linear map $\mathscr{L}$ and the nonhomogeneous system, see (6.66), with two significant properties: (i) the maximal linear relation does not determine a linear operator and (ii) the domain of the pre-minimal linear relation is not dense in $\tilde{\ell}_\psi^2$. In addition, let us consider system (6.22) from Example 6.2.4 with the coefficients $p_k \equiv 1$, $q_k \equiv 0$, and $w_k \equiv 1$ on $\mathcal{I}_{\mathbb{Z}}$ with $N \neq 0$, which imply the definiteness of system (6.22) on $\mathcal{I}_{\mathbb{Z}}$. If we put $f_k = \left(f_k^{[1]}, f_k^{[2]}\right)^\top$ with $f_0 = (1,0)^\top$ and $f_k = (0,0)^\top$ for all $k \in \mathcal{I}_{\mathbb{Z}} \setminus \{0\}$, then the corresponding nonhomogeneous system $z_k = \mathcal{S}_k z_{k+1} - \mathcal{J} \psi_k f_k$, i.e.,

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} + \begin{pmatrix} 0 \\ f_k^{[1]} \end{pmatrix},$$

possesses the solution $z_k = (x_k, u_k)^\top$, where $z_0 = (0,1)^\top$ and $z_k \equiv (0,0)^\top$ on $\mathcal{I}_{\mathbb{Z}}^+ \setminus \{0\}$, i.e., there exists $\tilde{f} \neq \tilde{0}$ such that $\{\tilde{0}, \tilde{f}\} \in T_{\max}$. This shows that even the definiteness of system ($\mathcal{S}_\lambda$) does not suffice to get $\operatorname{mul} T_{\max} = \{\tilde{0}\}$; cf. [147].

Thus, to guarantee that the maximal linear relation defines an operator, we need to assume explicitly that $\operatorname{mul} T_{\max} = \{\tilde{0}\}$, i.e., if there exists $z \in \tilde{0}$ such that $\mathscr{L}(z) = \psi f$ for some $f \in \ell_\psi^2$, then $z \equiv 0$; cf. [108, pg. 666]. In other words, we assume the "definiteness" of system ($\mathcal{S}_0^f$) for every $f \in \ell_\psi^2$. Then system ($\mathcal{S}_\lambda$) is definite as it follows by the choice $f \equiv 0$, see Theorem 6.2.11(iii) and Corollary 6.2.12. Moreover, in the next theorem we prove that this assumption even yields the density of $\operatorname{dom} T_0$ in $\tilde{\ell}_\psi^2$; cf. [108, Theorem 7.6]. As noted in [10, pg. 3], a similar condition is also needed for the study of operators associated with system (2.6) in [142].

Theorem 6.4.5. *If system* ($\mathcal{S}_0^f$) *is definite on* $\mathcal{I}_\mathbb{Z}$, *then* $\operatorname{dom} T_0$ *is dense in* $\tilde{\ell}^2_\psi$.

Proof. Assume that $\operatorname{dom} T_0$ is not dense in $\tilde{\ell}^2_\psi$, i.e., there exists $\tilde{f} \in (\operatorname{dom} T_0)^\perp$ such that $\|f\|_\psi \neq 0$. Let $\tilde{z} \in \operatorname{dom} T_0$ be such that $\mathscr{L}(z) = \psi f$ and $\tilde{v} \in \operatorname{dom} T_0$ be such that $\mathscr{L}(v) = \psi g$ for some $g \in \ell^2_\psi$. Then, by identity (6.14), we obtain

$$\langle z, g\rangle_\psi = \langle f, v\rangle_\psi = 0, \tag{6.68}$$

because $z, v \in \ell^2_{\psi,0}$. Since $g \in \ell^2_\psi$ was chosen arbitrarily and $z \in \ell^2_{\psi,0}$, we can take $g = z$ and the solution of $\mathscr{L}(v) = \psi z$ can be obtained as $v_k = \Theta_k \mathcal{J} \sum_{j=0}^{k-1} \Theta_j^* \psi_j z_j$, where $\Theta := \Theta(0)$ means a fundamental matrix of ($\mathcal{S}_0$) satisfying (6.15) for any $s \in \mathcal{I}_\mathbb{Z}^+$. Then equality (6.68) implies that $\langle z, z\rangle_\psi = 0$, i.e., $\psi_k z_k = 0$ on $\mathcal{I}_\mathbb{Z}$. Thus we have $\mathscr{L}(z) = \psi f$ and $z \in \tilde{0}$, which yields $z \equiv 0$ by the definiteness assumption for system ($\mathcal{S}_0^f$). So $\psi_k f_k = 0$ on $\mathcal{I}_\mathbb{Z}$, i.e., $f \in \tilde{0}$, and the density of $\operatorname{dom} T_0$ in $\tilde{\ell}^2_\psi$ is thus established. ∎

6.4.2 Orthogonal decomposition of sequence spaces

In this subsection we introduce a linear map which will allow orthogonal decompositions of $\tilde{\ell}^2_\psi$ and $\tilde{\ell}^2_{\psi,1}$ depending on the cardinality of the discrete interval $\mathcal{I}_\mathbb{Z}$. In particular, let us denote by $\mathcal{K}_\lambda$ the linear map defined by

$$\mathcal{K}_\lambda : \begin{cases} \tilde{\ell}^2_\psi \to \mathbb{C}^{2n} & \text{if } \mathcal{I}_\mathbb{Z} \text{ is a finite discrete interval,} \\ \tilde{\ell}^2_{\psi,1} \to \mathbb{C}^{2n} & \text{if } \mathcal{I}_\mathbb{Z} = [0,\infty)_\mathbb{Z}, \end{cases} \qquad \mathcal{K}_\lambda(\tilde{z}) := \sum_{k\in\mathcal{I}} \Theta_k^*(\bar{\lambda})\, \psi_k z_k, \tag{6.69}$$

where $\Theta(\lambda)$ is the fundamental matrix of system ($\mathcal{S}_\lambda$) specified in the paragraph preceding Lemma 6.2.6 with $k_0 \in \mathcal{I}_\mathbb{Z}^+$. If $\lambda = 0$ we write only $\mathcal{K}(\cdot)$ instead of $\mathcal{K}_0(\cdot)$. For completeness, we note that the sum $\sum_{k\in\mathcal{I}} \Theta_k^*(\bar{\lambda})\, \psi_k z_k$ does not depend on the choice of the representative $z \in \tilde{z} \in \tilde{\ell}^2_\psi$ or $z \in \tilde{z} \in \tilde{\ell}^2_{\psi,1}$, i.e., the map $\mathcal{K}_\lambda$ is defined correctly. In the following statement we utilize the matrix $\vartheta(\cdot)$ defined in (6.31).

Lemma 6.4.6. *Let* $\mathcal{I}_\mathbb{Z}$ *be a finite discrete interval and* $\lambda \in \mathbb{C}$. *Then* $\operatorname{Ran} \mathcal{K}_\lambda$ *is independent of* $\lambda \in \mathbb{C}$*; in particular,*

$$\operatorname{Ran} \mathcal{K}_\lambda = \left\{\xi \in \mathbb{C}^{2n} \mid \psi_k \Theta_k(\bar{\lambda})\, \xi = 0 \ \textit{for all } k \in \mathcal{I}_\mathbb{Z}\right\}^\perp = \operatorname{Ran} \vartheta(\mathcal{I}_\mathbb{Z}). \tag{6.70}$$

Furthermore, the space $\tilde{\ell}^2_\psi$ *admits the following orthogonal sum decomposition*

$$\tilde{\ell}^2_\psi = \operatorname{Ker} \mathcal{K}_\lambda \oplus \left\{\tilde{z} \in \tilde{\ell}^2_\psi \mid z = \Theta(\bar{\lambda})\, \xi,\ \xi \in \operatorname{Ran} \vartheta(\mathcal{I}_\mathbb{Z})\right\}. \tag{6.71}$$

Proof. For any $\xi \in \mathbb{C}^{2n}$ and $\tilde{v} \in \tilde{\ell}^2_\psi$, we have

$$\langle \xi, \mathcal{K}_\lambda(\tilde{v})\rangle_{\mathbb{C}^{2n}} = \sum_{k\in\mathcal{I}} \xi^*\, \Theta_k^*(\bar{\lambda})\, \psi_k v_k = \langle \Theta(\bar{\lambda})\, \xi, v\rangle_\psi.$$

Hence we have $\mathcal{K}_\lambda^* : \mathbb{C}^{2n} \to \tilde{\ell}^2_\psi$ with $\mathcal{K}_\lambda^*(\xi) := \tilde{z}$ for $\xi \in \mathbb{C}^{2n}$, where $\tilde{z}$ is the equivalence class corresponding to $z = \Theta(\bar{\lambda})\, \xi$. In particular,

$$\operatorname{Ran} \mathcal{K}_\lambda^* = \left\{\tilde{z} \in \tilde{\ell}^2_\psi \mid z = \Theta(\bar{\lambda})\, \xi \in \tilde{z},\ \xi \in \mathbb{C}^{2n}\right\}$$

and

$$\operatorname{Ker}\mathcal{K}_\lambda^* = \left\{\xi \in \mathbb{C}^{2n} \mid \|\Theta(\bar\lambda)\xi\|_\psi = 0\right\} = \left\{\xi \in \mathbb{C}^{2n} \mid \psi_k\Theta_k(\bar\lambda)\xi = 0 \text{ for all } k \in \mathcal{I}_\mathbb{Z}\right\}.$$

Thus the first equality in (6.70) follows from the fact that $\operatorname{Ran}\mathcal{K}_\lambda = (\operatorname{Ker}\mathcal{K}_\lambda^*)^\perp$.

Next, let us define the sequences $z := \Theta(\bar\lambda)\xi \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$, $v := \Theta(\bar\lambda)\eta \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$, and $r := \Theta(\bar\lambda)\zeta \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n}$, where $\xi,\eta,\zeta \in \mathbb{C}^{2n}$ are such that $\xi = \eta + \zeta$ with $\eta \in \operatorname{Ran}\mathcal{K}_\lambda$ and $\zeta \in (\operatorname{Ran}\mathcal{K}_\lambda)^\perp = \operatorname{Ker}\mathcal{K}_\lambda^*$. Then $\|r\|_\psi = \|\Theta(\bar\lambda)\zeta\|_\psi = 0$, i.e., $\tilde r = \tilde 0 \in \tilde\ell_\psi^2$. Thus $\tilde z = \tilde v$ and

$$\operatorname{Ran}\mathcal{K}_\lambda^* = \left\{\tilde z \in \tilde\ell_\psi^2 \mid z = \Theta(\bar\lambda)\eta,\ \eta \in \operatorname{Ran}\mathcal{K}_\lambda\right\}.$$

So, to complete the demonstration of equalities (6.70) and (6.71), it remains to show that $\operatorname{Ran}\mathcal{K}_\lambda = \operatorname{Ran}\vartheta(\mathcal{I}_\mathbb{Z})$, because $\tilde\ell_\psi^2 = \operatorname{Ker}\mathcal{K}_\lambda \oplus (\operatorname{Ker}\mathcal{K}_\lambda)^\perp = \operatorname{Ker}\mathcal{K}_\lambda \oplus \operatorname{Ran}\mathcal{K}_\lambda^*$.

Hence, let now $\tilde z \in \tilde\ell_\psi^2$. Then by the previous part $\tilde z = \tilde r + \tilde v$, where $\tilde r \in \operatorname{Ker}\mathcal{K}_\lambda$ and the equivalence class $\tilde v$ corresponds to $v = \Theta(\bar\lambda)\eta$ with $\eta \in \operatorname{Ran}\mathcal{K}_\lambda$. Therefore

$$\mathcal{K}_\lambda(\tilde z) = \mathcal{K}_\lambda(\tilde v) = \sum_{k\in\mathcal{I}} \Theta_k^*(\bar\lambda)\psi_k\Theta_k(\bar\lambda)\eta = \vartheta(\lambda,\mathcal{I}_\mathbb{Z})\eta,$$

which implies $\operatorname{Ran}\mathcal{K}_\lambda \subseteq \operatorname{Ran}\vartheta(\mathcal{I}_\mathbb{Z})$. On the other hand, if $\xi \in \mathbb{C}^{2n}$ and $z = \Theta(\bar\lambda)\xi$, then

$$\vartheta(\lambda,\mathcal{I}_\mathbb{Z})\xi = \sum_{k\in\mathcal{I}} \Theta_k^*(\bar\lambda)\psi_k\Theta_k(\bar\lambda)\xi = \sum_{k\in\mathcal{I}} \Theta_k^*(\bar\lambda)\psi_k z_k = \mathcal{K}_\lambda(\tilde z);$$

thereby showing that $\operatorname{Ran}\mathcal{K}_\lambda = \operatorname{Ran}\vartheta(\mathcal{I}_\mathbb{Z})$. ∎

In the next two lemmas we establish similar results in the case $\mathcal{I}_\mathbb{Z} = [0,\infty)_\mathbb{Z}$. Let us recall that then there is a finite discrete interval $\widehat{\mathcal{I}}_\mathbb{Z} \subset [0,\infty)_\mathbb{Z}$ for which the value of $\operatorname{rank}\vartheta(\widehat{\mathcal{I}}_\mathbb{Z})$ is maximal by Lemma 6.2.10, i.e., $\operatorname{rank}\vartheta(\widehat{\mathcal{I}}_\mathbb{Z}) = \operatorname{rank}\vartheta(\widetilde{\mathcal{I}}_\mathbb{Z})$ for any finite discrete interval $\widetilde{\mathcal{I}}_\mathbb{Z}$ such that $\widehat{\mathcal{I}}_\mathbb{Z} \subseteq \widetilde{\mathcal{I}}_\mathbb{Z}$, viz. (6.32). Moreover, if $\widehat{\mathcal{I}}_\mathbb{Z}$ is a finite discrete interval for which $\operatorname{rank}\vartheta(\widehat{\mathcal{I}}_\mathbb{Z})$ is maximal, then we have $\operatorname{Ran}\mathcal{K}_{\lambda,\widetilde{\mathcal{I}}} = \operatorname{Ran}\mathcal{K}_\lambda$ for any finite discrete interval $\widetilde{\mathcal{I}}_\mathbb{Z}$ such that $\widehat{\mathcal{I}}_\mathbb{Z} \subseteq \widetilde{\mathcal{I}}_\mathbb{Z} \subset [0,\infty)_\mathbb{Z}$, see (6.70). Here $\mathcal{K}_{\lambda,\widetilde{\mathcal{I}}}$ means the map $\mathcal{K}_\lambda$ defined as in (6.69) with $\mathcal{I}_\mathbb{Z}$ replaced by $\widetilde{\mathcal{I}}_\mathbb{Z}$. Especially, $\operatorname{dom}\mathcal{K}_{\lambda,\widetilde{\mathcal{I}}} = \tilde\ell^2_{\psi,\widetilde{\mathcal{I}}}$, where $\tilde\ell^2_{\psi,\widetilde{\mathcal{I}}}$ is given as in (6.60) for the discrete interval $\widetilde{\mathcal{I}}_\mathbb{Z}$ instead of $\mathcal{I}_\mathbb{Z}$.

Lemma 6.4.7. *Let $\mathcal{I}_\mathbb{Z} = [0,\infty)_\mathbb{Z}$ and $\widehat{\mathcal{I}}_\mathbb{Z} \subset [0,\infty)_\mathbb{Z}$ be a finite discrete interval for which* $\operatorname{rank}\vartheta(\widehat{\mathcal{I}}_\mathbb{Z})$ *is maximal. Then* $\operatorname{Ran}\mathcal{K}_\lambda = \operatorname{Ran}\vartheta(\widehat{\mathcal{I}}_\mathbb{Z})$ *for all $\lambda \in \mathbb{C}$; in particular,* $\operatorname{Ran}\mathcal{K}_\lambda$ *is independent of λ.*

Proof. Let $\tilde z \in \operatorname{dom}\mathcal{K}_\lambda = \tilde\ell^2_{\psi,1}$. Then there exists $K \in [0,\infty)_\mathbb{Z}$ such that $\psi_k z_k = 0$ for all $k \in [K,\infty)_\mathbb{Z}$. Hence, $z \in \ell_\psi^2$ and $\tilde z \in \tilde\ell^2_{\psi,[0,K]_\mathbb{Z}} = \operatorname{dom}\mathcal{K}_{\lambda,[0,K]}$, where $\tilde\ell^2_{\psi,[0,K]_\mathbb{Z}}$ is defined similarly as $\tilde\ell_\psi^2$ in (6.60) with $\mathcal{I}_\mathbb{Z} = [0,K]_\mathbb{Z}$. In particular, $\mathcal{K}_{\lambda,[0,N]}(\tilde z) = \mathcal{K}_\lambda(\tilde z)$, and thus $\operatorname{Ran}\mathcal{K}_\lambda \subseteq \operatorname{Ran}\mathcal{K}_{\lambda,[0,N]}$. Without loss of generality, we may assume that $\widehat{\mathcal{I}}_\mathbb{Z} \subseteq [0,K]_\mathbb{Z}$, and hence that $\operatorname{Ran}\mathcal{K}_{\lambda,\widehat{\mathcal{I}}} = \operatorname{Ran}\mathcal{K}_{\lambda,[0,N]} \supseteq \operatorname{Ran}\mathcal{K}_\lambda$.

Conversely, suppose that $\tilde z \in \tilde\ell^2_{\psi,\widehat{\mathcal{I}}} = \operatorname{dom}\mathcal{K}_{\lambda,\widehat{\mathcal{I}}}$. Let $\tilde v \in \tilde\ell^2_{\psi,1}$ be determined by the sequence $v \in \mathbb{C}([0,\infty)_\mathbb{Z})^{2n}$ with the terms

$$v_k = \begin{cases} z_k, & k \in \widehat{\mathcal{I}}_\mathbb{Z}, \\ 0, & k \notin \widehat{\mathcal{I}}_\mathbb{Z}. \end{cases}$$

Then $\mathcal{K}_{\lambda,\widehat{\mathcal{I}}}(\tilde z) = \sum_{k\in\widehat{\mathcal{I}}} \Theta_k^*(\bar\lambda)\psi_k z_k = \sum_{k=0}^\infty \Theta_k^*(\bar\lambda)\psi_k v_k = \mathcal{K}_\lambda(\tilde v)$. Hence $\operatorname{Ran}\mathcal{K}_{\lambda,\widehat{\mathcal{I}}} \subseteq \operatorname{Ran}\mathcal{K}_\lambda$, which upon combining with the first part of the proof yields $\operatorname{Ran}\mathcal{K}_\lambda = \operatorname{Ran}\mathcal{K}_{\lambda,\widehat{\mathcal{I}}}$. Therefore $\operatorname{Ran}\mathcal{K}_\lambda = \operatorname{Ran}\vartheta(\widehat{\mathcal{I}}_\mathbb{Z})$ by Lemma 6.4.6. ∎

Lemma 6.4.8. *Let $\mathcal{I}_\mathbb{Z} = [0,\infty)_\mathbb{Z}$ and $\widehat{\mathcal{I}}_\mathbb{Z} \subset [0,\infty)_\mathbb{Z}$ be a finite discrete interval for which* $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_\mathbb{Z})$ *is maximal. Then*

$$\tilde{\ell}^2_{\psi,1} = \operatorname{Ker} \mathcal{K}_\lambda \oplus \left\{ \tilde{z} \in \tilde{\ell}^2_{\psi,1} \mid z = \Theta(\bar{\lambda})\xi,\ \xi \in \operatorname{Ran} \vartheta(\widehat{\mathcal{I}}_\mathbb{Z}) \right\} \tag{6.72}$$

and $\operatorname{codim}(\operatorname{Ker} \mathcal{K}_\lambda) = \operatorname{rank} \vartheta(\widehat{\mathcal{I}}_\mathbb{Z})$.

Proof. In the complete analogy with the arguments given in the proof of Lemma 6.4.6, one can show that

$$\tilde{\ell}^2_{\psi,1} = \operatorname{Ker} \mathcal{K}_\lambda \oplus (\operatorname{Ker} \mathcal{K}_\lambda)^\perp = \operatorname{Ker} \mathcal{K}_\lambda \oplus \operatorname{Ran} \mathcal{K}^*_\lambda,$$

where

$$\operatorname{Ran} \mathcal{K}^*_\lambda = \left\{ \tilde{z} \in \tilde{\ell}^2_{\psi,1} \mid z = \Theta(\bar{\lambda})\xi,\ \xi \in \operatorname{Ran} \vartheta(\widehat{\mathcal{I}}_\mathbb{Z}) \right\},$$

and that

$$\begin{aligned}(\operatorname{Ran} \mathcal{K}_\lambda)^\perp = \operatorname{Ker} \mathcal{K}^*_\lambda &= \left\{ \xi \in \mathbb{C}^{2n} \mid \|\Theta(\bar{\lambda})\xi\|_\psi = 0 \right\} \\ &= \left\{ \xi \in \mathbb{C}^{2n} \mid \psi_k \Theta_k(\bar{\lambda})\xi = 0 \text{ for all } k \in [0,\infty)_\mathbb{Z} \right\}.\end{aligned}$$

Finally, let $\xi_1, \dots, \xi_m$ denote a basis for $\operatorname{Ran} \vartheta(\widehat{\mathcal{I}}_\mathbb{Z}) = \operatorname{Ran} \mathcal{K}_\lambda$ and put $v^{[j]}_k := \Theta_k(\bar{\lambda})\xi_j$ for all $k \in [0,\infty)_\mathbb{Z}$. Then $\tilde{v}^{[1]}, \dots, \tilde{v}^{[m]} \in \operatorname{Ran} \mathcal{K}^*_\lambda$ and suppose that $\sum_{j=1}^m c_j \tilde{v}^{[j]} = \tilde{0} \in \tilde{\ell}^2_{\psi,1}$ for some numbers $c_1, \dots, c_m \in \mathbb{C}$. Then $\psi_k \Theta_k(\bar{\lambda}) (\sum_{j=1}^m c_j \xi_j) = 0$ for all $k \in [0,\infty)_\mathbb{Z}$, which implies

$$\sum_{j=1}^m c_j \xi_j \in \operatorname{Ran} \mathcal{K}_\lambda \cap (\operatorname{Ran} \mathcal{K}_\lambda)^\perp = \{0\}$$

by the first part. Hence $c_1 = \cdots = c_m = 0$, i.e., $\tilde{v}^{[1]}, \dots, \tilde{v}^{[m]} \in \operatorname{Ran} \mathcal{K}^*_\lambda$ are linearly independent in $\tilde{\ell}^2_{\psi,1}$. Thus $\operatorname{codim}(\operatorname{Ker} \mathcal{K}_\lambda) = \dim(\operatorname{Ker} \mathcal{K}_\lambda)^\perp = \dim(\operatorname{Ran} \mathcal{K}^*_\lambda) = \operatorname{rank} \vartheta(\widehat{\mathcal{I}}_\mathbb{Z})$. ■

6.4.3 Minimal linear relation and its deficiency indices

Before we define the minimal linear relation $T_{\min}$ and establish the fundamental relation between $T_{\max}$ and $T_{\min}$, we prove two auxiliary lemmas.

Lemma 6.4.9. *We have* $\operatorname{Ker} \mathcal{K}_\lambda \subseteq \operatorname{Ran}(T_0 - \lambda I)$ *for every* $\lambda \in \mathbb{C}$.

Proof. Let $\lambda \in \mathbb{C}$ and $\tilde{f} \in \operatorname{Ker} \mathcal{K}_\lambda$, i.e., $\mathcal{K}_\lambda(\tilde{f}) = \sum_{k \in \mathcal{I}} \Theta^*_k(\bar{\lambda}) \psi_k z_k = 0$. Assume that $\mathcal{I}_\mathbb{Z}$ is a finite discrete interval, i.e., $\mathcal{I}_\mathbb{Z} = [0,N]_\mathbb{Z}$ for some $N \in \mathbb{N} \cup \{0\}$. Then $\tilde{f} \in \tilde{\ell}^2_\psi$ by (6.69) and for any $g \in \tilde{f}$ the sequence $z \in \ell^2_{\psi,0}$ with the terms

$$z_k := \begin{cases} 0, & k = 0, \\ \Theta_k(\lambda) \mathcal{J} \sum_{j=0}^{k-1} \Theta^*_j(\bar{\lambda}) \psi_j g_j, & k \in \mathcal{I}^+_\mathbb{Z} \setminus \{0\}, \end{cases} \tag{6.73}$$

solves the nonhomogeneous system $\mathcal{L}(z) = \lambda \psi z + \psi g$ on $\mathcal{I}_\mathbb{Z}$, see also [61, Theorem 3.17]. Thus, $\tilde{f} \in \operatorname{Ran}(T_0 - \lambda I)$ and the conclusion follows; cf. (6.64).

On the other hand, if $\mathcal{I}_\mathbb{Z} = [0,\infty)_\mathbb{Z}$, then $\tilde{f} \in \tilde{\ell}^2_{\psi,1}$ by (6.69), i.e., there exists $K \in [0,\infty)_\mathbb{Z}$ such that $\psi_k g_k = 0$ for all $k \in [K,\infty)_\mathbb{Z}$ and all $g \in \tilde{f}$. Hence for any $g \in \tilde{f}$ the sequence $z \in \ell^2_\psi$ with the terms

$$z_k := -\Theta_k(\lambda) \mathcal{J} \sum_{j=k}^{\infty} \Theta^*_j(\bar{\lambda}) \psi_j g_j = \begin{cases} -\Theta_k(\lambda) \mathcal{J} \sum_{j=k}^{K-1} \Theta^*_j(\bar{\lambda}) \psi_j g_j, & k \in [0, K-1]_\mathbb{Z}, \\ 0, & k \in [K,\infty)_\mathbb{Z}, \end{cases} \tag{6.74}$$

satisfies $\mathscr{L}(z) = \lambda\psi z + \psi g$ on $[0,\infty)_{\mathbb{Z}}$. In addition, $z_0 = -\Theta_0(\lambda)\mathcal{J}\mathcal{K}_\lambda(\tilde{f}) = 0$, which implies $z \in \ell^2_{\psi,0}$. Thus again $\tilde{f} \in \operatorname{Ran}(T_0 - \lambda I)$ and the proof is complete. ∎

Lemma 6.4.10. *We have* $\langle \tilde{f}, \tilde{y}\rangle_\psi = \langle \tilde{z}, \tilde{g}\rangle_\psi$ *for every* $\{\tilde{z}, \tilde{f}\} \in T_0$ *and* $\{\tilde{v}, \tilde{g}\} \in T_{\max}$.

Proof. Let $\{\tilde{z}, \tilde{f}\} \in T_0$ and $\{\tilde{v}, \tilde{g}\} \in T_{\max}$, i.e., there are $z \in \ell^2_{\psi,0}$ and $v \in \ell^2_\psi$ such that $\mathscr{L}(z) = \psi f$ and $\mathscr{L}(v) = \psi g$ on $\mathcal{I}_{\mathbb{Z}}$. Then, similarly as in (6.14), we get

$$\sum_{j=0}^{k} z_j^* \psi_j g_j = -(z,v)_j\Big|_0^{k+1} + \sum_{j=0}^{k} f_j^* \psi_j v_j.$$

However, $z \in \ell^2_{\psi,0}$ implies $z_0 = 0$ and the existence of $K \in \mathcal{I}_{\mathbb{Z}}^+$ such that $z_k = 0$ for all $k \in [K,\infty)_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}^+$. As a consequence we see that $\langle z, g\rangle_\psi = \langle f, v\rangle_\psi$, and thus $\langle z, g\rangle_\psi = \langle f, v\rangle_\psi$ for any $z \in \tilde{z}$ and $v \in \tilde{v}$. ∎

By Lemma 6.4.10 and the definition of the adjoint linear relation, see (A.4), one obtains

$$T_0 \subseteq T_{\max} \subseteq T_0^*, \tag{6.75}$$

from which we conclude that T_0 is symmetric in $\tilde{\ell}^{2\times 2}_\psi$, and hence closable. We then define the *minimal linear relation* $T_{\min}$ by

$$T_{\min} := \overline{T_0}. \tag{6.76}$$

Another approach to the definition of a minimal linear relation is to put $T_{\min} := T^*_{\max}$; cf. [116, Definition 2.3] and [13, Identity (4.2)]. In the next theorem we establish the equivalence of these approaches for the linear relations at hand. We note also that an alternative demonstration of the next statement can be patterned after that presented in [13, pp. 1354–1355] by using the results in Section 6.3 and [13, Proposition A.2].

Theorem 6.4.11. *The linear relations* T_0, $T_{\max}$, *and* $T_{\min}$ *as defined in* (6.63), (6.62), *and* (6.76), *respectively, satisfy*

$$T_0^* = T^*_{\min} = T_{\max}. \tag{6.77}$$

Proof. By (6.75), note that $T_{\max} \subseteq T_0^* = \overline{T_0}^* = T^*_{\min}$. Thus it remains to show that $T_0^* \subseteq T_{\max}$; or equivalently, given $\{\tilde{z}, \tilde{f}\} \in T_0^*$, that there is a $z \in \tilde{z} \in \tilde{\ell}^2_\psi$ such that $\mathscr{L}(z) = \psi f$.

Let $\{\tilde{z}, \tilde{f}\} \in T_0^*$ be given and $r \in \mathbb{C}(\mathcal{I}_{\mathbb{Z}}^+)^{2n}$ satisfy $\mathscr{L}(r) = \psi f$. Whenever $\{\tilde{v}, \tilde{g}\} \in T_0$, i.e., there exists $v \in \tilde{v}$ such that $v \in \ell^2_{\psi,0}$ and $\mathscr{L}(v) = \psi g$, we obtain from (6.14) that $\langle r, g\rangle_\psi = \langle f, v\rangle_\psi$, because $v_0 = 0$ and $v_k = 0$ for all sufficiently large $k \in \mathcal{I}_{\mathbb{Z}}^+$. Simultaneously, from the definition of T_0^*, we have $\langle \tilde{z}, \tilde{g}\rangle_\psi = \langle \tilde{f}, \tilde{v}\rangle_\psi$. Therefore $\langle z - r, g\rangle_\psi = 0$ for any $z \in \tilde{z} \in \operatorname{dom} T_0^*$ and $g \in \tilde{g} \in \operatorname{Ran} T_0$. Consequently, for any $\hbar \in \tilde{\hbar} \in \operatorname{Ker}\mathcal{K} \subseteq \operatorname{Ran} T_0$, see Lemma 6.4.9 with $\lambda = 0$, we get from the previous part that $\langle z - r, \hbar\rangle_\psi = 0$. Thus, for any $\xi \in \mathbb{C}^{2n}$ we have

$$\sum_{k\in\mathcal{I}} (z_k - r_k - \Theta_k \xi)^* \psi_k \hbar_k = 0, \tag{6.78}$$

because $\mathcal{K}(\tilde{\hbar}) = \sum_{k\in\mathcal{I}} \Theta_k^* \psi_k \hbar_k = 0$ by (6.69).

Let $\mathcal{I}_{\mathbb{Z}} = [0,N]_{\mathbb{Z}}$ with $N \in \mathbb{N} \cup \{0\}$, then obviously $\tilde{z}, \tilde{r} \in \tilde{\ell}^2_\psi$. Hence by (6.71) there exists $\eta \in \operatorname{Ran}\vartheta(\mathcal{I}_{\mathbb{Z}})$ such that the equivalence class corresponding to $z - r - \Theta\eta$ belongs to $\operatorname{Ker}\mathcal{K}$. But in that case it is equal to $\tilde{0}$ by (6.78). Therefore $\|z - r - \Theta\eta\|_\psi = 0$, i.e., $r + \Theta\eta \in \tilde{z}$, and $\mathscr{L}(r + \Theta\eta) = \mathscr{L}(r) = \psi f$, which implies $T_0^* \subseteq T_{\max}$.

In the opposite case, i.e., $\mathcal{I}_{\mathbb{Z}} = [0,\infty)_{\mathbb{Z}}$, let $\widehat{\mathcal{I}}_{\mathbb{Z}} \subset [0,\infty)_{\mathbb{Z}}$ be a finite discrete interval such that $k_0 \in \widehat{\mathcal{I}}_{\mathbb{Z}}^{+}$ and the value of $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}) = m \leq 2n$ is maximal as noted in Lemma 6.2.10. Then $\tilde{\ell}^2_{\psi,1} = \operatorname{Ker}\mathcal{K} \oplus \operatorname{Ker}\mathcal{K}^{\perp}$ and by Lemma 6.4.8 we see that there exists a basis $\tilde{v}^{[1]},\dots,\tilde{v}^{[m]} \in \tilde{\ell}^2_{\psi,1}$ of the space $\operatorname{Ker}\mathcal{K}^{\perp}$, in which $v_k^{[j]} := \Theta_k \xi^{[j]}$ on $[0,\infty)_{\mathbb{Z}}$ for $j \in \{1,\dots,m\}$ with $\xi^{[1]},\dots,\xi^{[m]}$ forming a basis of $\operatorname{Ran}\vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$. Moreover, it follows from the definition of $\tilde{\ell}^2_{\psi,1}$ that there is a finite discrete interval $\widetilde{\mathcal{I}}_{\mathbb{Z}} = [0,K]_{\mathbb{Z}}$ such that $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \widetilde{\mathcal{I}}_{\mathbb{Z}}$ and

$$\psi_k v_k^{[j]} = 0 \quad \text{for all } k \in [K,\infty)_{\mathbb{Z}} \text{ and } j \in \{1,\dots,m\}. \tag{6.79}$$

If $\tilde{d} \in \operatorname{Ker}\mathcal{K}_{\widetilde{\mathcal{I}}} := \operatorname{Ker}\mathcal{K}_{0,\widetilde{\mathcal{I}}}$, then for all $j \in \{1,\dots,m\}$ we get

$$\langle v^{[j]}, d\rangle_{\psi,\widetilde{\mathcal{I}}} := \sum_{k\in\widetilde{\mathcal{I}}} \xi^{[j]*}\Theta_k^*\psi_k d_k = \xi^{[j]*}\mathcal{K}_{\widetilde{\mathcal{I}}}(\tilde{d}) = 0,$$

which implies $\tilde{v}^{[1]},\dots,\tilde{v}^{[m]} \in \operatorname{Ker}\mathcal{K}^{\perp}_{\widetilde{\mathcal{I}}}$. Now, let $s \in \ell^2_{\psi,1}$ be defined by

$$s_k := \begin{cases} z_k - r_k, & k \in \widetilde{\mathcal{I}}_{\mathbb{Z}}, \\ 0, & k \in [0,\infty)_{\mathbb{Z}} \setminus \widetilde{\mathcal{I}}_{\mathbb{Z}}, \end{cases}$$

where $\{\tilde{z},\tilde{f}\}$ and r are the same as in the first part. Obviously $\tilde{s} \in \tilde{\ell}^2_{\psi}$ and, by Lemma 6.4.6, there is $\tilde{p} \in \operatorname{Ker}\mathcal{K}^{\perp}_{\widetilde{\mathcal{I}}}$, where $p := \Theta\eta$ for some $\eta \in \operatorname{Ran}\vartheta(\widetilde{\mathcal{I}}_{\mathbb{Z}})$, such that $\tilde{s}-\tilde{p} \in \operatorname{Ker}\mathcal{K}_{\widetilde{\mathcal{I}}}$. As a consequence, we obtain $\langle \tilde{s}-\tilde{p}, \tilde{v}^{[j]}\rangle_{\psi,\widetilde{\mathcal{I}}} = 0$ for all $j \in \{1,\dots,m\}$. Then, by (6.79), we have for all $j \in \{1,\dots,m\}$ that

$$\sum_{k\in\mathcal{I}} (z_k - r_k - \Theta_k\eta)^*\psi_k v_k^{[j]} = \sum_{k\in\widetilde{\mathcal{I}}} (z_k - r_k - \Theta_k\eta)^*\psi_k v_k^{[j]} = \langle \tilde{s}-\tilde{p}, \tilde{v}^{[j]}\rangle_{\psi,\widetilde{\mathcal{I}}} = 0. \tag{6.80}$$

Hence the direct sum decomposition of $\tilde{\ell}^2_{\psi,1}$ in (6.72) and identities (6.78) and (6.80) show that $\langle z-r-\Theta\eta, q\rangle_{\psi} = 0$ for any $q \in \ell^2_{\psi,1}$; in particular, for

$$q_k := \begin{cases} z_{k_1} - r_{k_1} - \Theta_{k_1}\eta, & k = k_1, \\ 0, & k \in [0,\infty)_{\mathbb{Z}} \setminus \{k_1\}. \end{cases}$$

Since $k_1 \in [0,\infty)_{\mathbb{Z}}$ can be chosen arbitrarily, it follows $\psi_k(z_k - r_k - \Theta_k\eta) \equiv 0$ on $[0,\infty)_{\mathbb{Z}}$, i.e., $\|z-(r+\Theta\eta)\|_{\psi} = 0$. Therefore, $r+\Theta\eta \in \tilde{z}$ and it satisfies $\mathscr{L}(r+\Theta\eta) = \mathscr{L}(r) = \psi f$, which again implies $T_0^* \subseteq T_{\max}$. ■

Moreover, the following theorem provides an explicit characterization of the minimal linear relation $T_{\min}$; cf. [135, Theorem 3.2].

Theorem 6.4.12. *Let system ($\mathcal{S}_\lambda$) be definite on a finite discrete interval $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}} := [a,b]_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$, where $a,b \in \mathcal{I}_{\mathbb{Z}}$. Then*

$$T_{\min} = \Big\{\{\tilde{z},\tilde{f}\} \in T_{\max} \mid \hat{z}_0 = 0 = (\hat{z},\hat{v})_{N+1} \text{ for all } \tilde{v} \in \operatorname{dom} T_{\max}\Big\}, \tag{6.81}$$

which in the case of $\mathcal{I}_{\mathbb{Z}}$ being a finite discrete interval reduces to

$$T_{\min} = \Big\{\{\tilde{z},\tilde{f}\} \in T_{\max} \mid \hat{z}_0 = 0 = \hat{z}_{N+1}\Big\}. \tag{6.82}$$

Proof. Since $T_{\min} = \overline{(T_{\min})}$ by the definition, identities (A.11), (6.67), and (6.77) yield

$$T_{\min} = \Big\{\{\tilde{z}, \tilde{f}\} \in T_{\max} \mid (\hat{z}, \hat{v})_k\big|_0^{N+1} = 0 \ \text{ for all } \hat{v} \in \operatorname{dom} T_{\max}\Big\}. \tag{6.83}$$

Let T be the linear relation on the right-hand side of (6.81). Then obviously $T \subseteq T_{\min}$. On the other hand, let $\{\tilde{z}, \tilde{f}\} \in T_{\min}$ be fixed. Then, $(\hat{z}, \hat{v})_k\big|_0^{N+1} = 0$ for all $\hat{v} \in \operatorname{dom} T_{\max}$ by (6.83). By Lemma 6.4.3, for any $\{\tilde{v}, \tilde{g}\} \in T_{\max}$ there exists $\{\tilde{r}, \tilde{h}\} \in T_{\max}$ such that $\hat{v}_k = 0$ for $k \in [0, c]_{\mathbb{Z}}$ and $\hat{r}_k = \hat{v}_k$ for $k \in [d+1, \infty)_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}^+$. Hence $(\hat{z}, \hat{v})_0 = (\hat{z}, \hat{v})_{N+1} = 0$ for all $\hat{v} \in \operatorname{dom} T_{\max}$. From the second part of Lemma 6.4.3 we get $\hat{z}_0 = 0$, because there exists $\{\tilde{z}^{[i]}, \tilde{f}^{[i]}\} \in T_{\max}$ such that $\hat{z}_0^{[i]} = e_i$. Therefore $T = T_{\min}$. If, in addition, $\mathcal{I}_{\mathbb{Z}}$ is a finite discrete interval, i.e., $\mathcal{I}_{\mathbb{Z}} = [0, N]_{\mathbb{Z}}$ with $N \in \mathbb{N} \cup \{0\}$, then $\operatorname{dom} T_{\max}$ contains also $\tilde{r}$ such that $\hat{r}_{N+1} = e_i$, $i \in \{1, \dots, 2n\}$, by the last part of Lemma 6.4.3. Hence equality (6.82) holds. ∎

Finally, we define the *defect subspace* and *defect index* for the minimal linear relation $T_{\min}$ according to (A.7) and (A.8), respectively, i.e., by using (6.77) we put

$$\widetilde{\mathcal{N}}_\lambda = \widetilde{\mathcal{N}}_\lambda(T_{\min}) := \Big\{\tilde{z} \in \tilde{\ell}^2_\Psi \mid \{\tilde{z}, \lambda\tilde{z}\} \in T^*_{\min} = T_{\max}\Big\}, \quad \tilde{n}_\lambda := \dim \widetilde{\mathcal{N}}_\lambda.$$

Since $T_{\min}$ is a closed, symmetric linear relation by Theorem 6.4.11, it follows that the value of the deficiency indices $\tilde{n}_\lambda$ is constant on each of the open upper and lower half-planes of $\mathbb{C}$, i.e., $\tilde{n}_{\pm\lambda} = \tilde{n}_{\pm i}$ for all $\lambda \in \mathbb{C}_+$; cf. [143, Theorem 2.13]. Thus we let $\tilde{n}_\pm := \tilde{n}_{\pm i}$. In addition, we introduce the following two subspaces

$$\mathcal{N}_\lambda := \Big\{z \in \ell^2_\Psi \mid \mathcal{L}(z)_k = \lambda \Psi_k z_k \text{ for all } k \in \mathcal{I}_{\mathbb{Z}}\Big\}, \quad n_\lambda := \dim \mathcal{N}_\lambda,$$
$$\widetilde{\mathfrak{N}}_\lambda = \widetilde{\mathfrak{N}}_\lambda(T_{\min}) := \Big\{\{\tilde{z}, \lambda\tilde{z}\} \mid \{\tilde{z}, \lambda\tilde{z}\} \in T_{\max}\Big\}.$$

In other words, the subspace $\mathcal{N}_\lambda \subseteq \ell^2_\Psi$ consists of all square summable solutions of system ($\mathcal{S}_\lambda$) and n_λ denotes the number of linearly independent square summable solutions of system ($\mathcal{S}_\lambda$), while $\widetilde{\mathfrak{N}}_\lambda$ is a subspace associated with the defect subspace $\widetilde{\mathcal{N}}_\lambda$. Then we obtain the following von Neumann decomposition of the maximal linear relation $T_{\max}$ in terms of the minimal linear relation $T_{\min}$ and the subspaces $\widetilde{\mathfrak{N}}_\lambda$ and $\widetilde{\mathfrak{N}}_{\bar{\lambda}}$, i.e.,

$$T_{\max} = T_{\min} \dotplus \widetilde{\mathfrak{N}}_\lambda \dotplus \widetilde{\mathfrak{N}}_{\bar{\lambda}}, \tag{6.84}$$

where the direct sum $\dotplus$ is orthogonal if $\lambda = \pm i$, see (A.9).

Assuming that $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$ is maximal, as described in Lemma 6.2.10, we next show a relationship between the numbers n_λ and $\tilde{n}_\lambda$.

Theorem 6.4.13. *Let $\widehat{\mathcal{I}}_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ be a finite discrete interval such that the value of* $\operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$ *is maximal. Then, for any $\lambda \in \mathbb{C}$,*

$$n_\lambda = \tilde{n}_\lambda + 2n - \operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}}). \tag{6.85}$$

Proof. Let $\lambda \in \mathbb{C}$ be given and π_1 denote the restriction of the quotient space map π (introduced at the beginning of this section) given by $\pi_1 := \pi|_{\mathcal{N}_\lambda} : \mathcal{N}_\lambda \to \widetilde{\mathcal{N}}_\lambda$. Since for any $\tilde{z} \in \widetilde{\mathcal{N}}_\lambda$ there exists $z \in \tilde{z}$ solving system ($\mathcal{S}_\lambda$), i.e., $z \in \mathcal{N}_\lambda$, the map π_1 is surjective. Thus

$$\dim \mathcal{N}_\lambda = \dim \widetilde{\mathcal{N}}_\lambda + \dim(\operatorname{Ker} \pi_1). \tag{6.86}$$

Moreover, we note that

$$\operatorname{Ker} \pi_1 = \Big\{z \in \ell^2_\Psi \mid \ \mathcal{L}(z)_k = \lambda \Psi_k z_k \text{ and } \Psi_k z_k = 0 \text{ for all } k \in \mathcal{I}_{\mathbb{Z}}\Big\},$$

which shows that $z \in \operatorname{Ker} \pi_1$ if and only if $z = \Theta \xi$ for some $\xi \in \bigcap_{K=0}^{N} \operatorname{Ker} \vartheta([0,K]_{\mathbb{Z}}) = \operatorname{Ker} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$. Therefore $\dim(\operatorname{Ker} \pi_1) = \dim[\operatorname{Ker} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})] = 2n - \operatorname{rank} \vartheta(\widehat{\mathcal{I}}_{\mathbb{Z}})$, which together with equality (6.86) implies (6.85). ∎

As a direct consequence of equality (6.85) we get the following properties of the numbers n_λ and $\tilde{n}_\lambda$.

Corollary 6.4.14. *The following statements hold true:*

(i) $n_\lambda - \tilde{n}_\lambda$ *is nonnegative and constant for all* $\lambda \in \mathbb{C}$*;*

(ii) $n_\lambda = \tilde{n}_\lambda$ *for some (and hence for any)* $\lambda \in \mathbb{C}$ *if and only if system* ($\mathcal{S}_\lambda$) *is definite on* $[0,\infty)_{\mathbb{Z}}$*;*

(iii) n_λ *is constant in the half-planes* $\mathbb{C}_+$ *and* $\mathbb{C}_-$*.*

Proof. The first statement follows from equality (6.85) and Lemma 6.2.9, while for the second statement it suffices to combine (6.85), Theorem 6.2.11, and Lemma 6.2.3. Finally, since the minimal linear relation $T_{\min}$ is symmetric, see (6.75) and (6.76), the number $\tilde{n}_\lambda$ is constant in the half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$ by [143, Theorem 2.13]. Hence the value of n_λ is also constant in $\mathbb{C}_+$ and $\mathbb{C}_-$ by the first part. ∎

Remark 6.4.15. The last statement of Corollary 6.4.14 extends the enumeration and analysis of linearly independent square summable solutions of system ($\mathbb{S}_\lambda$) provided in Section 2.4, see Theorem 2.4.8 and also equality (6.3). More precisely, since the number of linearly independent square summable solutions of system ($\mathcal{S}_\lambda$) or ($\mathbb{S}_\lambda$) is constant in $\mathbb{C}_+$ and $\mathbb{C}_-$, the the same property possesses also the function $r(\lambda)$ defined in (2.56) as it was already mentioned in Remark 2.4.21. Moreover, in the final result of this chapter we complete this analysis by a basic estimate, which shows that the number of linearly independent square summable solutions on the real line cannot exceed the same number calculated for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 6.4.16. *For any* $\lambda \in \mathbb{C}$ *we have*

$$\operatorname{Ker}(T_{\min} - \lambda I) = \{\tilde{0}\}. \tag{6.87}$$

Moreover, for every $\lambda \in \mathbb{R}$ *we have* $\tilde{n}_\lambda \le \tilde{n}_\pm$ *and* $n_\lambda \le n_{\pm i}$*.*

Proof. First, suppose that $\mathcal{I}_{\mathbb{Z}} = [0,N]_{\mathbb{Z}}$ for some $N \in \mathbb{N} \cup \{0\}$. Let $\lambda \in \mathbb{C}$, $\tilde{f} \in \tilde{\ell}^2_\Psi$, and $\tilde{z} \in \tilde{\ell}^2_\Psi$ be determined by $z \in \mathbb{C}(\mathcal{I}^+_{\mathbb{Z}})^{2n}$ constructed as in (6.73). Then $\mathscr{L}(z) = \lambda \Psi z + \Psi f$ on $\mathcal{I}_{\mathbb{Z}}$, which yields $\{\tilde{z}, \tilde{f}\} \in T_{\max} - \lambda I$, i.e., $\tilde{f} \in \operatorname{Ran}(T_{\max} - \lambda I)$. Hence $\tilde{\ell}^2_\Psi = \operatorname{Ran}(T_{\max} - \lambda I)$, because λ and $\tilde{f}$ were chosen arbitrarily. Thus

$$\operatorname{Ker}(T_{\min} - \lambda I) \overset{(A.5)}{=} [\operatorname{Ran}(T^*_{\min} - \bar{\lambda} I)]^\perp \overset{(6.77)}{=} [\operatorname{Ran}(T_{\max} - \bar{\lambda} I)]^\perp = (\tilde{\ell}^2_\Psi)^\perp = \{\tilde{0}\}.$$

On the other hand, let now $\mathcal{I}_{\mathbb{Z}} = [0,\infty)_{\mathbb{Z}}$. Then for $\lambda \in \mathbb{C}$, $\tilde{f} \in \tilde{\ell}^2_{\Psi,1}$, and $\tilde{z} \in \tilde{\ell}^2_{\Psi,1}$ determined by the sequence $z \in \mathbb{C}([0,\infty)_{\mathbb{Z}})^{2n}$ constructed as in (6.74) we obtain that $\{\tilde{z}, \tilde{f}\} \in T_{\max} - \lambda I$, see the proof of Lemma 6.4.9. Hence $\tilde{\ell}^2_{\Psi,1} \subseteq \operatorname{Ran}(T_{\max} - \lambda I)$. Since $\tilde{\ell}^2_{\Psi,1}$ is a dense subspace of $\tilde{\ell}^2_\Psi$, it follows that $\operatorname{Ran}(T_{\max} - \lambda I)$ is also dense in $\tilde{\ell}^2_\Psi$ for any $\lambda \in \mathbb{C}$. Thus, for the proof of equality (6.87) it suffices to utilize an analogous calculation as in the previous part. The remaining assertions follow from (A.10) and Corollary 6.4.14(i). ∎

6.5 Bibliographical notes

The results of this chapter were published in [A18] except for Remark 6.1.4, Lemmas 6.2.8 and 6.4.3, and Theorem 6.4.12, which were published later in [A21]. Their generalization to symplectic systems on time scales is one of the goals of the current research.

... when America's National Academy of Science asked shortly before his death what he thought were his three greatest achievements ... Johnny replied to the academy that he considered his most important contributions to have been on the theory of self-adjoint operators in Hilbert space, and on the mathematical foundations of quantum theory and the ergodic theorem.

John von Neumann, see [118, pg. 24]

Chapter 7

Self-adjoint extensions

This final chapter is devoted to the characterization of all self-adjoint extensions of the minimal linear relation $T_{\min}$ defined in (6.76). The description of self-adjoint extensions and their particular cases is a classic problem in the theory of differential and difference equations, see [14,27,44,51,60,80,81,95,98,119,124,126,127,147,159,160,162,169] and [A5]. As in [135,160,162,169], our main result is obtained by using square summable solutions of system ($\mathcal{S}_\lambda$) and the Glazman–Krein–Naimark theory, see again the Appendix of this thesis for general results from the latter theory.

Throughout this chapter we consider system ($\mathcal{S}_\lambda$) with the coefficients specified in Notation 6.1.2 and it is organized as follows. In Section 7.1 we establish a limit point criterion for system ($\mathcal{S}_\lambda$), see Theorem 7.1.1. In Section 7.2 we present the main result, Theorem 7.2.1, concerning the characterization of self-adjoint extensions of the minimal linear relation associated with system ($\mathcal{S}_\lambda$). We apply this to a consideration of the 2×2 (scalar) case for a finite discrete interval and describe the Krein–von Neumann extension explicitly, see Theorems 7.2.7 and 7.2.9, and Example 7.2.8. We note that there is no analogue of Theorems 7.1.1, 7.2.7, 7.2.9 and Example 7.2.8 in the setting of system (2.6). Finally, Section 7.3 is devoted to the proof of Theorem 7.2.1.

7.1 Limit point criterion

As noted in the Appendix, the equality $\tilde{n}_+ = \tilde{n}_-$ is necessary and sufficient for the existence of a self-adjoint extension of the minimal linear relation $T_{\min}$. Since the latter equality is equivalent with $n_+ = n_-$ by Corollary 6.4.14(i), we need to guarantee that the number of linearly independent square summable solutions of system ($\mathcal{S}_\lambda$) is constant on the set $\mathbb{C}\backslash\mathbb{R}$. This condition is trivially satisfied if the discrete interval $\mathcal{I}_\mathbb{Z}$ is finite or if system ($\mathcal{S}_\lambda$) is in the limit circle case for some (and hence for all) $\lambda\in\mathbb{C}$, i.e., $n_\lambda = 2n$, while it can be violated in any other case $n_\lambda < 2n$ with $\mathcal{I}_\mathbb{Z} = [0,\infty)_\mathbb{Z}$, see Remarks 2.4.21 and 6.4.15. Moreover, it was also discussed in Remark 2.4.16 that the classical limit point criterion for linear Hamiltonian differential and difference systems (2.5) and (2.6) utilizes the minimal eigenvalue of the corresponding weight matrix and a similar criterion cannot be derived in the current setting. Nevertheless, in the following theorem we give conditions, which imply the invariance of the limit point case on $\mathbb{C}\setminus\mathbb{R}$ for system ($\mathcal{S}_\lambda$) with the special linear dependence on λ specified in (6.25), i.e., for system (6.26); cf. [134, Theorem 6.2]. This statement is a discrete analogue of [116, Theorem 5.6].

Theorem 7.1.1. *Let $\mathcal{I}_\mathbb{Z} = [0,\infty)_\mathbb{Z}$ and consider system ($\mathcal{S}_\lambda$) with the coefficients given in (6.25), where $\mathcal{B}_k^*\mathcal{C}_k \equiv 0$, $\mathcal{B}_k^*\mathcal{D}_k > 0$, and $\mathcal{W}_k > 0$ for all $k \in \mathcal{I}_\mathbb{Z}$. If there exists $h \in \mathbb{C}(\mathcal{I}_\mathbb{Z})^1$ such that $h_k \geq h > 0$ and*

$$\mathcal{A}_k^*\mathcal{C}_k \geq -h_k\mathcal{W}_{k+1}, \qquad \sum_{k=0}^{\infty} \frac{1}{g_k\sqrt{h_k}} = \infty, \tag{7.1}$$

where $g_k := \max\left\{1, \left\|\mathcal{W}_{k+1}^{-1/2}(\mathcal{B}_k^*\mathcal{D}_k)^{-1/2}\right\|_2\right\}$, and a constant $T \geq 0$ such that

$$\Delta\left(\frac{1}{h_k}\right) g_k \leq \frac{T}{\sqrt{h_k}} \quad \text{for all } k \in \mathcal{I}_\mathbb{Z}, \tag{7.2}$$

then system ($\mathcal{S}_\lambda$) is in the limit point case for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$, i.e., $n_\lambda = n$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

Proof. System ($\mathcal{S}_\lambda$) with the coefficients from (6.25) can be written as (6.26) and the invertibility of $\mathcal{B}_k$ and $\mathcal{W}_k$ on $\mathcal{I}_\mathbb{Z}$ implies the definiteness of this system by Theorem 6.2.5. In accordance with Theorem 2.4.3, we have $n_\lambda = n$ if and only if $\widetilde{\mathcal{Z}}_k(\lambda)\xi \notin \ell^2_\Psi$ for any $\xi \in \mathbb{C}^n\setminus\{0\}$, where $\widetilde{\mathcal{Z}}(\lambda)$ is the same as in Section 6.3, i.e., the $2n \times n$ solution of system ($\mathcal{S}_\lambda$) determined by the initial condition $\widetilde{Z}_0(\lambda) = -\mathcal{J}\alpha^*$ with a given $\alpha \in \Gamma$. Then $z_k := \binom{x_k}{u_k} = \widetilde{\mathcal{Z}}_k(\lambda)\xi$ with $x, u \in \mathbb{C}([0,\infty)_\mathbb{Z})^n$ satisfies $z_0^*\mathcal{J}z_0 = 0$. Moreover, it is sufficient to consider only $\lambda = \pm i$, because the number $n_\lambda \geq n$ is constant in $\mathbb{C}_+$ and $\mathbb{C}_-$ by Corollary 6.4.14(iii). Hence, let $\xi \in \mathbb{C}^n\setminus\{0\}$ and $\lambda \in \{\pm i\}$ be fixed. We show that under the current assumptions we have $z \notin \ell^2_\Psi$.

Let us assume that $z \in \ell^2_\Psi$. By a direct calculation, we obtain from the block structure of the system at hand, see (6.26), and from the symplecticity of the matrix $\mathcal{S}_k$, see identities (1.17) and (1.18), that

$$\Delta(x_k^* u_k) = -x_{k+1}^*\mathcal{A}_k^*\mathcal{C}_k x_{k+1} - x_{k+1}^*\mathcal{C}_k^*\mathcal{B}_k u_{k+1} - u_{k+1}^*\mathcal{B}_k^*\mathcal{C}_k x_{k+1} - u_{k+1}^*\mathcal{B}_k^*\mathcal{D}_k u_{k+1} - \lambda x_k^*\mathcal{W}_k x_k.$$

Since $\mathcal{B}_k^*\mathcal{D}_k > 0$ and $h_k > 0$, the quantity $\mathcal{F}_k(x,u) := \left(\sum_{j=0}^k \frac{1}{h_j} u_{j+1}^*\mathcal{B}_j^*\mathcal{D}_j u_{j+1}\right)^{1/2} \geq 0$ is well-defined. Then the latter equality and the assumption $\mathcal{B}_k^*\mathcal{C}_k \equiv 0$ yield

$$\mathcal{F}_k^2(x,u) = -\sum_{j=0}^k \frac{1}{h_j} x_{j+1}^*\mathcal{A}_j^*\mathcal{C}_j x_{j+1} - \lambda\sum_{j=0}^k \frac{1}{h_j} x_j^*\mathcal{W}_j x_j - \sum_{j=0}^k \frac{1}{h_j}\Delta(x_j^* u_j). \tag{7.3}$$

From the Hermitian property and positive definiteness of $\mathcal{W}_k$ and $\mathcal{B}_k^*\mathcal{D}_k$, the Cauchy–Schwarz inequality, inequality (1.8), and the definition of g_k we obtain

$$\begin{aligned} |x_{k+1}^* u_{k+1}| &= \left|\left(\mathcal{W}_{k+1}^{1/2} x_{k+1}\right)^* \mathcal{W}_{k+1}^{-1/2}(\mathcal{B}_k^*\mathcal{D}_k)^{-1/2}(\mathcal{B}_k^*\mathcal{D}_k)^{1/2}u_{k+1}\right| \\ &\leq \left\|\mathcal{W}_{k+1}^{1/2} x_{k+1}\right\|_2 \times \left\|\mathcal{W}_{k+1}^{-1/2}(\mathcal{B}_k^*\mathcal{D}_k)^{-1/2}(\mathcal{B}_k^*\mathcal{D}_k)^{1/2}u_{k+1}\right\|_2 \\ &\leq \left\|\mathcal{W}_{k+1}^{1/2} x_{k+1}\right\|_2 \times \left\|\mathcal{W}_{k+1}^{-1/2}(\mathcal{B}_k^*\mathcal{D}_k)^{-1/2}\right\|_\sigma \times \left\|(\mathcal{B}_k^*\mathcal{D}_k)^{1/2}u_{k+1}\right\|_2 \\ &\leq g_k\left\|\mathcal{W}_{k+1}^{1/2} x_{k+1}\right\|_2 \times \left\|(\mathcal{B}_k^*\mathcal{D}_k)^{1/2}u_{k+1}\right\|_2. \end{aligned} \tag{7.4}$$

Hence the latter inequality, assumption (7.2), the Cauchy–Schwarz inequality, and the inequality of arithmetic and geometric means $\sqrt{ab} \le \frac{a+b}{2}$ yield

$$\begin{aligned}
\left|\sum_{j=0}^{k} \Delta\left(\frac{1}{h_j}\right) x_{j+1}^* u_{j+1}\right| &\le \sum_{j=0}^{k} \Delta\left(\frac{1}{h_j}\right) |x_{j+1}^* u_{j+1}| \le \sum_{j=0}^{k} \Delta\left(\frac{1}{h_j}\right) g_j \left\|\mathscr{W}_{j+1}^{1/2} x_{j+1}\right\|_2 \times \left\|(\mathscr{B}_j^* \mathscr{D}_j)^{1/2} u_{j+1}\right\|_2 \\
&\le \sum_{j=0}^{k} T \left\|\mathscr{W}_{j+1}^{1/2} x_{j+1}\right\|_2 \times h_j^{-1/2} \left\|(\mathscr{B}_j^* \mathscr{D}_j)^{1/2} u_{j+1}\right\|_2 \\
&\le \left(T^2 \sum_{j=0}^{k} \left\|\mathscr{W}_{j+1}^{1/2} x_{j+1}\right\|_2^2\right)^{1/2} \times \left(\sum_{j=0}^{k} h_j^{-1} \left\|(\mathscr{B}_j^* \mathscr{D}_j)^{1/2} u_{j+1}\right\|_2^2\right)^{1/2} \\
&\le \frac{1}{2}\left(T^2 \sum_{j=0}^{k} \left\|\mathscr{W}_{j+1}^{1/2} x_{j+1}\right\|_2^2 + \sum_{j=0}^{k} h_j^{-1} \left\|(\mathscr{B}_j^* \mathscr{D}_j)^{1/2} u_{j+1}\right\|_2^2\right) \\
&\le \tfrac{1}{2}\left(T^2 \|z\|_\psi^2 + \mathscr{F}_k^2(x,u)\right). \qquad (7.5)
\end{aligned}$$

By using the summation by parts together with the assumption $h_k \ge h$ and the inequalities from (7.4) and (7.5) we get

$$\begin{aligned}
\left|\operatorname{re}\left[\sum_{j=0}^{k} \frac{1}{h_j} \Delta(x_j^* u_j)\right]\right| &\le \left|\sum_{j=0}^{k} \frac{1}{h_j} \Delta(x_j^* u_j)\right| \le \left|\left[x_j^* u_j / h_j\right]_0^{k+1} - \sum_{j=0}^{k} \Delta\left(\frac{1}{h_j}\right) x_{j+1}^* u_{j+1}\right| \\
&\le |x_0^* u_0 / h_0| + |x_{k+1}^* u_{k+1} / h_{k+1}| + \left|\sum_{j=0}^{k} \Delta\left(\frac{1}{h_j}\right) x_{j+1}^* u_{j+1}\right| \\
&\le T_1 + \frac{1}{h} g_k \left\|\mathscr{W}_{k+1}^{1/2} x_{k+1}\right\|_2 \times \left\|(\mathscr{B}_k^* \mathscr{D}_k)^{1/2} u_{k+1}\right\|_2 + \tfrac{1}{2}\left(T^2 \|z\|_\psi^2 + \mathscr{F}_k^2(x,u)\right), \quad (7.6)
\end{aligned}$$

where $T_1 := |x_0^* u_0 / h_0|$. Since

$$\operatorname{re}\left(\mathscr{F}_k^2(x,u)\right) = -\sum_{j=0}^{k} \frac{1}{h_j} x_{j+1}^* \mathscr{A}_j^* \mathscr{C}_j x_{j+1} - \operatorname{re}\left(\sum_{j=0}^{k} \frac{1}{h_j} \Delta(x_j^* u_j)\right)$$

and the inequality in (7.1) implies

$$-\sum_{j=0}^{k} \frac{1}{h_j} x_{j+1}^* \mathscr{A}_j^* \mathscr{C}_j x_{j+1} \le \sum_{j=0}^{k} x_{j+1}^* \mathscr{W}_j x_{j+1} \le \|z\|_\psi^2,$$

it follows from (7.3) and (7.6) that

$$\frac{1}{2} \sum_{j=0}^{k} g_j^{-1} h_j^{-1/2} \mathscr{F}_j^2(x,u) \le T_2 \sum_{j=0}^{k} g_j^{-1} h_j^{-1/2} + \frac{1}{h} \sum_{j=0}^{k} h_j^{-1/2} \left\|\mathscr{W}_{j+1}^{1/2} x_{j+1}\right\|_2 \times \left\|(\mathscr{B}_j^* \mathscr{D}_j)^{1/2} u_{j+1}\right\|_2,$$

where we put $T_2 := T_1 + (1 + T^2/2)\|z\|_\psi^2$. Then with the aid of the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
G_k := \frac{1}{2} \sum_{j=0}^{k} g_j^{-1} h_j^{-1/2} [\mathscr{F}_j^2(x,u) - 2T_2] &\le \frac{1}{h}\left(\sum_{j=0}^{k} \left\|\mathscr{W}_{j+1}^{1/2} x_{j+1}\right\|_2^2\right)^{1/2} \times \left(\sum_{j=0}^{k} \left\|(\mathscr{B}_j^* \mathscr{D}_j)^{1/2} u_{j+1}\right\|_2^2\right)^{1/2} \\
&\le \frac{1}{h} \|z\|_\psi \mathscr{F}_k(x,u). \qquad (7.7)
\end{aligned}$$

In the next part we show that $\mathcal{F}_k^2(x,u) \leq 2T_2$ for all $k \in \mathcal{I}_{\mathbb{Z}}$. Assume that there exists an index $m \in \mathcal{I}_{\mathbb{Z}}$ such that $\mathcal{F}_m^2(x,u) > 2T_2$. Since $\mathcal{F}_k^2(x,u)$ is nondecreasing, we have $\mathcal{F}_k^2(x,u) - 2T_2 > t$ for all $k \in [m,\infty)_{\mathbb{Z}}$, where $t := \mathcal{F}_m^2(x,u) - 2T_2$. Also G_k is nondecreasing for all $k \in [m-1,\infty)_{\mathbb{Z}}$ and for all $k \in [m,\infty)_{\mathbb{Z}}$ we get from inequality (7.7) and the relation $\mathcal{F}_k^2(x,u) = 2g_k h_k^{1/2}\,\Delta G_{k-1} + 2T_2$ that

$$h^2 - 2\|z\|_\psi^2\, G_k^{-2}\, T_2 \leq 2G_k^{-2}\|z\|_\psi^2\, g_k h_k^{1/2}\,\Delta G_{k-1}. \tag{7.8}$$

In addition, $G_k \geq \frac{t}{2}\sum_{j=0}^k g_j^{-1} h_j^{-1/2} \to \infty$ as $k \to \infty$ by the second part of (7.1). Now, let $0 < a < 2h^2$ be arbitrary and $\ell \in [m,\infty)_{\mathbb{Z}}$ be such that $G_\ell \geq 2\|z\|_\psi T_2^{1/2}/\sqrt{2h^2 - a}$. Then we have $a/2 \leq h^2 - 2G_k^{-2}T_2\|z\|_\psi^2$ for all $k \in [\ell,\infty)_{\mathbb{Z}}$, which together with (7.8) yields for any $k \in [\ell+1,\infty)_{\mathbb{Z}}$ that

$$\begin{aligned}
\frac{a}{2}\sum_{j=\ell+1}^{k}\frac{1}{g_j h_j^{1/2}} &\leq \sum_{j=\ell+1}^{k}\frac{1}{g_j h_j^{1/2}}\Big(h^2 - 2G_j^{-2}T_2\|z\|_\psi^2\Big) \leq \sum_{j=\ell+1}^{k} 2G_j^{-2}\|z\|_\psi^2\,\Delta G_{j-1} \\
&\leq 2\|z\|_\psi^2 \sum_{j=\ell+1}^{k}\frac{\Delta G_{j-1}}{G_j G_{j-1}} \leq -2\|z\|_\psi^2 \sum_{j=\ell+1}^{k}\Delta\Big(\frac{1}{G_{j-1}}\Big) \leq 2\|z\|_\psi^2\,\frac{1}{G_\ell} < \infty.
\end{aligned}$$

But it contradicts the second assumption from (7.1) as $k \to \infty$. Thus it holds $\mathcal{F}_k^2(x,u) \leq 2T_2$ for all $k \in \mathcal{I}_{\mathbb{Z}}$, i.e.,

$$\sum_{j=0}^{\infty} h_j^{-1} u_{j+1}^* \mathcal{B}_j^* \mathcal{D}_j u_{j+1} \leq 2T_2 < \infty. \tag{7.9}$$

Since system ($\mathcal{S}_\lambda$) is definite on $[0,\infty)_{\mathbb{Z}}$, there exists $s \in \mathcal{I}_{\mathbb{Z}}$ such that $\sum_{j=0}^s z_k^* \Psi_k z_k = T_3 > 0$. Hence the positive definiteness of $\mathcal{W}_k$ and the extended Lagrange identity in (6.11) yield

$$\big|z_{k+1}^* \mathcal{J} z_{k+1}\big| = \Big|z_0^* \mathcal{J} z_0 \pm 2i\sum_{j=0}^{k} z_j^* \Psi_j z_j\Big| = 2\Big|\sum_{j=0}^{k} z_j^* \Psi_j z_j\Big| \geq 2\Big|\sum_{j=0}^{s} z_j^* \Psi_j z_j\Big| = 2T_3, \tag{7.10}$$

for any $k \in [s,\infty)_{\mathbb{Z}}$. Simultaneously, we get from (7.4) the estimate

$$\big|z_{k+1}^* \mathcal{J} z_{k+1}\big| \leq 2\big|x_{k+1}^* u_{k+1}\big| \leq 2g_k h_k^{1/2}\,\|\mathcal{W}_{k+1}^{1/2} x_{k+1}\|_2 \times h_j^{-1/2}\,\|(\mathcal{B}_k^* \mathcal{D}_k)^{1/2} u_{k+1}\|_2. \tag{7.11}$$

Inequalities (7.9), (7.10), (7.11) together with the Cauchy–Schwarz inequality imply for any $k \in [s,\infty)_{\mathbb{Z}}$ that

$$\begin{aligned}
\sum_{j=s}^{k}\frac{1}{g_j h_j^{1/2}} &\leq \sum_{j=s}^{k}\frac{2}{\big|z_{j+1}^* \mathcal{J} z_{j+1}\big|}\Big\|\mathcal{W}_{k+1}^{1/2} x_{k+1}\Big\|_2 \times h_j^{-1/2}\Big\|(\mathcal{B}_k^* \mathcal{D}_k)^{1/2} u_{k+1}\Big\|_2 \\
&\leq \frac{1}{T_3}\sum_{j=s}^{k}\Big\|\mathcal{W}_{k+1}^{1/2} x_{k+1}\Big\|_2 \times h_j^{-1/2}\Big\|(\mathcal{B}_k^* \mathcal{D}_k)^{1/2} u_{k+1}\Big\|_2 \\
&\leq \frac{1}{T_3}\Big(\sum_{j=s}^{k}\Big\|\mathcal{W}_{k+1}^{1/2} x_{k+1}\Big\|_2^2\Big)^{1/2} \times \Big(\sum_{j=s}^{k} h_j^{-1}\Big\|(\mathcal{B}_k^* \mathcal{D}_k)^{1/2} u_{k+1}\Big\|_2^2\Big)^{1/2} \\
&\leq \frac{1}{T_3}\|z\|_\psi \sqrt{2}\, T_2^{1/2} < \infty,
\end{aligned}$$

which (again) contradicts the second condition in (7.1) as $k \to \infty$. Hence $z \notin \ell^2_\psi$. Since ξ and λ were chosen arbitrarily, it follows that $\widetilde{Z}(\lambda)\xi \notin \ell^2_\psi$ for any $\xi \in \mathbb{C}^n \setminus \{0\}$. Therefore, system ($\mathcal{S}_\lambda$) is in the limit point case for $\lambda \in \{\pm i\}$ and consequently for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. ■

Upon applying Theorem 7.1.1 to system (6.22) with $q_k \equiv 0$ we obtain the following corollary for a special case of the second order Sturm–Liouville difference equation (6.24), because one easily observes that $z(\lambda) \in \ell^2_\psi$ if and only if $\sum_{k=0}^\infty |y_k(\lambda)|^2 w_k < \infty$, where $z_k(\lambda)$ are ψ_k are as in Example 6.2.4.

Corollary 7.1.2. *Let $\mathcal{I}_\mathbb{Z} = [0,\infty)_\mathbb{Z}$ and consider equation (6.24) with $q_k \equiv 0$, $p_k < 0$ and $w_k > 0$ for all $k \in \mathcal{I}_\mathbb{Z}$. If there exist $h_k \in \mathbb{C}(\mathcal{I}_\mathbb{Z})^1$ and a constant $T \geq 0$ such that $h_k \geq h > 0$ and*

$$\sum_{k=0}^{\infty} \frac{1}{g_k\sqrt{h_k}} = \infty, \qquad \Delta\Big(\frac{1}{h_k}\Big) g_k \leq \frac{T}{\sqrt{h_k}}, \qquad k \in \mathcal{I}_\mathbb{Z},$$

where $g_k := \max\Big\{1, \Big(-\frac{p_{k+1}}{w_{k+1}}\Big)^{1/2}\Big\}$, then equation (6.24) is in the limit point case for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$, i.e., there exists only one nontrivial solution satisfying $\sum_{k=0}^\infty |y_k(\lambda)|^2 w_k < \infty$.

It was shown in [96, Theorem 10], see also [157, Corollary 3.1], that equation (6.24) with $p_k \neq 0$ and $w_k > 0$ is in the limit point case for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ if $\sum_{k=0}^\infty \frac{(w_k w_{k+1})^{1/2}}{|p_{k+1}|} = \infty$. Corollary 7.1.2 partially generalizes this classical limit-point criterion as shown in the following example; cf. [A23, Example 3.4].

Example 7.1.3. Let us consider the equation

$$(6.24), \quad p_k \equiv -1, \quad q_k \equiv 0, \quad w_k = 1/(k+1)^2, \quad \mathcal{I}_\mathbb{Z} = [0,\infty)_\mathbb{Z}. \tag{7.12}$$

Then the criterion from [96, Theorem 10] cannot be applied, because

$$\sum_{k=0}^{\infty} \frac{\big(w_k w_{k+1}\big)^{1/2}}{|p_{k+1}|} = \sum_{k=0}^{\infty} \sqrt{\frac{1}{(k+1)^2(k+2)^2}} = 1 < \infty.$$

On the other hand, the assumptions of Corollary 7.1.2 are satisfied with $h_k \equiv 1$, $g_k = k+2$, and $T = 0$, i.e., equation (7.12) is in the limit point case for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. This fact can be verified directly by using the Weyl alternative, see e.g. [9, Theorem 5.6.1] or Corollary 2.4.23. More precisely, equation (7.12) with $\lambda = 0$ possesses two linearly independent solutions $y_k^{[1]} \equiv 1$ and $y_k^{[2]} = k$ for $k \in \mathcal{I}_\mathbb{Z} \cup \{-1\}$. Since only $y^{[1]}$ is square summable with respect to w_k, it follows from the Weyl alternative that equation (7.12) has to be in the limit point case for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. ▲

7.2 Main results

According to Corollary 6.4.14(iii), the number of linearly independent square summable solutions of system ($\mathcal{S}_\lambda$) is constant in the upper and lower half-planes of $\mathbb{C}$. Hence the numbers $n_+ := n_\lambda$ for $\lambda \in \mathbb{C}_+$ and $n_- := n_\lambda$ for $\lambda \in \mathbb{C}_-$ are well-defined. Let $\lambda_0 \in \mathbb{C}_+$ be fixed. Then system ($\mathcal{S}_{\lambda_0}$) has n_+ linearly independent square summable solutions, which we denote by $s^{[1]}(\lambda_0), \dots, s^{[n_+]}(\lambda_0)$, and similarly system ($\mathcal{S}_{\bar\lambda_0}$) has n_- linearly independent square summable solutions, which we denote by $s^{[1]}(\bar\lambda_0), \dots, s^{[n_-]}(\bar\lambda_0)$. Let

$$\begin{gathered}\varphi_k^{[i]} := s_k^{[i]}(\lambda_0), \quad \varphi_k^{[j+n_+]} := s_k^{[j]}(\bar\lambda_0), \quad i \in \{1,\dots,n_+\}, \quad j \in \{1,\dots,n_-\}, \quad k \in \mathcal{I}_\mathbb{Z}^+, \\ \Theta_k^+ := (\varphi_k^{[1]},\dots,\varphi_k^{[n_+]}), \quad \Theta_k^- := (\varphi_k^{[1+n_+]},\dots,\varphi_k^{[p]}), \quad \text{and} \quad p := n_+ + n_-.\end{gathered} \tag{7.13}$$

Note that $2n \leq \mathscr{n} \leq 4n$. Moreover, if system ($\mathcal{S}_\lambda$) is definite on $\mathcal{I}_\mathbb{Z}$, then the solutions $\varphi^{[1]}, \dots, \varphi^{[\mathscr{n}]}$ belong to different equivalence classes. Hence for all $i \in \{1, \dots, \mathscr{n}_+\}$ and $j \in \{\mathscr{n}_+ + 1, \dots, \mathscr{n}\}$ we have $\{\widetilde{\varphi}^{[i]}, \lambda_0 \widetilde{\varphi}^{[i]}\} \in T_{\max}$ and $\{\widetilde{\varphi}^{[j]}, \bar{\lambda}_0 \widetilde{\varphi}^{[j]}\} \in T_{\max}$ with $\widetilde{\varphi}^{[\ell]} \equiv \varphi^{[\ell]}$ for $\ell \in \{1, \dots, \mathscr{n}\}$. We also define the matrix $\Omega := \big((\Theta^+, \Theta^-), (\Theta^+, \Theta^-)\big)_{N+1}$, i.e.,

$$\Omega = \begin{pmatrix} \Omega^{[1,1]} & \Omega^{[1,2]} \\ \Omega^{[2,1]} & \Omega^{[2,2]} \end{pmatrix} = \begin{pmatrix} (\varphi^{[1]}, \varphi^{[1]})_{N+1} & \dots & (\varphi^{[1]}, \varphi^{[\mathscr{n}]})_{N+1} \\ \vdots & \ddots & \vdots \\ (\varphi^{[\mathscr{n}]}, \varphi^{[1]})_{N+1} & \dots & (\varphi^{[\mathscr{n}]}, \varphi^{[\mathscr{n}]})_{N+1} \end{pmatrix} \in \mathbb{C}^{\mathscr{n} \times \mathscr{n}}, \tag{7.14}$$

where $\Omega^{[1,2]} \in \mathbb{C}^{\mathscr{n}_+ \times \mathscr{n}_-}$. The elements $\omega_{ij} := (\varphi^{[i]}, \varphi^{[j]})_{N+1}$ exist finite for all $i, j \in \{1, \dots, \mathscr{n}\}$ by identity (6.14). Furthermore, from (6.12) one easily concludes that the matrix $\Omega^{[1,2]}$ consists of the elements $(\varphi^{[i]}, \varphi^{[j]})_{N+1} = (\varphi^{[i]}, \varphi^{[j]})_0$ for $i \in \{1, \dots, \mathscr{n}_+\}$ and $j \in \{\mathscr{n}_+ + 1, \dots, \mathscr{n}\}$.

Identity (6.84) implies that for any $\{\tilde{z}, \tilde{f}\} \in T_{\max}$ we have

$$\hat{z}_k = \hat{v}_k + \sum_{j=1}^{\mathscr{n}} \xi_j \varphi_k^{[j]}, \quad k \in \mathcal{I}_\mathbb{Z}^+, \tag{7.15}$$

where $\hat{v} \in \operatorname{dom} T_{\min}$ and $\xi_1, \dots, \xi_{\mathscr{n}} \in \mathbb{C}$ are determined uniquely. Especially, for the pairs $\{\tilde{z}^{[1]}, \tilde{f}^{[1]}\}, \dots, \{\tilde{z}^{[2n]}, \tilde{f}^{[2n]}\} \in T_{\max}$ (see Lemma 6.4.3), we get the unique expression

$$\hat{z}_k^{[i]} = \hat{v}_k^{[i]} + \sum_{j=1}^{\mathscr{n}} \xi_{i,j} \varphi_k^{[j]}, \quad k \in \mathcal{I}_\mathbb{Z}^+, \quad i \in \{1, \dots, 2n\}. \tag{7.16}$$

If we put $Z_k := (\hat{z}_k^{[1]}, \dots, \hat{z}_k^{[2n]})$ for $k \in \mathcal{I}_\mathbb{Z}^+$, then identity (7.16) implies

$$Z_k = V_k + (\Theta_k^+,\ \Theta_k^-)\, \Xi^\top, \tag{7.17}$$

where $V_k := (\hat{v}_k^{[1]}, \dots, \hat{v}_k^{[2n]}) \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n \times 2n}$ and the matrix $\Xi \in \mathbb{C}^{2n \times \mathscr{n}}$ consists of $\xi_{i,j}$. In particular, for $k = 0$ we obtain $I = V_0 + (\Theta_0^+,\ \Theta_0^-)\, \Xi^\top$, which together with (6.81) yields $I = (\Theta_0^+,\ \Theta_0^-)\, \Xi^\top$, i.e., $\operatorname{rank} \Xi = 2n$ by the second inequality in (1.3). From the definition of $\hat{z}^{[i]}$ (see Lemma 6.4.3), its expression in (7.16), and identity (6.81) we have

$$0 = (\hat{z}^{[i]}, \varphi^{[\ell]})_{N+1} = (\hat{v}^{[i]}, \varphi^{[\ell]})_{N+1} + \sum_{j=1}^{\mathscr{n}} \overline{\xi_{i,j}} (\varphi^{[j]}, \varphi^{[\ell]})_{N+1} = \sum_{j=1}^{\mathscr{n}} \overline{\xi_{i,j}} (\varphi^{[j]}, \varphi^{[\ell]})_{N+1}$$

for all $i \in \{1, \dots, 2n\}$ and any $\ell \in \{1, \dots, \mathscr{n}\}$, i.e., $\overline{\Xi}\, \Omega = 0$. Since $\operatorname{rank} \Xi = 2n$, the first inequality in (1.3) implies

$$\operatorname{rank} \Omega \leq \mathscr{n} - 2n.$$

On the other hand, the equality $\Omega^{[1,2]} = \Theta_0^{+*} \mathcal{J}\, \Theta_0^-$ and the first inequality in (1.3) yield

$$\operatorname{rank} \Omega^{[1,2]} \geq \mathscr{n} - 2n.$$

Therefore $\operatorname{rank} \Omega = \mathscr{n} - 2n = \operatorname{rank} \Omega^{[1,2]}$. Since $\mathscr{n} - 2n \leq \mathscr{n}_+$ and $\mathscr{n} - 2n \leq \mathscr{n}_-$, we may assume, without loss of generality, that $\varphi^{[1]}, \dots, \varphi^{[\mathscr{n}_+]}$ are arranged such that

$$\operatorname{rank} \Omega^{[1,2]}_{\mathscr{n}-2n,\, \mathscr{n}_-} = \mathscr{n} - 2n. \tag{7.18}$$

The main result concerning the characterization of all self-adjoint extension of $T_{\min}$ is stated in the following theorem and its proof is given in Section 7.3; cf. [135, Theorem 5.7]. Recall that for the existence of a self-adjoint extension it is essential to have $\mathscr{n}_+ = \mathscr{n}_-$.

Theorem 7.2.1. *Let system* ($\mathcal{S}_\lambda$) *be definite on the discrete interval* $\mathcal{I}_\mathbb{Z}$, *equality* $n_+ = n_- =: q$ *hold and assume that the solutions* $\varphi^{[1]}, \dots, \varphi^{[q]}$ *are arranged such that* (7.18) *holds. Then a linear relation* $T \subseteq \tilde{\ell}^{2\times 2}_{\Psi}$ *is a self-adjoint extension of* $T_{\min}$ *if and only if there exist matrices* $M \in \mathbb{C}^{q\times 2n}$ *and* $L \in \mathbb{C}^{q\times(2q-2n)}$ *such that*

$$\operatorname{rank}(M, L) = q, \quad M\mathcal{J}M^* - L\,\Omega_{2q-2n}L^* = 0, \tag{7.19}$$

and

$$T = \left\{ \{\tilde{z}, \tilde{f}\} \in T_{\max} \mid M\hat{z}_0 - L \begin{pmatrix} (\varphi^{[1]}, \hat{z})_{N+1} \\ \vdots \\ (\varphi^{[2q-2n]}, \hat{z})_{N+1} \end{pmatrix} = 0 \right\}. \tag{7.20}$$

Remark 7.2.2. If, in addition to the assumptions of Theorem 7.2.1, there exists $\nu \in \mathbb{R}$ such that ($\mathcal{S}_\nu$) has q linearly independent square summable solutions $\theta^{[1]}, \dots, \theta^{[q]}$ (we suppress the argument ν), then the statement of Theorem 7.2.1 can be formulated by using these solutions, which are (without loss of generality) arranged such that the submatrix Υ_{2q-2n} has the full rank, where

$$\Upsilon := \begin{pmatrix} (\theta^{[1]}, \theta^{[1]})_{N+1} & \dots & (\theta^{[1]}, \theta^{[q]})_{N+1} \\ \vdots & \ddots & \vdots \\ (\theta^{[q]}, \theta^{[1]})_{N+1} & \dots & (\theta^{[q]}, \theta^{[q]})_{N+1} \end{pmatrix},$$

see Lemma 7.3.3. Moreover, the Wronskian-type identity (6.12) yields that $\Upsilon = \Theta_0^*\mathcal{J}\Theta_0$, where $\Theta_k := (\theta_k^{[1]}, \dots, \theta_k^{[q]})$ for $k \in \mathcal{I}_\mathbb{Z}^+$.

In the next part we discuss several special cases of Theorem 7.2.1. If system ($\mathcal{S}_\lambda$) is in the limit point case for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$, i.e., $n_+ = n_- = n$, then the boundary conditions at $N+1$ (which is necessary equal to ∞) are superfluous as stated in the following corollary; cf. [135, Theorem 5.9]. This situation occurs, e.g., when the assumptions of Theorem 7.1.1 are satisfied. The proof follows directly from Theorem 7.2.1.

Corollary 7.2.3. *Let system* ($\mathcal{S}_\lambda$) *be definite on the discrete interval* $\mathcal{I}_\mathbb{Z}$ *and* $n_+ = n_- = n$ *hold. Then a linear relation* $T \subseteq \tilde{\ell}^{2\times 2}_{\Psi}$ *is a self-adjoint extension of* $T_{\min}$ *if and only if there exists a matrix* $M \in \mathbb{C}^{n\times 2n}$ *such that*

$$\operatorname{rank} M = n, \quad M\mathcal{J}M^* = 0,$$

and

$$T = \left\{ \{\tilde{z}, \tilde{f}\} \in T_{\max} \mid M\hat{z}_0 = 0 \right\}.$$

As it was already discussed in the previous chapters, if there exists $\lambda_0 \in \mathbb{C}$ with the property $n_{\lambda_0} = 2n$, then system ($\mathcal{S}_\lambda$) is in the limit circle case for all $\lambda \in \mathbb{C}$, i.e., $n_+ = n_- = 2n$. Hence for any $\nu \in \mathbb{R}$ there exist solutions $\theta^{[1]}, \dots, \theta^{[2n]}$ (we again suppress the argument ν) of system ($\mathcal{S}_\nu$), which are linearly independent, square summable, and the fundamental matrix $\Theta \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n\times 2n}$ satisfies $\Theta_0 = I$, which implies $\Upsilon = \mathcal{J}$, i.e., $\operatorname{rank}\Upsilon = 2n$, see Remark 7.2.2. Upon combining the latter remark and Theorem 7.2.1 we obtain the following result; cf. [135, Theorem 5.10].

Corollary 7.2.4. *Let system* ($\mathcal{S}_\lambda$) *be definite on the discrete interval* $\mathcal{I}_\mathbb{Z}$, $\nu \in \mathbb{R}$ *be fixed, and assume that there exists a number* $\lambda_0 \in \mathbb{C}$ *such that* $n_{\lambda_0} = 2n$. *Let* $\Theta \in \mathbb{C}(\mathcal{I}_\mathbb{Z}^+)^{2n\times 2n}$ *be the fundamental matrix of system* ($\mathcal{S}_\nu$) *satisfying* $\Theta_0 = I$ *and denote its columns by* $\theta^{[1]}, \dots, \theta^{[2n]}$, *i.e.,* $\Theta_k = (\theta_k^{[1]}, \dots, \theta_k^{[2n]})$. *Then a linear relation* $T \subseteq \tilde{\ell}^{2\times 2}_{\Psi}$ *is a self-adjoint extension of* $T_{\min}$ *if and only if there exist matrices* $M, L \in \mathbb{C}^{2n\times 2n}$ *such that*

$$\operatorname{rank}(M, L) = 2n, \quad M\mathcal{J}M^* - L\mathcal{J}L^* = 0, \tag{7.21}$$

and
$$T=\left\{\{\tilde z,\tilde f\}\in T_{\max}\mid M\hat z_0-L\begin{pmatrix}(\theta^{[1]},\hat z)_{N+1}\\ \vdots\\ (\theta^{[2n]},\hat z)_{N+1}\end{pmatrix}=0\right\}. \tag{7.22}$$

Especially, if $\mathcal{I}_{\mathbb{Z}}$ is a finite discrete interval, then the equality $n_\lambda=2n$ is trivially satisfied for any $\lambda\in\mathbb{C}$. Therefore we get from Corollary 7.2.4 yet one more special case of Theorem 7.2.1.

Corollary 7.2.5. *Let $\mathcal{I}_{\mathbb{Z}}$ be a finite discrete interval and system ($\mathcal{S}_\lambda$) be definite on $\mathcal{I}_{\mathbb{Z}}$. Then a linear relation $T\subseteq\tilde\ell_\Psi^{2\times 2}$ is a self-adjoint extension of $T_{\min}$ if and only if there exist matrices $M,L\in\mathbb{C}^{2n\times 2n}$ such that*
$$\operatorname{rank}(M,L)=2n,\quad M\mathcal{J}M^*-L\mathcal{J}L^*=0, \tag{7.23}$$
and
$$T=T_{M,L}:=\left\{\{\tilde z,\tilde f\}\in T_{\max}\mid M\hat z_0-L\hat z_{N+1}=0\right\}. \tag{7.24}$$

Proof. By Corollary 7.2.4 every self-adjoint extension of $T_{\min}$ can be expressed as in (7.22) with matrices $M,L\in\mathbb{C}^{2n\times 2n}$ satisfying (7.21). If we put $\tilde L:=L\Theta^*_{N+1}\mathcal{J}\in\mathbb{C}^{2n\times 2n}$, then the matrices $M,\tilde L$ satisfies (7.23) and the linear relation in (7.22) can be written as $T_{M,\tilde L}$. ∎

One can easily observe that a linear relation $T_{M,L}$, i.e., the linear relation given by (7.24) with $M,L\in\mathbb{C}^{2n\times 2n}$ satisfying (7.23), is the same as a linear relation $T_{\mathcal{M},\mathcal{L}}$ determined by the matrices $\mathcal{M}:=CM$ and $\mathcal{L}:=CL$ for an arbitrary invertible matrix $C\in\mathbb{C}^{2n\times 2n}$. We show that the converse is also true, see Remark 7.2.10(i). Moreover, it is well known that all self-adjoint extensions of operators associated with the regular second order Sturm–Liouville differential equations can be expressed by using the separated or coupled boundary conditions, see e.g. [38]. In the last part of this section we show similar results for scalar symplectic systems on a finite interval, i.e., $n=1$ and $\mathcal{I}_{\mathbb{Z}}=[0,N]_{\mathbb{Z}}$ with $N\in\mathbb{N}\cup\{0\}$, and provide a unique representation of all self-adjoint extensions of $T_{\min}$. The main assumptions for this treatment are summarized in the following hypothesis.

Hypothesis 7.2.6. The discrete interval $\mathcal{I}_{\mathbb{Z}}$ is finite, i.e., there exists $N\in\mathbb{N}\cup\{0\}$ such that $\mathcal{I}_{\mathbb{Z}}=[0,N]_{\mathbb{Z}}$, we have $n=1$, system ($\mathcal{S}_\lambda$) is definite on $\mathcal{I}_{\mathbb{Z}}$, and the matrices $M,L\in\mathbb{C}^{2\times 2}$ are such that (7.23) holds.

If Hypothesis 7.2.6 is satisfied, then identity (7.23) implies that
$$\text{either}\quad \operatorname{rank}M=\operatorname{rank}L=2\quad\text{or}\quad \operatorname{rank}M=\operatorname{rank}L=1.$$
Hence we get the following dichotomy on the boundary conditions in (7.24).

Theorem 7.2.7. *Let Hypothesis 7.2.6 be satisfied. Then the following statements hold.*

(i) *A linear relation $T_{M,L}$ given through $M,L\in\mathbb{C}^{2\times 2}$ with $\operatorname{rank}M=1=\operatorname{rank}L$ is a self-adjoint extension of $T_{\min}$ if and only if $T_{M,L}=T_{P,Q}:=\left\{\{\tilde z,\tilde f\}\in T_{\max}\mid P\hat z_0=0=Q\hat z_{N+1}\right\}$, where*
$$P=\begin{pmatrix}\cos\alpha_0 & \sin\alpha_0\\ 0 & 0\end{pmatrix}\quad\text{and}\quad Q=\begin{pmatrix}0 & 0\\ -\sin\alpha_{N+1} & \cos\alpha_{N+1}\end{pmatrix} \tag{7.25}$$
for a unique pair $\alpha_0,\alpha_{N+1}\in[0,\pi)$.

(ii) *A linear relation $T_{M,L}$ given through $M,L\in\mathbb{C}^{2\times 2}$ with $\operatorname{rank}M=2=\operatorname{rank}L$ is a self-adjoint extension of $T_{\min}$ if and only if $T_{M,L}=T_{R,\beta}:=\left\{\{\tilde z,\tilde f\}\in T_{\max}\mid e^{i\beta}R\hat z_0=\hat z_{N+1}\right\}$ with a unique $\beta\in[0,\pi)$ and a symplectic matrix $R\in\mathbb{R}^{2\times 2}$.*

Proof. Since the pairs of matrices P, Q and $\mathrm{e}^{i\beta}R, I$ satisfy (7.23), Corollary 7.2.5 implies that the linear relations $T_{P,Q}$ and $T_{R,\beta}$ are self-adjoint extensions of $T_{\min}$.

(i) Let $T_{M,L}$ be a linear relation given through $M, L \in \mathbb{C}^{2\times 2}$ satisfying (7.23) and with $\operatorname{rank} M = 1 = \operatorname{rank} L$. Since by (1.4) we have $\dim[\operatorname{Ran} M \cap \operatorname{Ran} L] = 0$, it follows that $M\xi = L\eta$ for some $\xi, \eta \in \mathbb{C}^2$ if and only if $M\xi = 0 = L\eta$. Therefore the boundary conditions in (7.24) can be expressed as $M\hat{z}_0 = 0 = L\hat{z}_{N+1}$. The rank condition implies that $M = ab^\top$ and $L = cd^\top$ for some vectors $a, b, c, d \in \mathbb{C}^2 \setminus \{0\}$. Then the equality $M\mathcal{J}M^* = 0 = L\mathcal{J}L^*$ does not depend on the vectors a, c and it is equivalent to $b^\top\mathcal{J}b = 0 = d^\top\mathcal{J}d$, which implies that b and d are (scalar) complex multiples of vectors from $\mathbb{R}^2$. Thus, without loss of generality, the vectors a, c may be chosen such that M, L can be written in the form as in (7.25) for some $\alpha_0, \alpha_{N+1} \in [0, \pi)$. The uniqueness follows from the fact that $\operatorname{cotan}\beta = \operatorname{cotan}\gamma$ with $\beta, \geq \in (0, \pi)$ if and only if $\beta = \gamma$.

(ii) Finally, let $T_{M,L}$ be a linear relation given through $M, L \in \mathbb{C}^{2\times 2}$ satisfying (7.23) and with $\operatorname{rank} M = 2 = \operatorname{rank} L$. Then the boundary conditions in (7.24) can be written as $\hat{z}_{N+1} = K\hat{z}_0$, where $K := L^{-1}M$. Upon applying the second equality in (7.23) we obtain that the matrix K is symplectic, i.e., $K\mathcal{J}K^* = \mathcal{J}$. Therefore, $K^{-1} = -\mathcal{J}K^*\mathcal{J}$ and $|\det K| = 1$, i.e., $\det K = \mathrm{e}^{i\varepsilon}$ for some $\varepsilon \in [0, 2\pi)$, which implies $K^{-1} = \mathrm{e}^{-i\varepsilon}K^{\mathrm{adj}} = -\mathrm{e}^{i\varepsilon}\mathcal{J}K^\top\mathcal{J}$, i.e., $K^{*\top} = \overline{K} = \mathrm{e}^{i\varepsilon}K$. If we put $R := \mathrm{e}^{-i\varepsilon/2}K$, i.e., $K = \mathrm{e}^{i\varepsilon/2}R$, then $\overline{R} = R$ and $\det R = 1$, i.e., $R \in \mathbb{R}^{2\times 2}$ is a symplectic matrix. Uniqueness can be verified by a direct calculation. ■

As an illustration of the last theorem we provide a description of the Krein–von Neumann extension of the minimal linear relation $T_{\min}$ under Hypothesis 7.2.6.

Example 7.2.8. Assume that system ($\mathcal{S}_\lambda$) is such that Hypothesis 7.2.6 holds and that the minimal linear relation $T_{\min}$ is positive, i.e., there exists $c > 0$ such that $\langle \tilde{z}, \tilde{f} \rangle_\Psi \geq c\|\tilde{z}\|_\Psi$ for all $\{\tilde{z}, \tilde{f}\} \in T_{\min}$. Then the Krein–von Neumann self-adjoint extension extension of $T_{\min}$ admits the representation given in (A.14), i.e.,

$$T_K = T_{\min} \dotplus (\operatorname{Ker} T_{\max} \times \{0\}).$$

We show that T_K can be also expressed as in the second part of Theorem 7.2.7 with a suitable matrix R and a number $\beta \in [0, 2\pi)$. By definition,

$$\operatorname{Ker} T_{\max} = \left\{\tilde{z} \in \ell^2_\Psi \mid \{\tilde{z}, \tilde{0}\} \in T_{\max}\right\},$$

i.e., $\hat{z}$ solves system ($\mathcal{S}_0$), i.e., $\mathcal{L}(\hat{z})_k = 0$ on $[0, N]_{\mathbb{Z}}$. Because all solutions of system ($\mathcal{S}_0$) are square summable in this case, the assumption of the definiteness of system ($\mathcal{S}_\lambda$) implies that $\dim \operatorname{Ker} T_{\max} = 2$. If $\tilde{z} \in \operatorname{dom} T_K$, then there exist $\tilde{v} \in \operatorname{dom} T_{\min}$ and $\tilde{r} \in \operatorname{Ker} T_{\max}$ such that $\tilde{z} = \tilde{v} + \tilde{r}$ or

$$\hat{z}_k = \hat{v}_k + \hat{r}_k \quad \text{for all } k \in [0, N+1]_{\mathbb{Z}}, \tag{7.26}$$

where $\hat{z} \in \tilde{z}$, $\hat{v} \in \tilde{v}$, and $\hat{r} \in \tilde{r}$ are the uniquely determined elements. Moreover, we have $\hat{v}_0 = 0 = \hat{v}_{N+1}$ by (6.82) and $\hat{r}_k = \alpha^{[1]}\hat{r}^{[1]}_k + \alpha^{[2]}\hat{r}^{[2]}_k$ for all $k \in [0, N+1]_{\mathbb{Z}}$, where $\hat{r}^{[1]}$ and $\hat{r}^{[2]}$ form a basis of $\operatorname{Ker} T_{\max}$.

Let us define the matrix $\mathcal{G} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := (\mathcal{S}_0 \times \mathcal{S}_1 \times \cdots \times \mathcal{S}_N)^{-1} \in \mathbb{C}^{2\times 2}$. Then one can easily conclude that the matrix $\mathcal{G}$ is symplectic and every solution $z \in \mathbb{C}([0, N+1]_{\mathbb{Z}})^2$ of system ($\mathcal{S}_0$) satisfies

$$z_{N+1} = \mathcal{G} z_0. \tag{7.27}$$

In the following construction we consider two cases: either $b \neq 0$ or $b = 0$.

First, assume that $b \neq 0$. Then there exist two solutions of system ($\mathcal{S}_0$) such that

$$\hat{r}_0^{[1]} = \begin{pmatrix} 0 \\ 1/b \end{pmatrix} \quad \text{and} \quad \hat{r}_0^{[2]} = \begin{pmatrix} 1 \\ -a/b \end{pmatrix}.$$

These solutions are obviously linearly independent and by (7.27) we have

$$\hat{r}_{N+1}^{[1]} = \begin{pmatrix} 1 \\ a/b \end{pmatrix} \quad \text{and} \quad \hat{r}_{N+1}^{[2]} = \begin{pmatrix} 0 \\ c - da/b \end{pmatrix}.$$

If we take these two solutions as a basis of $\operatorname{Ker} T_{\max}$, then (7.26) yields

$$\hat{z}_k = \hat{v}_k + \alpha^{[1]} \hat{r}_k^{[1]} + \alpha^{[2]} \hat{r}_k^{[2]} \quad \text{for all } k \in [0, N+1]_{\mathbb{Z}}.$$

Upon evaluating $\hat{z}_k$ at $k = 0$ and $k = N+1$ we obtain

$$\hat{z}_0 = \begin{pmatrix} \alpha^{[2]} \\ \alpha^{[1]}/b - \alpha^{[2]} a/b \end{pmatrix} \quad \text{and} \quad \hat{z}_{N+1} = \begin{pmatrix} \alpha^{[1]} \\ \alpha^{[1]} d/b + \alpha^{[2]} c - \alpha^{[2]} da/b \end{pmatrix},$$

which for $\hat{z}_k = \left(\begin{smallmatrix} \hat{x}_k \\ \hat{u}_k \end{smallmatrix}\right)$ implies $\alpha^{[1]} = \hat{x}_{N+1}$ and $\alpha^{[2]} = \hat{x}_0$. Therefore

$$\begin{pmatrix} \hat{x}_{N+1} \\ \hat{x}_{N+1}\, d/b + \hat{x}_0\, c - \hat{x}_0\, da/b \end{pmatrix} = \hat{z}_{N+1} = \mathcal{G}\hat{z}_0 = \mathcal{G} \begin{pmatrix} \hat{x}_0 \\ \hat{x}_{N+1}/b - \hat{x}_0\, a/b \end{pmatrix}.$$

It means that $\hat{z} \in \operatorname{dom} T_{R,\beta}$, where $\beta \in [0, \pi)$ is such that $\mathrm{e}^{i\beta} = \sqrt{ad - bc}$, and $R = \mathrm{e}^{-i\beta}\, \mathcal{G}$, i.e., $T_K \subseteq T_{R,\beta}$. On the other hand, the linear relations T_K and $T_{R,\beta}$ are self-adjoint extensions of $T_{\min}$, thus $T_K = T_{R,\beta}$. Especially, if the coefficients a, b, c, d are real, then $T_{R,\beta} = T_{\mathcal{G},0}$.

If $b = 0$, then $\mathcal{G} = \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$ with $|ad| = 1$, i.e., $d \neq 0$. In this case we proceed in the same way with the basis of $\operatorname{Ker} T_{\max}$ given by the solutions $\hat{r}^{[1]}$ and $\hat{r}^{[2]}$ of ($\mathcal{S}_0$) such that

$$\hat{r}_0^{[1]} = \begin{pmatrix} 0 \\ 1/d \end{pmatrix}, \quad \hat{r}_0^{[2]} = \begin{pmatrix} 1 \\ -c/d \end{pmatrix}.$$

Then $\left(\begin{smallmatrix} \hat{x}_0\, a \\ \hat{u}_{N+1} \end{smallmatrix}\right) = \hat{z}_{N+1} = \mathcal{G}\hat{z}_0 = \mathcal{G}\left(\begin{smallmatrix} \hat{x}_0 \\ \hat{u}_{N+1}/d - \hat{x}_0\, c/d \end{smallmatrix}\right)$. This shows (again) that $T_K = T_{R,\beta}$ with $\beta \in [0, \pi)$ being such that $\mathrm{e}^{i\beta} = \sqrt{ad}$, and $R = \mathrm{e}^{-i\beta}\, \mathcal{G}$.

In particular, let $\mathcal{S}_k = \left(\begin{smallmatrix} 1 & -b_k \\ 0 & 1 \end{smallmatrix}\right)$ and $\Psi_k = \left(\begin{smallmatrix} w_k & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $b_k > 0$ and $w_k > 0$ on $[0, N]_{\mathbb{Z}}$. This system is definite on $[0, N]_{\mathbb{Z}}$ and corresponds to the second order Sturm–Liouville difference equation $-\Delta[p_k \Delta y_{k-1}(\lambda)] = \lambda w_k y_k(\lambda)$ with $b_k = 1/p_{k+1}$, see Example 6.2.4. Then $\mathcal{G} = \left(\begin{smallmatrix} 1 & \sum_{k=0}^N b_k \\ 0 & 1 \end{smallmatrix}\right)$ and by the previous part we have

$$T_K = \left\{ \{\hat{z}, \tilde{f}\} \in T_{\max} \mid \hat{z} = \left(\begin{smallmatrix} \hat{x} \\ \hat{u} \end{smallmatrix}\right) \in \mathbb{C}([0, N+1]_{\mathbb{Z}})^2,\ \hat{u}_0 = \hat{u}_{N+1} = \left(\sum_{k=0}^N b_k \right)^{-1} \times (\hat{x}_{N+1} - \hat{x}_0) \right\}.$$

▲

The boundary conditions in Theorem 7.2.7 include four particular cases. Namely, with the notation $\hat{z}_k = (\hat{x}_k, \hat{u}_k)^\top$ we get for $\alpha_0 = 0$ and $\alpha_{N+1} = \pi/2$ the Dirichlet boundary conditions $\hat{x}_0 = 0 = \hat{x}_{N+1}$, while for $\alpha_0 = \pi/2$ and $\alpha_{N+1} = 0$ we have the Neumann boundary conditions $\hat{u}_0 = 0 = \hat{u}_{N+1}$. The choice $R = I$ and $\beta = 0$ yields the periodic boundary conditions $\hat{z}_0 = \hat{z}_{N+1}$ and the choice $R = I$ and $\beta = \pi$ leads to the antiperiodic boundary conditions $\hat{z}_0 = -\hat{z}_{N+1}$.

In the first part of the following theorem we show that any self-adjoint extension of $T_{\min}$ can be described by using the matrices determining the Dirichlet and Neumann boundary conditions. For convenience, we introduce the *general boundary trace map* $\gamma_{M,L}$: $\mathbb{C}(\mathcal{I}_{\mathbb{Z}}^{+})^2 \to \mathbb{C}^2$ as

$$\gamma_{M,L}(\hat{z}) := M\hat{z}_0 - L\hat{z}_{N+1},$$

see also [38]. Then $T_{M,L} = \left\{\{\tilde{z}, \tilde{f}\} \in T_{\max} \mid \gamma_{M,L}(\hat{z}) = 0\right\}$. Especially, for P, Q given in (7.25) we denote $\gamma_x := \gamma_{P,Q}$ for $\alpha_0 = 0$, $\alpha_{N+1} = \pi/2$, i.e., $\gamma_x(\hat{z}) = 0$ abbreviates the Dirichlet boundary conditions, and similarly $\gamma_u := \gamma_{P,Q}$ for $\alpha_0 = \pi/2$, $\alpha_{N+1} = 0$, i.e., $\gamma_u(\hat{z}) = 0$ abbreviates the Neumann boundary conditions. In the second part of this theorem we derive yet another equivalent representation of $T_{M,L}$, which possesses the uniqueness property.

Theorem 7.2.9. *Let Hypothesis 7.2.6 be satisfied. Then the following hold.*

(i) *A linear relation T is a self-adjoint extension of $T_{\min}$ if and only if there exist matrices $F, G \in \mathbb{C}^{2\times2}$ such that*

$$\operatorname{rank}(F, G) = 2, \quad FG^* = GF^* \tag{7.28}$$

and

$$T = T_{F,G} := \left\{\{\tilde{z}, \tilde{f}\} \in T_{\max} \mid F\gamma_x(\hat{z}) + G\gamma_u(\hat{z}) = 0\right\}. \tag{7.29}$$

(ii) *We have $T_{F,G} = T_{\mathcal{F},\mathcal{G}}$, where $\mathcal{F}, \mathcal{G}$ satisfy (7.28), if and only if $\mathcal{F} = CF$ and $\mathcal{G} = CG$ for some invertible matrix $C \in \mathbb{C}^{2\times2}$.*

(iii) *A linear relation T is a self-adjoint extension of $T_{\min}$ if and only if there exists a unitary matrix $V \in \mathbb{C}^{2\times2}$ such that*

$$T = T_V := \left\{\{\tilde{z}, \tilde{f}\} \in T_{\max} \mid i(V - I)\gamma_x(\hat{z}) = (V + I)\gamma_u(\hat{z})\right\}. \tag{7.30}$$

(iv) *We have $T_V = T_{\mathcal{V}}$, where $\mathcal{V} \in \mathbb{C}^{2\times2}$ is a unitary matrix, if and only if $\mathcal{V} = V$.*

Proof. (i) Let T be given by (7.29) with the matrices $F, G \in \mathbb{C}^{2\times2}$ satisfying (7.28). If we put $M := FP_0 + GP_{\pi/2}$ and $L := FQ_{\pi/2} + GQ_0$, where P_0, $P_{\pi/2}$ and Q_0, $Q_{\pi/2}$ are the matrices corresponding to P, Q defined in (7.25). Then $M\mathcal{J}M^* - L\mathcal{J}L^* = FG^* - GF^* = 0$ and $\operatorname{rank}(F, G) = 2$ is equivalent to $\operatorname{rank}(M, L) = 2$. Hence M, L satisfy (7.23). Moreover, for the left-hand side of the boundary conditions in (7.29) we have $F\gamma_x(\hat{z}) + G\gamma_u(\hat{z}) = \gamma_{M,L}(\hat{z})$. Therefore $\{\tilde{z}, \tilde{f}\} \in T_{M,L}$ if and only if $\{\tilde{z}, \tilde{f}\} \in T_{F,G}$, i.e., $T_{F,G}$ is a self-adjoint extension of $T_{\min}$ by Corollary 7.2.5. On the other hand, let T be a self-adjoint extension of $T_{\min}$, i.e., $T = T_{M,L}$ with $M, L \in \mathbb{C}^{2\times2}$ satisfying (7.23). If we put $F := MP_0 - LP_{\pi/2}$ and $G := LQ_0 - MQ_{\pi/2}$, then the conditions in (7.28) hold and $\gamma_{M,L}(\hat{z})$ can be written as in (7.29).

(ii) Sufficiency is clear. Assume that $T_{F,G} = T_{\mathcal{F},\mathcal{G}}$ for two pairs of matrices F, G and $\mathcal{F}, \mathcal{G}$ satisfying (7.28). Then, by (7.29), we have for any $\{\tilde{z}, \tilde{f}\} \in T_{\max}$ that $F\gamma_x(\hat{z}) + G\gamma_u(\hat{z}) = 0$ if and only if $\mathcal{F}\gamma_x(\hat{z}) + \mathcal{G}\gamma_u(\hat{z}) = 0$. It means that $\hat{z}_0, \hat{z}_{N+1}$ solve simultaneously the both systems of algebraic equations with the coefficient matrices F, G and $\mathcal{F}, \mathcal{G}$. It yields the equivalence of systems, which implies an existence of an invertible matrix $C \in \mathbb{C}^{2\times2}$ such that $\mathcal{F} = CF$ and $\mathcal{G} = CG$.

(iii) Let T be given by (7.30) with a unitary matrix $V \in \mathbb{C}^{2\times2}$. If we put $F := \frac{i}{2}(I - V)$ and $G := \frac{1}{2}(I + V)$. Then $FG^* = GF^*$ and, by (1.2), $\operatorname{rank}(F, G) = 2$, i.e., the matrices F, G satisfy (7.28). Since the boundary conditions in (7.30) are equivalent to the boundary conditions in (7.29) with F, G defined above, i.e., $\{\tilde{z}, \tilde{f}\} \in T_{F,G}$ if and only if $\{\tilde{z}, \tilde{f}\} \in T_V$, it follows from the previous part that the linear relation T_V is a self-adjoint extension of $T_{\min}$. On the other hand, let T be a self-adjoint extension of $T_{\min}$. Then, by the part (i), we have $T = T_{F,G}$ with $F, G \in \mathbb{C}^{2\times 2}$ satisfying (7.28). Since by (1.2) and (7.28) we have $\operatorname{rank}(F+iG) = 2$, the matrix $V := (F+iG)^{-1}(iG-F)$ is well-defined. One can directly verify that V is a unitary matrix and the boundary conditions $F\gamma_x(\hat{z}) + G\gamma_u(\hat{z}) = 0$ are satisfied if and only if $i(V-I)\gamma_x(\hat{z}) - (V+I)\gamma_u(\hat{z}) = 0$, i.e., $T_{F,G} = T_V$.

(iv) If $V = \mathcal{V}$, then $T_V = T_{\mathcal{V}}$. On the other hand, assume that $T_V = T_{\mathcal{V}}$ for two unitary matrices $V, \mathcal{V} \in \mathbb{C}^{2\times 2}$. Then $T_{F,G} = T_V = T_{\mathcal{V}} = T_{\mathcal{F},\mathcal{G}}$ with the pairs of matrices F, G and $\mathcal{F}, \mathcal{G}$ being given as in the previous part. Then $V = (F+iG)^{-1}(iG-F)$ and $\mathcal{V} = (\mathcal{F}+i\mathcal{G})^{-1}(i\mathcal{G}-\mathcal{F})$ and by the part (ii) there exists an invertible matrix $C \in \mathbb{C}^{2\times 2}$ such that $\mathcal{F} = CF$ and $\mathcal{G} = CG$. Upon combining these facts we obtain $V = \mathcal{V}$. ∎

Remark 7.2.10.

(i) As a consequence of Theorem 7.2.7(i)-(ii) we obtain that $T_{M,L} = T_{\mathcal{M},\mathcal{L}}$ if and only if $\mathcal{M} = CM$ and $\mathcal{L} = CL$ for some invertible matrix $C \in \mathbb{C}^{2\times 2}$.

(ii) Theorem 7.2.7(iii)-(iv) shows that the map from the set of all 2×2 unitary matrices to the set of all self-adjoint extensions expressed as in (7.30) is a bijection.

7.3 Proof of main result

In this section, a proof is given for Theorem 7.2.1 which utilizes several arguments from the linear algebra and whose main idea goes back to [162]. It is based on a construction of a suitable GKN-set (see Theorem A.1) and on a more convenient expression than that given in (7.15) for elements in $\operatorname{dom} T_{\max}$. Similar results for system (2.6) can be found in [135, Section 4].

Lemma 7.3.1. *Let system* ($\mathcal{S}_\lambda$) *be definite on the discrete interval* $\mathcal{I}_\mathbb{Z}$, $\{\tilde{z}, \tilde{f}\} \in T_{\max}$ *be arbitrary, and* $\varphi^{[1]}, \dots, \varphi^{[n+1]}$ *be arranged such that equality* (7.18) *holds. Then the element* $\hat{z}$ *can be uniquely expressed as*

$$\hat{z}_k = \hat{v}_k + \sum_{i=1}^{2n} \eta_i \hat{z}_k^{[i]} + \sum_{j=1}^{p-2n} \zeta_j \varphi_k^{[j]}, \quad k \in \mathcal{I}_\mathbb{Z}^+, \tag{7.31}$$

where $\hat{v} \in \operatorname{dom} T_{\min}$, $\hat{z}^{[1]}, \dots, \hat{z}^{[2n]}$ *are specified in Lemma* 6.4.3, *and the numbers* $\eta_i, \zeta_j \in \mathbb{C}$ *for all* $i \in \{1, \dots, 2n\}$ *and* $j \in \{1, \dots, p-2n\}$. *Moreover,*

$$\operatorname{rank} \Omega_{p-2n} = p - 2n, \tag{7.32}$$

where Ω *was defined in* (7.14).

Proof. Since (7.18) is satisfied, there exists an invertible matrix $P \in \mathbb{C}^{p\times p}$ such that

$$\Omega P = \begin{pmatrix} I_{p-2n} & 0_{(p-2n)\times 2n} \\ Q & R \end{pmatrix}, \tag{7.33}$$

where $0_{(p-2n)\times 2n}$ stands for the $(p-2n)\times 2n$ zero matrix. If we put $\Xi = (\Xi^{[1]}, \Xi^{[2]})$, where $\Xi^{[1]} \in \mathbb{C}^{2n\times(p-2n)}$ and $\Xi^{[2]} \in \mathbb{C}^{2n\times 2n}$, and multiply (7.33) by $\overline{\Xi}$ from the left, we obtain

$$\overline{\Xi^{[1]}} = -\overline{\Xi^{[2]}}\,Q,$$

i.e., $\overline{\Xi} = \big(-\overline{\Xi^{[2]}}\,Q, \overline{\Xi^{[2]}}\big)$. It implies that $\operatorname{rank} \Xi^{[2]} = 2n$ by the second inequality in (1.3), because $\operatorname{rank} \Xi = 2n$. If we multiply equality (7.17) by $(\Xi^{[2]})^{\top -1}$ from the right, we get

$$Z_k (\Xi^{[2]})^{\top -1} = V_k (\Xi^{[2]})^{\top -1} + \Theta_k^{[1]} \,\Xi^{[1]\top} (\Xi^{[2]})^{\top -1} + \Theta_k^{[2]},$$

where $\Theta_k^{[1]} \in \mathbb{C}^{2n\times(\mathfrak{p}-2n)}$ and $\Theta_k^{[2]} \in \mathbb{C}^{2n\times 2n}$ are such that $(\Theta_k^+, \Theta_k^-) = (\Theta_k^{[1]}, \Theta_k^{[2]})$ for all $k \in \mathcal{I}_{\mathbb{Z}}^+$. It shows that every solution $\varphi^{[\mathfrak{p}-2n+1]}, \dots, \varphi^{[\mathfrak{p}]}$ can be uniquely expressed with $\hat{\vartheta}^{[1]}, \dots, \hat{\vartheta}^{[2n]}$, $\hat{z}^{[1]}, \dots, \hat{z}^{[2n]}$, and $\varphi^{[1]}, \dots, \varphi^{[\mathfrak{p}-2n]}$, i.e.,

$$\varphi_k^{[j]} = \hat{r}_k^{[j]} + \sum_{\ell=1}^{2n} \eta_{j,\ell}\, \hat{z}_k^{[\ell]} + \sum_{s=1}^{\mathfrak{p}-2n} \zeta_{j,s}\, \varphi_k^{[s]}, \quad k \in \mathcal{I}_{\mathbb{Z}}^+, \quad j \in \{\mathfrak{p}-2n+1, \dots, \mathfrak{p}\}, \tag{7.34}$$

for some $\hat{r}^{[j]} \in \operatorname{dom} T_{\min}$ and $\eta_{j,\ell}, \zeta_{j,s} \in \mathbb{C}$. Therefore the expression in (7.31) follows from equality (7.15). Moreover, if we multiply both sides of (7.34) by $\varphi_k^{[i]*}\mathcal{J}$ from the left, where $i \in \{1, \dots, \mathfrak{p}-2n\}$, then

$$(\varphi^{[i]}, \varphi^{[j]})_{N+1} = (\varphi^{[i]}, \hat{\vartheta}^{[j]})_{N+1} + \sum_{\ell=1}^{2n} \eta_{j,\ell}\, (\varphi^{[i]}, \hat{z}^{[\ell]})_{N+1} + \sum_{s=1}^{\mathfrak{p}-2n} \zeta_{j,s}\, (\varphi^{[i]}, \varphi^{[s]})_{N+1}.$$

Hence from (6.81) and the definition of $\hat{z}^{[i]}$ we have

$$\Omega^{[1,2]}_{\mathfrak{p}-2n,\, \mathfrak{n}_-} = \Omega_{\mathfrak{p}-2n}\, T^\top, \tag{7.35}$$

where $T \in \mathbb{C}^{\mathfrak{n}_-\times(\mathfrak{p}-2n)}$ is a matrix consisting of the elements $\zeta_{j,s}$ for $j \in \{\mathfrak{n}_+ + 1, \dots, \mathfrak{p}\}$ and $s \in \{1, \dots, \mathfrak{p}-2n\}$. Since the solutions are arranged such that $\operatorname{rank} \Omega^{[1,2]}_{\mathfrak{p}-2n,\, \mathfrak{n}_-} = \mathfrak{p}-2n$, equality (7.32) follows from identity (7.35) and the second inequality in (1.3). ■

Remark 7.3.2. If we switch the role of $s^{[\cdot]}(\lambda_0)$ and $s^{[\cdot]}(\bar{\lambda}_0)$ in the definition of $\varphi^{[1]}, \dots, \varphi^{[\mathfrak{p}]}$ in (7.13), i.e., we put $\varphi^{[i]} = w^{[i]}(\bar{\lambda}_0)$ for $i \in \{1, \dots, \mathfrak{n}_-\}$ and $\varphi^{[j+\mathfrak{n}_-]} = v^{[j]}(\lambda_0)$ for $j \in \{1, \dots, \mathfrak{n}_+\}$, then the solutions $\varphi^{[1]}, \dots, \varphi^{[\mathfrak{n}_-]}$ can be arranged such that (7.31) and (7.32) hold.

Now we give the proof of Theorem 7.2.1.

Proof of Theorem 7.2.1. Assume that T is a self-adjoint extension of $T_{\min}$. Then, by Theorem A.1, there exists a GKN-set $\{\beta_j\}_{j=1}^q$ for $(T_{\min}, T_{\max})$ such that (A.12) holds. Since $\beta_j \in T_{\max}$, they may be identified as $\beta_j = \{\widehat{w}^{[j]}, \tilde{h}^{[j]}\} \in T_{\max}$. By Lemma 7.3.1, the elements $\widehat{w}^{[j]}$ can be uniquely expressed as

$$\widehat{w}_k^{[j]} = \hat{\vartheta}_k^{[j]} + \sum_{i=1}^{2n} \eta_{j,i}\, \hat{z}_k^{[i]} + \sum_{l=1}^{2q-2n} \zeta_{j,l}\, \varphi_k^{[l]}, \quad k \in \mathcal{I}_{\mathbb{Z}}^+, \tag{7.36}$$

where $\hat{\vartheta}^{[j]} \in \operatorname{dom} T_{\min}$ and $\eta_{j,i}, \zeta_{j,l} \in \mathbb{C}$. We next show that the matrices

$$M := \left(\widehat{w}_0^{[1]}, \dots, \widehat{w}_0^{[q]}\right)^* \mathcal{J} \in \mathbb{C}^{q\times 2n} \quad \text{and} \quad L := \begin{pmatrix} \overline{\zeta_{1,1}} & \cdots & \overline{\zeta_{1,2q-2n}} \\ \vdots & \ddots & \vdots \\ \overline{\zeta_{q,1}} & \cdots & \overline{\zeta_{q,2q-2n}} \end{pmatrix} \in \mathbb{C}^{q\times(2q-2n)}$$

satisfy the relations in (7.19).

Since $\operatorname{rank}(M, L) \le q$, let us assume that $\operatorname{rank}(M, L) < q$. Then there exists a vector $c = (c_1, \dots, c_q)^\top \in \mathbb{C}^q \setminus \{0\}$ such that $c^*(M, L) = 0$, i.e., $c^* M = 0 = c^* L$. If $\widehat{w}_k := \sum_{j=1}^q c_j \widehat{w}_k^{[j]}$ for $k \in \mathcal{I}_{\mathbb{Z}}^+$, then we have $\widehat{w}_0 = \mathcal{J} M^* c = 0$ and also $(\widehat{w}, \varphi^{[i]})_{N+1} = \sum_{j=1}^q \overline{c_j}\, (\widehat{w}^{[j]}, \varphi^{[i]})_{N+1}$ for all $i \in \{1, \dots, 2q-2n\}$. Hence by (7.36) and (6.81) we have

$$\left((\widehat{w}, \varphi^{[1]})_{N+1}, \dots, (\widehat{w}, \varphi^{[2q-2n]})_{N+1}\right) = c^* L\, \Omega_{2q-2n} = 0.$$

But then $(\widetilde{w}, \hat{v})_{N+1} = 0$ for any $\hat{v} \in \operatorname{dom} T_{\max}$, because it can be written as in (7.31). It means that $\widetilde{w} \in \operatorname{dom} T_{\min}$ by (6.81) and hence $\beta_1, \dots, \beta_q$ are linearly dependent in $T_{\max}$ modulo $T_{\min}$, which contradicts the assumption that that $\{\beta_j\}_{j=1}^q$ is a GKN-set. Therefore, the first condition in (7.19) is satisfied.

Next, we see that

$$\begin{pmatrix} (\widetilde{w}^{[1]}, \widetilde{w}^{[1]})_0 & \cdots & (\widetilde{w}^{[1]}, \widetilde{w}^{[q]})_0 \\ \vdots & \ddots & \vdots \\ (\widetilde{w}^{[q]}, \widetilde{w}^{[1]})_0 & \dots & (\widetilde{w}^{[q]}, \widetilde{w}^{[q]})_0 \end{pmatrix} = M \mathcal{J} M^* \tag{7.37}$$

and by using (7.36), (6.81), and the definition of $\hat{z}^{[i]}$, also see that

$$\begin{pmatrix} (\widetilde{w}^{[1]}, \widetilde{w}^{[1]})_{N+1} & \cdots & (\widetilde{w}^{[1]}, \widetilde{w}^{[q]})_{N+1} \\ \vdots & \ddots & \vdots \\ (\widetilde{w}^{[q]}, \widetilde{w}^{[1]})_{N+1} & \dots & (\widetilde{w}^{[q]}, \widetilde{w}^{[q]})_{N+1} \end{pmatrix} = L\, \Omega_{2q-2n} L^*. \tag{7.38}$$

Since $\{\beta_j\}_{j=1}^q$ is a GKN-set, we obtain from (6.67) that

$$0 = [\beta_i : \beta_j] = (\widetilde{w}^{[i]}, \widetilde{w}^{[j]})_k\Big|_0^{N+1}$$

for all $i, j \in \{1, \dots, q\}$. By (7.37) and (7.38), this implies that $M \mathcal{J} M^* - L\, \Omega_{2q-2n} L^* = 0$, and so the second condition in (7.19) is also satisfied.

For any $\hat{z} \in \operatorname{dom} T_{\max}$, we can write

$$\begin{pmatrix} (\widetilde{w}^{[1]}, \hat{z})_0 \\ \vdots \\ (\widetilde{w}^{[q]}, \hat{z})_0 \end{pmatrix} = M \hat{z}_0 \quad \text{and} \quad \begin{pmatrix} (\widetilde{w}^{[1]}, \hat{z})_{N+1} \\ \vdots \\ (\widetilde{w}^{[q]}, \hat{z})_{N+1} \end{pmatrix} = L \begin{pmatrix} (\varphi^{[1]}, \hat{z})_{N+1} \\ \vdots \\ (\varphi^{[2q-2n]}, \hat{z})_{N+1} \end{pmatrix}, \tag{7.39}$$

where the second equality follows from (7.36), (6.81), and the definition of $\hat{z}^{[i]}$. Upon combining (A.12), (6.67), (7.39), we obtain that the linear relation T can be expressed as

$$\begin{aligned} T &= \left\{ \{\tilde{z}, \tilde{f}\} \in T_{\max} \;\middle|\; (\hat{z}, \widetilde{w}^{[j]})_k\Big|_0^{N+1} = 0 \text{ for all } j \in \{1, \dots, q\} \right\} \\ &= \left\{ \{\tilde{z}, \tilde{f}\} \in T_{\max} \;\middle|\; \widetilde{w}_k^{[j]*} \mathcal{J} \hat{z}_k\Big|_0^{N+1} = 0 \text{ for all } j \in \{1, \dots, q\} \right\} \\ &= \left\{ \{\tilde{z}, \tilde{f}\} \in T_{\max} \;\middle|\; M \hat{z}_0 - L \begin{pmatrix} (\varphi^{[1]}, \hat{z})_{N+1} \\ \vdots \\ (\varphi^{[2q-2n]}, \hat{z})_{N+1} \end{pmatrix} = 0 \right\}, \end{aligned}$$

i.e., as written in (7.20).

On the other hand, let $M \in \mathbb{C}^{q \times 2n}$ and $L \in \mathbb{C}^{q \times (2q-2n)}$ satisfy (7.19) and T be the linear relation given by (7.20). We then must show that there exists a GKN-set $\{\beta_j\}_{j=1}^q$ for $(T_{\min}, T_{\max})$ such that T can be expressed as in (A.12). Denote the columns of $\mathcal{J} M^* \in \mathbb{C}^{2n \times q}$ as $\rho_1, \dots, \rho_q$ and the columns of the matrix $(\varphi_k^{[1]}, \dots, \varphi_k^{[2q-2n]}) L^* \in \mathbb{C}^{2n \times q}$ as $w_k^{[1]}, \dots, w_k^{[q]}$, i.e.,

$$\rho_i := \mathcal{J} M^* e_i \quad \text{and} \quad w_k^{[i]} := \sum_{l=1}^{2q-2n} \overline{\eta_{i,l}}\, \varphi_k^{[l]} \quad \text{for all } i \in \{1, \dots, q\}, \tag{7.40}$$

where e_i is the i-th canonical unit vector in $\mathbb{C}^q$ and $\eta_{i,j}$ are the elements of the matrix L for $i \in \{1, \dots, q\}$ and $j \in \{1, \dots, 2q-2n\}$. Then $w^{[i]} \in T_{\max}$ for all $i \in \{1, \dots, q\}$ and, by Lemma 6.4.3, there exist $\beta_i := \{\tilde{r}^{[i]}, \tilde{h}^{[i]}\} \in T_{\max}$ such that

$$\hat{v}_0^{[i]} = \rho_i \quad \text{and} \quad \hat{v}_k^{[i]} = w_k^{[i]}, \quad k \in [b+1, \infty)_{\mathbb{Z}} \cap \mathcal{I}_{\mathbb{Z}}^{+}$$

for all $i \in \{1, \dots, q\}$, where the number b is determined by the finite discrete interval $\mathcal{I}_{\mathbb{Z}}^{\mathrm{D}} := [a, b]_{\mathbb{Z}} \subseteq \mathcal{I}_{\mathbb{Z}}$ with $a, b \in \mathcal{I}_{\mathbb{Z}}$ on which system ($\mathcal{S}_\lambda$) is definite. We next show that $\{\beta_i\}_{i=1}^q$ form a GKN-set for $(T_{\min}, T_{\max})$.

Since the linear independence of $\beta_1, \dots, \beta_q$ in $T_{\max}$ modulo $T_{\min}$ is equivalent to the linear independence of $\hat{v}^{[1]}, \dots, \hat{v}^{[q]}$ in $\operatorname{dom} T_{\max}$ modulo $T_{\min}$, we assume that there exists $C = (c_1, \dots, c_q)^{\top} \in \mathbb{C}^q \setminus \{0\}$ such that

$$\hat{v} := \sum_{j=1}^{q} c_j \hat{v}^{[j]} \in \operatorname{dom} T_{\min}.$$

Then, from (6.81) and (7.40), we have for all $\varphi^{[1]}, \dots, \varphi^{[2q-2n]} \in T_{\max}$ that

$$0 = \left((\hat{v}, \varphi^{[1]})_{N+1}, \dots, (\hat{v}, \varphi^{[2q-2n]})_{N+1}\right) = C^* L \Omega_{2q-2n}.$$

This implies $C^* L = 0$, because Ω_{2q-2n} is assumed to be invertible, see (7.32). Simultaneously we have $\hat{v}_0 = 0$, which yields

$$0 = \hat{v}_0 = \sum_{j=1}^{q} c_j \hat{v}_0^{[j]} = \mathcal{J} M^* C,$$

i.e., $C^* M = 0$, because the matrix $\mathcal{J}$ is invertible. But this means $C^*(M, L) = 0$, which contradicts the first assumption in (7.19).

Next, let

$$Y_k := \begin{pmatrix} (\hat{v}^{[1]}, \hat{v}^{[1]})_k & \cdots & (\hat{v}^{[1]}, \hat{v}^{[q]})_k \\ \vdots & \ddots & \vdots \\ (\hat{v}^{[q]}, \hat{v}^{[1]})_k & \cdots & (\hat{v}^{[q]}, \hat{v}^{[q]})_k \end{pmatrix}.$$

Since it can be directly calculated that $Y_0 = M \mathcal{J} M^*$ and $Y_{N+1} = L \Omega_{2q-2n} L^*$, the second equality in (7.19) implies $Y_0 - Y_{N+1} = 0$. Therefore, by using (6.67), we get

$$[\beta_i : \beta_j] = (\hat{v}^{[i]}, \hat{v}^{[j]})_k \Big|_0^{N+1} = 0,$$

which shows that $\{\beta_i\}_{i=1}^q$ is a GKN-set for $(T_{\min}, T_{\max})$ as defined in the Appendix.

Finally, let $\{\widetilde{w}, \tilde{g}\} \in T_{\max}$ be arbitrary, then

$$M \widetilde{w}_0 = \begin{pmatrix} (\hat{v}^{[1]}, \widetilde{w})_0 \\ \vdots \\ (\hat{v}^{[q]}, \widetilde{w})_0 \end{pmatrix} \quad \text{and} \quad L \begin{pmatrix} (\varphi^{[1]}, \widetilde{w})_{N+1} \\ \vdots \\ (\varphi^{[2q-2n]}, \widetilde{w})_{N+1} \end{pmatrix} = \begin{pmatrix} (\hat{v}^{[1]}, \widetilde{w})_{N+1} \\ \vdots \\ (\hat{v}^{[q]}, \widetilde{w})_{N+1} \end{pmatrix}. \tag{7.41}$$

By (6.67) the condition $[\{\widetilde{w}, \tilde{g}\} : \beta_i] = 0$ is equivalent to

$$(\widetilde{w}, \hat{v}^{[i]})_k \Big|_0^{N+1} = 0 = -(\hat{v}^{[i]}, \widetilde{w})_k \Big|_0^{N+1} \tag{7.42}$$

for all $i \in \{1,\dots,q\}$. Hence, by (7.41), we see that (7.42) can be written as

$$M\widetilde{w}_0 - L\begin{pmatrix} (\varphi^{[1]},\widetilde{w})_{N+1} \\ \vdots \\ (\varphi^{[2q-2n]},\widetilde{w})_{N+1} \end{pmatrix} = 0.$$

Therefore the linear relation T given in (7.20) can be equivalently expressed as in (A.12), which means that T is a self-adjoint extension of $T_{\min}$. ∎

The simplification of Theorem 7.2.1 in the limit circle case is based on the following lemma.

Lemma 7.3.3. *Let system* ($\mathcal{S}_\lambda$) *be definite on the discrete interval $\mathcal{I}_\mathbb{Z}$ and $\varphi^{[1]},\dots,\varphi^{[\mathfrak{n}_+]}$ be arranged as in Lemma 7.3.1. Assume that there exists a number $\nu \in \mathbb{R}$ such that system* ($\mathcal{S}_\nu$) *possesses $q := \max\{\mathfrak{n}_+,\mathfrak{n}_-\}$ linearly independent square summable solutions (suppressing the argument ν) given by $\theta^{[1]},\dots,\theta^{[q]}$. Then these solutions can be arranged such that $\operatorname{rank}\Upsilon_{\mathfrak{p}-2n} = \mathfrak{p}-2n$, where*

$$\Upsilon := \begin{pmatrix} (\theta^{[1]},\theta^{[1]})_{N+1} & \dots & (\theta^{[1]},\theta^{[q]})_{N+1} \\ \vdots & \ddots & \vdots \\ (\theta^{[q]},\theta^{[1]})_{N+1} & \dots & (\theta^{[q]},\theta^{[q]})_{N+1} \end{pmatrix} \in \mathbb{C}^{q\times q}.$$

Moreover, for any $\{\tilde{z},\tilde{f}\} \in T_{\max}$ the element $\hat{z}$ can be uniquely expressed as

$$\hat{z}_k = \hat{v}_k + \sum_{i=1}^{2n} \alpha_i \hat{z}_k^{[i]} + \sum_{j=1}^{\mathfrak{p}-2n} \beta_j \theta_k^{[j]}, k \in \mathcal{I}_\mathbb{Z}^+,$$

where $\hat{v} \in \operatorname{dom} T_{\min}$, $\hat{z}^{[1]},\dots,\hat{z}^{[2n]}$ are given in Lemma 6.4.3, and $\alpha_i,\beta_j \in \mathbb{C}$ for all $i \in \{1,\dots,2n\}$ and $j \in \{1,\dots,\mathfrak{p}-2n\}$.

Proof. Since $\theta^{[1]},\dots,\theta^{[q]} \in \operatorname{dom} T_{\max}$, Lemma 7.3.1 implies that there exist unique numbers $\alpha_{i,j},\beta_{i,\ell} \in \mathbb{C}$ such that

$$\theta_k^{[i]} = \hat{v}_k^{[i]} + \sum_{j=1}^{2n} \alpha_{i,j}\hat{z}_k^{[j]} + \sum_{\ell=1}^{\mathfrak{p}-2n} \beta_{i,\ell}\varphi_k^{[\ell]}, \quad k \in \mathcal{I}_\mathbb{Z}^+, \tag{7.43}$$

where $i \in \{1,\dots,q\}$. Then the definition of $\hat{z}^{[i]}$ and identity (6.81) yield

$$\Upsilon = B\,\Omega_{\mathfrak{p}-2n}\,B^*, \tag{7.44}$$

where the matrix $B = \big(\overline{\beta_{i,j}}\big) \in \mathbb{C}^{q\times(\mathfrak{p}-2n)}$. Hence $\operatorname{rank}\Upsilon \le \mathfrak{p}-2n$ by the first inequality in (1.3). On the other hand, by the Wronskian-type identity in (6.12) we have $\Upsilon = \Theta_0^*\mathcal{J}\Theta_0$, where $\Theta_k := (\theta_k^{[1]},\dots,\theta_k^{[q]})$. Since the solutions $\theta_k^{[1]},\dots,\theta_k^{[q]}$ are linearly independent, we have $\operatorname{rank}\Theta_k = q$ for all $k \in \mathcal{I}_\mathbb{Z}^+$, and hence $\operatorname{rank}\Upsilon \ge \mathfrak{p}-2n$ by the second inequality in (1.3). Therefore $\operatorname{rank}\Upsilon = \mathfrak{p}-2n$, which implies that the solutions $\Theta_k := (\theta_k^{[1]},\dots,\theta_k^{[q]})$ can be arranged such that $\operatorname{rank}\Upsilon_{\mathfrak{p}-2n} = \mathfrak{p}-2n$. In this case, the invertibility of the submatrix $B_{\mathfrak{p}-2n}$ follows from the equality $\Upsilon_{\mathfrak{p}-2n} = B_{\mathfrak{p}-2n}\,\Omega_{\mathfrak{p}-2n}\,B^*_{\mathfrak{p}-2n}$, which is obtained analogously to (7.44). Since from (7.43) we have

$$(\theta_k^{[1]},\dots,\theta_k^{[\mathfrak{p}-2n]}) = (\hat{v}_k^{[1]},\dots,\hat{v}_k^{[\mathfrak{p}-2n]}) + (\hat{z}_k^{[1]},\dots,\hat{z}_k^{[2n]})A^*_{2n,\mathfrak{p}-2n} + (\varphi_k^{[1]},\dots,\varphi_k^{[\mathfrak{p}-2n]})B^*_{q-2n},$$

where $A = \big(\overline{\alpha_{i,j}}\big) \in \mathbb{C}^{q\times 2n}$, the invertibility of $B_{\mathfrak{p}-2n}$ means that $\varphi_k^{[1]},\dots,\varphi_k^{[\mathfrak{p}-2n]}$ can be uniquely expressed by using $\theta_k^{[1]},\dots,\theta_k^{[\mathfrak{p}-2n]}$, $\hat{v}_k^{[1]},\dots,\hat{v}_k^{[\mathfrak{p}-2n]}$, and $\hat{z}_k^{[1]},\dots,\hat{z}_k^{[2n]}$. Upon combining these expressions with (7.31) we obtain the second part of the statement. ∎

7.4 Bibliographical notes

The results of this chapter were published in [A18]. Their generalization to symplectic systems on time scales is one of the goals of the current research as well as an extension of Theorem 7.1.1 to the discrete symplectic systems with $\mathcal{B}_k^* \mathcal{C}_k \not\equiv 0$. The topic of the present section is also closely related to the characterization of the spectrum of self-adjoint linear relations, which represents another goal of our future research.

... the theory of linear relation would seem to have the potential for contributing to the enrichment and clarification of many aspects of operator theory, including those concerned with non-closable or non-densely-defined linear operators.

Ronald Cross, see [45, pg. iii]

Appendix

Linear relations

In this supplementary chapter we recall several results from the theory of linear relations, which are relevant to the content of Chapters 6 and 7. The theory of linear relations has been established as a suitable tool for the study of multivalued or non-densely defined linear operators in a Hilbert space. Its history goes back to [8] and the results were further developed, e.g., in [42,45,46,82]. A *(closed) linear relation* $\mathcal{T}$ in a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathbb{C}$ with the inner product $\langle\cdot,\cdot\rangle$ is a (closed) linear subspace of the product space $\mathcal{H}^2 := \mathcal{H}\times\mathcal{H}$, i.e., the Hilbert space of all ordered pairs $\{z,f\}$ such that $z,f\in\mathcal{H}$. The *domain*, *range*, *kernel*, and the *multivalued part* of $\mathcal{T}$ are respectively defined as

$$\operatorname{dom}\mathcal{T} := \{z\in\mathcal{H} \mid \{z,f\}\in\mathcal{T}\}, \tag{A.1}$$

$$\operatorname{Ran}\mathcal{T} := \Big\{z\in\mathcal{H} \mid \text{there exists } f\in\mathcal{H} \text{ such that } \{z,f\}\in\mathcal{T}\Big\}, \tag{A.2}$$

$$\operatorname{Ker}\mathcal{T} := \{z\in\mathcal{H} \mid \{z,0\}\in\mathcal{T}\}, \quad \operatorname{mul}\mathcal{T} := \Big\{f\in\mathcal{H} \mid \{0,f\}\in\mathcal{T}\Big\}. \tag{A.3}$$

In general, we let $\mathcal{T}(z) := \{f\in\mathcal{H} \mid \{z,f\}\in\mathcal{T}\}$, and note that a linear relation $\mathcal{T}$ is the graph of a linear operator in $\mathcal{H}$ when $\mathcal{T}(0)=\{0\}$, i.e., when the subspace $\operatorname{mul}\mathcal{T}$ is trivial. The *inverse* of $\mathcal{T}$, denoted as $\mathcal{T}^{-1}$, is the linear relation

$$\mathcal{T}^{-1} := \Big\{\{f,z\} \mid \{z,f\}\in\mathcal{T}\Big\}$$

and it satisfies

$$\operatorname{dom}\mathcal{T}^{-1} = \operatorname{Ran}\mathcal{T}, \quad \operatorname{Ran}\mathcal{T}^{-1} = \operatorname{dom}\mathcal{T}, \quad \operatorname{Ker}\mathcal{T}^{-1} = \operatorname{mul}\mathcal{T}, \quad \text{and} \quad \operatorname{mul}\mathcal{T}^{-1} = \operatorname{Ker}\mathcal{T}.$$

By $\overline{\mathcal{T}}$ we mean the closure of $\mathcal{T}$. The *sum* $\mathcal{T}+\mathcal{U}$ and the *algebraic sum* $\mathcal{T}\dotplus\mathcal{U}$ are defined as

$$\mathcal{T}+\mathcal{U} := \Big\{\{z,f+g\} \mid \{z,f\}\in\mathcal{T},\ \{z,g\}\in\mathcal{U}\Big\},$$
$$\mathcal{T}\dotplus\mathcal{U} := \Big\{\{z+y,f+g\} \mid \{z,f\}\in\mathcal{T},\ \{y,g\}\in\mathcal{U}\Big\}.$$

The *adjoint* $\mathcal{T}^*$ of the linear relation $\mathcal{T}$ is the closed linear relation given by

$$\mathcal{T}^* := \Big\{\{y,g\} \in \mathcal{H}^2 \mid \langle z,g\rangle = \langle f,y\rangle \text{ for all } \{z,f\} \in \mathcal{T}\Big\}. \tag{A.4}$$

The definition of $\mathcal{T}^*$ reduces to the standard definition for the graph of the adjoint operator when $\mathcal{T}$ is a densely defined operator. The adjoint linear relation $\mathcal{T}^*$ satisfies

$$\mathcal{T}^* = \left(\overline{\mathcal{T}}\right)^*, \quad \mathcal{T}^{**} = \overline{\mathcal{T}}, \quad \operatorname{Ker} \mathcal{T}^* = (\operatorname{Ran} \mathcal{T})^\perp = \left(\operatorname{Ran} \overline{\mathcal{T}}\right)^\perp, \quad (\operatorname{dom} \mathcal{T})^\perp = \operatorname{mul} \mathcal{T}^*, \tag{A.5}$$

A linear relation $\mathcal{T}$ is said to be *symmetric* (or *Hermitian*) if $\mathcal{T} \subseteq \mathcal{T}^*$, and it is said to be *self-adjoint* if $\mathcal{T}^* = \mathcal{T}$. It is easily seen that $\mathcal{T}$ is a symmetric linear relation if and only if $\langle z,g\rangle = \langle f,y\rangle$ for all $\{z,f\},\{y,g\} \in T$. A symmetric linear relation $\mathcal{T}_1$ is said to be a *self-adjoint extension* of $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}_1$ and $\mathcal{T}_1^* = \mathcal{T}_1$.

For $\lambda \in \mathbb{C}$ and a linear relation $\mathcal{T}$ we define the linear relation

$$\mathcal{T} - \lambda I := \Big\{\{z, f - \lambda z\} \in \mathcal{H}^2 \mid \{z,f\} \in \mathcal{T}\Big\} \tag{A.6}$$

with the property $(\mathcal{T} - \lambda I)^* = \mathcal{T}^* - \bar{\lambda} I$. Then

$$M_\lambda(\mathcal{T}) := \operatorname{Ker}(\mathcal{T}^* - \lambda I) = \{z \in \mathcal{H} \mid \{z, \lambda z\} \in \mathcal{T}^*\} \tag{A.7}$$

is said to be the *defect subspace* of $\mathcal{T}$ and λ. Its dimension, i.e., the number

$$d_\lambda(\mathcal{T}) := \operatorname{Ker}(\mathcal{T}^* - \lambda I), \tag{A.8}$$

is said to be the *deficiency index* of $\mathcal{T}$ and λ. Since

$$\operatorname{Ran}(\mathcal{T} - \bar{\lambda} I)^\perp = \operatorname{Ker}(\mathcal{T}^* - \lambda I)$$

the deficiency indices of $\mathcal{T}$ and $\overline{\mathcal{T}}$ with the same λ are equal by (A.5), see [143, Lemma 2.4].

If $\mathcal{T}$ is a symmetric linear relation, the values of $d_\lambda(\mathcal{T})$ are constant in the open upper and lower half-planes of $\mathbb{C}$, see [143, Theorem 2.13]. Hence we define the *positive* and *negative deficiency* indices as $d_\pm(\mathcal{T}) := d_{\pm i}(\mathcal{T})$. If $\mathcal{T}$ is a closed symmetric linear relation, then for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the following direct sum decomposition (a generalization of the von Neumann formula)

$$\mathcal{T}^* = \mathcal{T} \dotplus \mathcal{M}_\lambda(\mathcal{T}) \dotplus \mathcal{M}_{\bar{\lambda}}(\mathcal{T}) \tag{A.9}$$

holds, where $\mathcal{M}_\lambda(\mathcal{T}) = \Big\{\{z,\lambda z\} \mid \{z,\lambda z\} \in \mathcal{T}^*\Big\}$ and the sum $\dotplus$ is orthogonal for $\lambda = \pm i$, see [116, Proposition 2.22]. A closed symmetric linear relation $\mathcal{T}$ possesses a self-adjoint extension if and only if the positive and negative deficiency indices are equal, i.e., $d_+(\mathcal{T}) = d_-(\mathcal{T})$, see [42, Corollary, pg. 34]. Moreover, it was shown in [116, Lemma 2.25] that

$$d_\lambda(T) \le d_\pm(T), \tag{A.10}$$

whenever $\lambda \in \mathbb{R}$ and $\operatorname{Ker}(\mathcal{T} - \lambda I) = \{0\}$.

Since the characterization of self-adjoint extensions of the minimal linear relation associated with system ($\mathcal{S}_\lambda$), see Chapter 7, is derived by applying the Glazman–Krein–Naimark theory for linear relations, we recall the most fundamental parts of this theory, see [143] for more details. A complex linear space $\mathcal{S}$ with a complex-valued function

$[:] : \mathscr{S} \times \mathscr{S} \to \mathbb{C}$ is called *pre-symplectic* if it possesses the conjugate bilinear and skew-Hermitian properties, i.e., for all $P, Q, R \in \mathscr{S}$ and $\alpha \in \mathbb{C}$ we have

$$[P : Q + R] = [P : Q] + [P : R], \quad [P + Q : R] = [P : R] + [Q : R],$$
$$[\alpha P : Q] = \alpha [P : Q], \quad [P : \alpha Q] = \bar{\alpha} [P : Q],$$
$$[P : Q] = -\overline{[Q : P]},$$

see [72] for more details. If we put $\mathscr{S} = \mathscr{H}^2$ and

$$[\{z, f\} : \{u, g\}] := \langle f, u \rangle - \langle z, g \rangle$$

for $\{z, f\}, \{u, g\} \in \mathscr{H}^2$, then $\mathscr{S}$ and $[:]$ form the pre-symplectic space.

For a symmetric linear relation $\mathcal{T} \subseteq \mathscr{H}^2$ we have

$$[\mathcal{T} : \mathcal{T}] = 0 = [\mathcal{T} : \mathcal{T}^*] \quad \text{and} \quad \overline{\mathcal{T}} = \Big\{ \{z, f\} \in \mathcal{T}^* \mid [\{z, f\} : \mathcal{T}^*] = 0 \Big\}, \tag{A.11}$$

see [143, Theorem 3.5]. If, in addition, the linear relation $\mathcal{T}$ is closed and $d := d_+(\mathcal{T}) = d_-(\mathcal{T})$, then the set $\{\beta_j\}_{j=1}^d$ with $\beta_j \in \mathcal{T}^*$ for $j \in \{1, \ldots, d\}$ such that

1. $\beta_1, \ldots, \beta_d$ are linearly independent in $\mathcal{T}^*$ modulo $\mathcal{T}$,
2. $[\beta_j : \beta_i] = 0$ for all $i, j \in \{1, \ldots, d\}$,

is called *GKN-set* for the pair of linear relations $(\mathcal{T}, \mathcal{T}^*)$. The following theorem provides the necessary and sufficient conditions for a linear relation $\mathcal{T}_1 \subseteq \mathscr{H}^2$ being a self-adjoint extension of $\mathcal{T}$, see [143, Theorem 4.7].

Theorem A.1. *Let $\mathcal{T} \subseteq \mathscr{H}^2$ be a closed symmetric linear relation such that $d_+(\mathcal{T}) = d_-(\mathcal{T}) = d$. A subspace $\mathcal{T}_1 \subseteq \mathscr{H}^2$ is a self-adjoint extension of $\mathcal{T}$ if and only if there exists GKN-set $\{\beta_j\}_{j=1}^d$ for $(\mathcal{T}, \mathcal{T}^*)$ such that*

$$\mathcal{T}_1 = \Big\{ F \in \mathcal{T}^* \mid [F : \beta_j] = 0 \ \textit{for all } j = 1, \ldots, d \Big\}. \tag{A.12}$$

Finally, a linear relation $\mathcal{T}$ is called *semibounded below*, if there exists $a \in \mathbb{R}$ such that

$$\langle z, f \rangle \geq a \langle z, z \rangle \ \text{ for all } \{z, f\} \in \mathcal{T}. \tag{A.13}$$

The number $m(\mathcal{T}) := \sup\{a \in \mathbb{R} \mid \text{inequality (A.13) holds}\}$ is called the *lower bound* of $\mathcal{T}$. If $m(\mathcal{T}) > 0$, the linear relation $\mathcal{T}$ is said to be *positive*. Then, by analogy with the case of densely defined positive symmetric operators (see [43, Theorem 5]), the smallest and largest self-adjoint extensions of a positive symmetric linear relation are respectively known as the *Krein–von Neumann* (or soft) *extension* $\mathcal{T}_K$ and the *Friedrichs* (or hard) *extension* $\mathcal{T}_F$. In particular, if $\mathcal{T}$ is closed and $m(\mathcal{T}) > 0$, then the Krein–von Neumann extension admits the representation

$$\mathcal{T}_K = \mathcal{T} \dotplus (\operatorname{Ker} \mathcal{T}^* \times \{0\}), \tag{A.14}$$

see [43, Corollary 1] and also [84].

We could, of course, use any notation we want; do not laugh at notations; invent them, they are powerful. In fact, mathematics is, to a large extent, invention of better notations.

Richard Phillips Feynman, see [78, pg. 17-7]

List of symbols

The items in the following list are sorted by their pronunciation or LaTeX command. The number refers to the page with the definition (or the first occurrence) of the symbol.

Every mathematical discipline goes through three periods of development: the naive, the formal, and the critical.

David Hilbert, see [131, p. 240]

List of author's publications (as of August 22, 2016)

Items [A1, A2, A4–A21] are indexed in the MathSciNet Database. Moreover, items [A1, A6–A9] were included in the author's doctoral dissertation [A24].

2009

[A1] R. Hilscher and P. Zemánek, *Trigonometric and hyperbolic systems on time scales*, Dynam. Systems Appl. **18** (2009), no. 3-4, 483–505. (Cited on page 145.)

[A2] R. Šimon Hilscher and P. Zemánek, *Definiteness of quadratic functionals for Hamiltonian and symplectic systems: a survey*, Int. J. Difference Equ. **4** (2009), no. 1, 49–67. (Cited on page 145.)

[A3] P. Zemánek, *Discrete trigonometric and hyperbolic systems: An overview*, Ulmer Seminare über Funktionalanalysis und Differentialgleichungen **14** (2009), 345–359. (Cited on page 6.)

2010

[A4] S. L. Clark and P. Zemánek, *On a Weyl–Titchmarsh theory for discrete symplectic systems on a half line*, Appl. Math. Comput. **217** (2010), no. 7, 2952–2976. (Cited on pages 1, 4, 13, 16, 18, 19, 22, 25, 26, 27, 33, 38, 41, 50, 100, 101, 102, 105, and 145.)

[A5] R. Šimon Hilscher and P. Zemánek, *Friedrichs extension of operators defined by linear Hamiltonian systems on unbounded interval*, Math. Bohem. **135** (2010), no. 2, 209–222. (Cited on pages 119 and 145.)

2011

[A6] P. Hasil and P. Zemánek, *Critical second order operators on time scales*, Discrete Contin. Dyn. Syst. **2011** (2011), suppl., 653–659. (Cited on page 145.)

[A7] R. Šimon Hilscher and P. Zemánek, *Weyl–Titchmarsh theory for time scale symplectic systems on half line*, Abstr. Appl. Anal. **2011** (2011), Art. ID 738520, 41 pp. (electronic). (Cited on pages 14 and 145.)

[A8] R. Šimon Hilscher and P. Zemánek, *Overview of Weyl–Titchmarsh theory for second order Sturm–Liouville equations on time scales*, Int. J. Difference Equ. **6** (2011), no. 1, 39–51. (Cited on page 145.)

[A9] P. Zemánek, *Krein-von Neumann and Friedrichs extensions for second order operators on time scales*, Int. J. Dyn. Syst. Differ. Equ. **3** (2011), no. 1–2, 132–144. (Cited on page 145.)

2012

[A10] R. Šimon Hilscher and P. Zemánek, *New results for time reversed symplectic dynamic systems and quadratic functionals*, Electron. J. Qual. Theory Differ. Equ. (2012), no. 15, 11 pp. (electronic). (Cited on page 145.)

[A11] P. Zemánek, *Rofe–Beketov formula for symplectic systems*, Adv. Difference Equ. **2012** (2012), no. 104, 9 pp. (electronic). (Cited on page 145.)

[A12] P. Zemánek and P. Hasil, *Friedrichs extension of operators defined by Sturm–Liouville equations of higher order on time scales*, Appl. Math. Comput. **218** (2012), no. 22, 10829–10842. (Cited on page 145.)

2013

[A13] R. Šimon Hilscher and P. Zemánek, *Weyl disks and square summable solutions for discrete symplectic systems with jointly varying endpoints*, Adv. Difference Equ. **2013** (2013), no. 232, 18 pp. (electronic). (Cited on pages 1, 2, 54, and 145.)

[A14] P. Zemánek, *A note on the equivalence between even-order Sturm–Liouville equations and symplectic systems on time scales*, Appl. Math. Lett. **26** (2013), no. 1, 134–139. (Cited on page 145.)

2014

[A15] R. Šimon Hilscher and P. Zemánek, *Weyl–Titchmarsh theory for discrete symplectic systems with general linear dependence on spectral parameter*, J. Difference Equ. Appl. **20** (2014), no. 1, 84–117. (Cited on pages 1, 39, and 145.)

[A16] R. Šimon Hilscher and P. Zemánek, *Limit point and limit circle classification for symplectic systems on time scales*, Appl. Math. Comput. **233** (2014), 623–646. (Cited on pages 1, 2, 14, 39, 54, and 145.)

[A17] R. Šimon Hilscher and P. Zemánek, *Generalized Lagrange identity for discrete symplectic systems and applications in Weyl–Titchmarsh theory*, in "Theory and Applications of Difference Equations and Discrete Dynamical Systems", Proceedings of the 19th International Conference on Difference Equations and Applications (Muscat, 2013),

Z. AlSharawi, J. Cushing, and S. Elaydi (editors), Springer Proceedings in Mathematics & Statistics, Vol. 102, pp. 187–202, Springer, Berlin, 2014. (Cited on pages 1, 2, 87, 90, and 145.)

2015

[A18] S. L. Clark and P. Zemánek, *On discrete symplectic systems: Associated maximal and minimal linear relations and nonhomogeneous problems*, J. Math. Anal. Appl. **421** (2015), no. 1, 779–805. (Cited on pages 1, 2, 89, 117, 135, and 145.)

[A19] R. Šimon Hilscher and P. Zemánek, *Limit circle invariance for two differential systems on time scales*, Math. Nachr. **288** (2015), no. 5-6, 696–709. (Cited on pages 1, 2, 63, and 145.)

[A20] R. Šimon Hilscher and P. Zemánek, *Time scale symplectic systems with analytic dependence on spectral parameter*, J. Difference Equ. Appl. **21** (2015), no. 3, 209–239. (Cited on pages 1, 2, 73, 87, and 145.)

2016

[A21] P. Zemánek and S. L. Clark, *Characterization of self-adjoint extensions for discrete symplectic systems*, J. Math. Anal. Appl. **440** (2016), no. 1, 323–350. (Cited on pages 1, 2, 89, 117, and 145.)

Accepted

[A22] P. Zemánek, *Limit point criteria for second order Sturm–Liouville equations on time scales*, in "Differential and Difference Equations with Applications", Proceedings of the International Conference on Differential & Difference Equations and Applications 2015 (Amadora, 2015), S. Pinelas, O. Došlý, Z. Došlá, and P. Kloeden (editors), Springer Proceedings in Mathematics & Statistics, Springer, Berlin, to appear. (Not cited.)

Submitted

[A23] P. Zemánek, *Principal solution in Weyl–Titchmarsh theory for second order Sturm–Liouville equation on time scales*, submitted. (Cited on pages 36 and 123.)

Doctoral dissertation

[A24] P. Zemánek, *New Results in Theory of Symplectic Systems on Time Scales*, doctoral dissertation, Masaryk University, Brno, 2011. ISBN 978-80-210-5515-5. (Cited on page 145.)

Bibliography

[1] *The Pentagon: A Mathematics Magazine for Students*, Vol. **XI** (1951), no. 1. (Cited on page 41.)

[2] C. D. Ahlbrandt, *Equivalence of discrete Euler equations and discrete Hamiltonian systems*, J. Math. Anal. Appl. **180** (1993), no. 2, 498–517. (Cited on page 10.)

[3] C. D. Ahlbrandt and A. C. Peterson, *The (n, n)-disconjugacy of a $2n$th order linear difference equation*, in "Advances in Differences Equations" (R. P. Agarwal, editor), Comput. Math. Appl. **28** (1994), no. 1-3, 1–9. (Cited on page 6.)

[4] C. D. Ahlbrandt and A. C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer Texts in the Mathematical Sciences, Vol. 16, Kluwer Academic Publishers Group, Dordrecht, 1996. ISBN 0-7923-4277-1. (Cited on pages 6, 10, and 95.)

[5] D. R. Anderson, *Discrete trigonometric matrix functions*, Panamer. Math. J. **7** (1997), no. 1, 39–54. (Cited on page 6.)

[6] D. R. Anderson, *Titchmarsh–Sims–Weyl theory for complex Hamiltonian systems on Sturmian time scales*, J. Math. Anal. Appl. **373** (2010), no. 2, 709–725. (Cited on page 14.)

[7] T. M. Apostol, *Mathematical Analysis*, second edition, Addison-Wesley Publishing, Reading, 1974. (Cited on page 97.)

[8] R. Arens, *Operational calculus of linear relations*, Pacific J. Math. **11** (1961), 9–23. (Cited on pages 90 and 137.)

[9] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Mathematics in Science and Engineering, Vol. 8, Academic Press, New York, 1964. (Cited on pages 5, 6, 7, 12, 13, 16, 17, 34, 35, 56, 89, and 123.)

[10] H. Behncke, *Spectral theory of Hamiltonian difference systems with almost constant coefficients*, J. Difference Equ. Appl. **19** (2013), no. 1, 1–12. (Cited on pages 13 and 109.)

[11] H. Behncke and F. O. Nyamwala, *Spectral theory of difference operators with almost constant coefficients*, J. Difference Equ. Appl. **17** (2011), no. 5, 677–695. (Cited on page 13.)

[12] H. Behncke and F. O. Nyamwala, *Spectral theory of difference operators with almost constant coefficients II*, J. Difference Equ. Appl. **17** (2011), no. 5, 821–829. (Cited on page 13.)

[13] J. Behrndt, S. Hassi, H. S. V. de Snoo, and R. Wietsma, *Square-integrable solutions and Weyl functions for singular canonical systems*, Math. Nachr. **284** (2011), no. 11-12, 1334–1384. (Cited on pages 90, 94, 96, 106, and 113.)

[14] M. B. Bekker, M. Bohner, and H. Voulov, *Extreme self-adjoint extensions of a semi-bounded q-difference operator*, Math. Nachr. **287** (2014), no. 8-9, 869–884. (Cited on page 119.)

[15] J. M. Berezans'kiĭ, *Expansions in Eigenfunctions of Selfadjoint Operators*, translated from the Russian by R. Bolstein, J. M. Danskin, J. Rovnyak and L. Shulman, Translations of Mathematical Monographs, Vol. 17, American Mathematical Society, Providence, 1968. (Cited on page 13.)

[16] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas*, second edition, Princeton University Press, Princeton, 2009. ISBN 978-0-691-14039-1. (Cited on pages 3, 4, 5, 77, and 78.)

[17] M. Bohner, *Linear Hamiltonian difference systems: Disconjugacy and Jacobi-type conditions*, J. Math. Anal. Appl. **199** (1996), no. 3, 804–826. (Cited on pages 41 and 54.)

[18] M. Bohner, *Symplectic systems and related discrete quadratic functionals*, in "Dedicated to Professor Dragoslav S. Mitrinović (1908–1995)" (Niš, 1996), Facta Univ. Ser. Math. Inform. (1997), no. 12, 143–156. (Cited on page 6.)

[19] M. Bohner, *Discrete linear Hamiltonian eigenvalue problems*, in "Advances in Difference Equations, II", Comput. Math. Appl. **36** (1998), no. 10-12, 179–192. (Cited on page 42.)

[20] M. Bohner and O. Došlý, *Disconjugacy and transformations for symplectic systems*, Rocky Mountain J. Math. **27** (1997), no. 3, 707–743. (Cited on pages 6, 10, and 38.)

[21] M. Bohner and O. Došlý, *Trigonometric transformations of symplectic difference systems*, J. Differential Equations **163** (2000), no. 1, 113–129. (Cited on page 6.)

[22] M. Bohner and O. Došlý, *Trigonometric systems in oscillation theory of difference equations*, in "Dynamic Systems and Aplications, Vol. 3." (Proceedings of the Third International Conference on Dynamic Systems and Applications, Atlanta, GA, 1999), pp. 99–104, Dynamic, Atlanta, 2001. (Cited on page 6.)

[23] M. Bohner, O. Došlý, and W. Kratz, *An oscillation theorem for discrete eigenvalue problems*, Rocky Mountain J. Math. **33** (2003), no. 4, 1233–1260. (Cited on page 13.)

[24] M. Bohner, O. Došlý, and W. Kratz, *Positive semidefiniteness of discrete quadratic functionals*, Proc. Edinb. Math. Soc. (2) **46** (2003), no. 3, 627–636. (Cited on page 6.)

[25] M. Bohner, O. Došlý, and W. Kratz, *Sturmian and spectral theory for discrete symplectic systems*, Trans. Amer. Math. Soc. **361** (2009), no. 6, 3109–3123. (Cited on pages 6 and 13.)

[26] M. Bohner and S. Sun, *Weyl–Titchmarsh theory for symplectic difference systems*, Appl. Math. Comput. **216** (2010), no. 10, 2855–2864. (Cited on pages 1, 13, 25, 26, 38, 41, and 97.)

[27] B. M. Brown and J. S. Christiansen, *On the Krein and Friedrichs extensions of a positive Jacobi operator*, Expo. Math. **23** (2005), no. 2, 179–186. (Cited on page 119.)

[28] B. M. Brown and W. D. Evans, *On an extension of Copson's inequality for infinite series*, Proc. Roy. Soc. Edinburgh Sect. A **121** (1992), no. 1-2, 169–183. (Cited on page 90.)

[29] B. M. Brown, W. D. Evans, and L. L. Littlejohn, *Orthogonal polynomials and extensions of Copson's inequality*, in "Proceedings of the Seventh Spanish Symposium on Orthogonal Polynomials and Applications (VII SPOA, Granada, 1991)", J. Comput. Appl. Math. **48** (1993), no. 1-2, 33–48. (Cited on page 90.)

[30] B. M. Brown, W. D. Evans, and M. Plum, *Titchmarsh–Sims–Weyl theory for complex Hamiltonian systems*, Proc. London Math. Soc. (3) **87** (2003), no. 2, 419–450. (Cited on page 12.)

[31] J. Candy and W. Rozmus, *A symplectic integration algorithm for separable Hamiltonian functions*, J. Comput. Phys. **92** (1991), no. 1, 230–256. (Cited on page 6.)

[32] J. Chen and Y. Shi, *The limit circle and limit point criteria for second-order linear difference equations*, Comput. Math. Appl. **47** (2004), no. 6-7, 967–976. (Cited on page 13.)

[33] S. L. Clark, *A spectral analysis for self-adjoint operators generated by a class of second order difference equations*, J. Math. Anal. Appl. **197** (1996), no. 1, 267–285. (Cited on page 13.)

[34] S. L. Clark and F. Gesztesy, *Weyl–Titchmarsh M-function asymptotics for matrix-valued Schrödinger operators*, Proc. London Math. Soc. (3) **82** (2001), no. 3, 701–724. (Cited on page 12.)

[35] S. L. Clark and F. Gesztesy, *Weyl–Titchmarsh M-function asymptotics, local uniqueness results, trace formulas, and Borg-type theorems for Dirac operators*, Trans. Amer. Math. Soc. **354** (2002), no. 9, 3475–3534 (electronic). (Cited on page 12.)

[36] S. L. Clark and F. Gesztesy, *On Weyl–Titchmarsh theory for singular finite difference Hamiltonian systems*, J. Comput. Appl. Math. **171** (2004), no. 1-2, 151–184. (Cited on pages 12, 26, and 72.)

[37] S. L. Clark, F. Gesztesy, and R. Nichols, *Principal solutions revisited*, in "Stochastic and Infinite Dimensional Analysis", C. C. Bernido, M. V. Carpio-Bernido, M. Grothaus, T. Kuna, M. J. Oliveira, and J. L. da Silva (editors), Trends in Mathematics, Birkhäuser, Basel, to appear. (Cited on page 36.)

[38] S. L. Clark, F. Gesztesy, R. Nichols, and M. Zinchenko, *Boundary data maps and Krein's resolvent formula for Sturm–Liouville operators on a finite interval*, Oper. Matrices **8** (2014), no. 1, 1–71. (Cited on pages 126 and 129.)

[39] S. L. Clark, F. Gesztesy, and M. Zinchenko, *Weyl–Titchmarsh theory and Borg–Marchenko-type uniqueness results for CMV operators with matrix-valued Verblunsky coefficients*, Oper. Matrices **1** (2007), no. 4, 535–592. (Cited on pages 12 and 13.)

[40] S. L. Clark, F. Gesztesy, and M. Zinchenko, *Borg–Marchenko-type uniqueness results for CMV operators*, Skr. K. Nor. Vidensk. Selsk. (2008), no. 1, 1–18. (Cited on page 13.)

[41] S. L. Clark, F. Gesztesy, and M. Zinchenko, *Minimal rank decoupling of full-lattice CMV operators with scalar- and matrix-valued Verblunsky coefficients*, in "Difference Equations and Applications" (Proceedings of the Fourteenth International Conference on Difference Equations and Applications, Istanbul, 2008), M. Bohner, Z. Došlá, G. Ladas, M. Ünal, and A. Zafer (editors), pp. 19–59, Uğur–Bahçeşehir University Publishing Company, 2009. (Cited on page 13.)

[42] E. A. Coddington, *Extension Theory of Formally Normal and Symmetric Subspaces*, Memoirs of the American Mathematical Society, Vol. 134, American Mathematical Society, Providence, 1973. (Cited on pages 90, 137, and 138.)

[43] E. A. Coddington and H. S. V. de Snoo, *Positive selfadjoint extensions of positive symmetric subspaces*, Math. Z. **159** (1978), no. 3, 203–214. (Cited on page 139.)

[44] E. A. Coddington and A. Dijksma, *Self-adjoint subspaces and eigenfunction expansions for ordinary differential subspaces*, J. Differential Equations **20** (1976), no. 2, 473–526. (Cited on page 119.)

[45] R. Cross, *Multivalued Linear Operators*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 213, Marcel Dekker, New York, 1998. ISBN 0-8247-0219-0. (Cited on pages 90 and 137.)

[46] A. Dijksma and H. S. V. de Snoo, *Self-adjoint extensions of symmetric subspaces*, Pacific J. Math. **54** (1974), 71–100. (Cited on pages 90 and 137.)

[47] D. Donnelly and E. Rogers, *Symplectic integrators: An introduction*, Amer. J. Phys. **73** (2005), no. 10, 938–945. (Cited on page 6.)

[48] O. Došlý, *Symplectic difference systems: Natural dependence on a parameter*, Adv. Dyn. Syst. Appl. **8** (2013), no. 2, 193–201. (Cited on page 72.)

[49] O. Došlý, *Symplectic difference systems with periodic coefficients: Krein's traffic rules for multipliers*, Adv. Difference Equ. **2013** (2013), no. 85, 13 pp. (electronic). (Cited on pages 65, 66, 67, 69, and 72.)

[50] O. Došlý and J. Elyseeva, *Singular comparison theorems for discrete symplectic systems*, J. Difference Equ. Appl. **20** (2014), no. 8, 1268–1288. (Cited on page 6.)

[51] O. Došlý and P. Hasil, *Friedrichs extension of operators defined by symmetric banded matrices*, Linear Algebra Appl. **430** (2009), no. 8-9, 1966–1975. (Cited on page 119.)

[52] O. Došlý and R. Hilscher, *Disconjugacy, transformations and quadratic functionals for symplectic dynamic systems on time scales*, J. Difference Equ. Appl. **7** (2001), no. 2, 265–295. (Cited on page 7.)

[53] O. Došlý, R. Hilscher, and V. Zeidan, *Nonnegativity of discrete quadratic functionals corresponding to symplectic difference systems*, Linear Algebra Appl. **375** (2003), 21–44. (Cited on page 6.)

[54] O. Došlý and W. Kratz, *Oscillation theorems for symplectic difference systems*, J. Difference Equ. Appl. **13** (2007), no. 7, 585–605. (Cited on pages 6 and 13.)

[55] O. Došlý and W. Kratz, *A Sturmian separation theorem for symplectic difference systems*, J. Math. Anal. Appl. **325** (2007), no. 1, 333–341. (Cited on page 6.)

[56] O. Došlý and W. Kratz, *Oscillation and spectral theory for symplectic difference systems with separated boundary conditions*, J. Difference Equ. Appl. **16** (2010), no. 7, 831–846. (Cited on pages 6 and 13.)

[57] O. Došlý and W. Kratz, *A remark on focal points of recessive solutions of discrete symplectic systems*, J. Math. Anal. Appl. **363** (2010), no. 1, 209–213. (Cited on page 6.)

[58] O. Došlý and Š. Pechancová, *Trigonometric recurrence relations and tridiagonal trigonometric matrices*, Int. J. Difference Equ. **1** (2006), no. 1, 19–29. (Cited on page 6.)

[59] O. Došlý and Z. Pospíšil, *Hyperbolic transformation and hyperbolic difference systems*, Fasc. Math. (2001), no. 32, 25–48. (Cited on page 6.)

[60] M. A. El-Gebeily, D. O'Regan, and R. P. Agarwal, *Characterization of self-adjoint ordinary differential operators*, Math. Comput. Modelling **54** (2011), no. 1-2, 659–672. (Cited on page 119.)

[61] S. Elaydi, *An Introduction to Difference Equations*, third edition, Undergraduate Texts in Mathematics, Springer, New York, 2005. ISBN 0-387-23059-9. (Cited on page 112.)

[62] Yu. V. Eliseeva, *Comparison theorems for symplectic systems of difference equations*, Differ. Equ. **46** (2010), 1339–1352. (Cited on page 6.)

[63] Yu. V. Eliseeva, *On the spectra of discrete symplectic boundary value problems with separated boundary conditions*, Russian Math. (Iz. VUZ) **55** (2011), no. 11, 71–75. Translated from: Izv. Vyssh. Uchebn. Zaved. Mat. **2011** (2011), no.11, 84–88 (Russian). (Cited on page 6.)

[64] J. V. Elyseeva, *Transformations and the number of focal points for conjoined bases of symplectic difference systems*, J. Difference Equ. Appl. **15** (2009), no. 11-12, 1055–1066. (Cited on page 6.)

[65] J. V. Elyseeva, *The comparative index and the number of focal points for conjoined bases of symplectic difference systems*, in "Discrete Dynamics and Difference Equations", S. Elaydi, H. Oliveira, J. M. Ferreira, and J. F. Alves (editors), pp. 231–238, World Scientific Publishing, London, 2010. (Cited on pages 6 and 13.)

[66] J. V. Elyseeva, *On relative oscillation theory for symplectic eigenvalue problems*, Appl. Math. Lett. **23** (2010), no. 10, 1231–1237. (Cited on pages 13 and 15.)

[67] J. V. Elyseeva, *A note on relative oscillation theory for symplectic difference systems with general boundary conditions*, Appl. Math. Lett. **25** (2012), no. 11, 1809–1814. (Cited on page 6.)

[68] J. V. Elyseeva, *Generalized oscillation theorems for symplectic difference systems with nonlinear dependence on spectral parameter*, Appl. Math. Comput. **251** (2015), 92–107. (Cited on page 6.)

[69] L. H. Erbe and P. X. Yan, *Disconjugacy for linear Hamiltonian difference systems*, J. Math. Anal. Appl. **167** (1992), no. 2, 355–367. (Cited on page 9.)

[70] L. H. Erbe and P. X. Yan, *Qualitative properties of Hamiltonian difference systems*, J. Math. Anal. Appl. **171** (1992), no. 2, 334–345. (Cited on page 9.)

[71] W. N. Everitt, *A personal history of the m-coefficient*, J. Comput. Appl. Math. **171** (2004), no. 1-2, 185–197. (Cited on page 12.)

[72] W. N. Everitt and L. Markus, *Boundary Value Problems and Symplectic Algebra for Ordinary Differential and Quasi-differential Operators*, Mathematical Surveys and Monographs, Vol. 61, American Mathematical Society, Providence, 1999. ISBN 0-8218-1080-4. (Cited on page 139.)

[73] K. Feng, *On difference schemes and symplectic geometry*, in "Proceedings of the 1984 Beijing Symposium on Differential Geometry and Differential Equations", K. Feng (editor), pp. 42–58, Science Press, Beijing, 1985. (Cited on page 6.)

[74] K. Feng, *The Hamiltonian way for computing Hamiltonian dynamics*, in "Applied and Industrial Mathematics" (Venice, 1989), Math. Appl., Vol. 56, pp. 17–35, Kluwer Acad. Publ., Dordrecht, 1991. (Cited on pages 1 and 6.)

[75] K. Feng and M. Qin, *Symplectic Geometric Algorithms for Hamiltonian Systems*, translated and revised from the Chinese original and with a foreword by F. Duan, Zhejiang Science and Technology Publishing House, Hangzhou; Springer, Heidelberg, 2010. ISBN 978-7-5341-3595-8; 978-3-642-01776-6. (Cited on page 6.)

[76] K. Feng and D. L. Wang, *A note on conservation laws of symplectic difference schemes for Hamiltonian systems*, J. Comput. Math. **9** (1991), no. 3, 229–237. (Cited on page 6.)

[77] K. Feng and D. L. Wang, *Symplectic difference schemes for Hamiltonian systems in general symplectic structure*, J. Comput. Math. **9** (1991), no. 1, 86–96. (Cited on page 6.)

[78] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics I: Mainly Mechanics, Radiation, and Heat*, Addison-Wesley Publishing, Reading, 1963. (Cited on page 141.)

[79] A. Fischer and Ch. Remling, *The absolutely continuous spectrum of discrete canonical systems*, Trans. Amer. Math. Soc. **361** (2009), no. 2, 793–818. (Cited on pages 90 and 93.)

[80] X. Hao, J. Sun, A. Wang, and A. Zettl, *Characterization of domains of self-adjoint ordinary differential operators II*, Results Math. **61** (2012), no. 3-4, 255–281. (Cited on page 119.)

[81] X. Hao, J. Sun, and A. Zettl, *Canonical forms of self-adjoint boundary conditions for differential operators of order four*, J. Math. Anal. Appl. **387** (2012), no. 2, 1176–1187. (Cited on page 119.)

[82] S. Hassi, H. S. V. de Snoo, and F. H. Szafraniec, *Componentwise and Cartesian decompositions of linear relations*, Dissertationes Math. (Rozprawy Mat.) **465** (2009), 59. (Cited on pages 90 and 137.)

[83] S. Hassi, H. S. V. de Snoo, and H. Winkler, *Boundary-value problems for two-dimensional canonical systems*, Integral Equations Operator Theory **36** (2000), no. 4, 445–479. (Cited on page 90.)

[84] S. Hassi, A. Sandovici, H. S. V. de Snoo, and H. Winkler, *A general factorization approach to the extension theory of nonnegative operators and relations*, J. Operator Theory **58** (2007), no. 2, 351–386. (Cited on page 139.)

[85] E. Hellinger, *Zur Stieltjesschen Kettenbruchtheorie* (in German), Math. Ann. **86** (1922), no. 1-2, 18–29. (Cited on page 12.)

[86] D. Hilbert, *Mathematical problems*, Bull. Amer. Math. Soc. **8** (1902), no. 10, 437–479. (Cited on page v.)

[87] R. Hilscher and V. Růžičková, *Implicit Riccati equations and quadratic functionals for discrete symplectic systems*, Int. J. Difference Equ. **1** (2006), no. 1, 135–154. (Cited on pages 6, 41, and 54.)

[88] R. Hilscher and V. Růžičková, *Riccati inequality and other results for discrete symplectic systems*, J. Math. Anal. Appl. **322** (2006), no. 2, 1083–1098. (Cited on pages 41 and 54.)

[89] R. Hilscher and V. Zeidan, *Discrete optimal control: The accessory problem and necessary optimality conditions*, J. Math. Anal. Appl. **243** (2000), no. 2, 429–452. (Cited on page 6.)

[90] R. Hilscher and V. Zeidan, *Discrete optimal control: Second order optimality conditions*, in "In honour of Professor Allan C. Peterson on the occasion of his 60th birthday, Pat I", J. Difference Equ. Appl. **8** (2002), no. 10, 875–896. (Cited on page 6.)

[91] R. Hilscher and V. Zeidan, *Second order sufficiency criteria for a discrete optimal control problem*, J. Difference Equ. Appl. **8** (2002), no. 6, 573–602. (Cited on page 6.)

[92] R. Hilscher and V. Zeidan, *Symplectic difference systems: Variable stepsize discretization and discrete quadratic functionals*, Linear Algebra Appl. **367** (2003), 67–104. (Cited on pages 6, 41, and 54.)

[93] R. Hilscher and V. Zeidan, *Equivalent conditions to the nonnegativity of a quadratic functional in discrete optimal control*, Math. Nachr. **266** (2004), 48–59. (Cited on page 6.)

[94] R. Hilscher and V. Zeidan, *Nonnegativity and positivity of quadratic functionals in discrete calculus of variations: Survey*, J. Difference Equ. Appl. **11** (2005), no. 9, 857–875. (Cited on page 6.)

[95] D. B. Hinton, A. M. Krall, and J. K. Shaw, *Boundary conditions for differential operators with intermediate deficiency index*, Appl. Anal. **25** (1987), no. 1-2, 43–53. (Cited on page 119.)

[96] D. B. Hinton and R. T. Lewis, *Spectral analysis of second order difference equations*, J. Math. Anal. Appl. **63** (1978), no. 2, 421–438. (Cited on pages 13, 90, and 123.)

[97] D. B. Hinton and A. N. Schneider, *On the Titchmarsh–Weyl coefficients for singular S-Hermitian systems I*, Math. Nachr. **163** (1993), 323–342. (Cited on page 12.)

[98] D. B. Hinton and A. N. Schneider, *On the Titchmarsh–Weyl coefficients for singular S-Hermitian systems II*, Math. Nachr. **185** (1997), 67–84. (Cited on pages 12 and 119.)

[99] D. B. Hinton and A. N. Schneider, *Titchmarsh–Weyl coefficients for odd-order linear Hamiltonian systems*, J. Spectral Math. Appl. (2006), 36 pp. (electronic). (Cited on page 12.)

[100] D. B. Hinton and J. K. Shaw, *On Titchmarsh–Weyl $M(\lambda)$-functions for linear Hamiltonian systems*, J. Differential Equations **40** (1981), no. 3, 316–342. (Cited on page 12.)

[101] D. B. Hinton and J. K. Shaw, *Titchmarsh–Weyl theory for Hamiltonian systems*, in "Spectral Theory of Differential Operators" (Proc. Conf., Birmingham, Ala., 1981), I. W. Knowles and R. T. Lewis (editors), North-Holland Math. Stud., Vol. 55, pp. 219–231, North-Holland, Amsterdam, 1981. (Cited on pages 7 and 12.)

[102] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985. ISBN 0-521-30586-1. (Cited on pages 3 and 4.)

[103] A. Jirari, *Second-Order Sturm–Liouville Difference Equations and Orthogonal Polynomials*, Memoirs of the American Mathematical Society, Vol. 113, no. 542, American Mathematical Society, Providence, 1995. (Cited on page 13.)

[104] I. S. Kac, *Linear relations generated by canonical differential equations*, Funct. Anal. Appl. **17** (1983), no. 4, 315–317. Translated from: Funktsional. Anal. i Prilozhen. **17** (1983), no. 4, 86–87 (Russian). (Cited on page 90.)

[105] W. G. Kelley and A. C. Peterson, *Difference Equations: An Introduction with Applications*, second edition, Harcourt/Academic Press, San Diego, 2001. ISBN 0-12-403330-X. (Cited on pages 13, 42, 43, and 48.)

[106] M. Kline, *Mathematics: The Loss of Certainty*, Oxford University Press, New York, 1980. ISBN 0-19-502754-X. (Cited on page 55.)

[107] V. I. Kogan and F. S. Rofe-Beketov, *On square-integrable solutions of symmetric systems of differential equations of arbitrary order*, Proc. Roy. Soc. Edinburgh Sect. A **74** (1974/75), 5–40 (1976). (Cited on page 101.)

[108] A. M. Krall, *$M(\lambda)$ theory for singular Hamiltonian systems with one singular point*, SIAM J. Math. Anal. **20** (1989), no. 3, 664–700. (Cited on pages 3, 12, 28, and 109.)

[109] A. M. Krall, *$M(\lambda)$ theory for singular Hamiltonian systems with two singular points*, SIAM J. Math. Anal. **20** (1989), no. 3, 701–715. (Cited on page 12.)

[110] A. M. Krall, *A limit-point criterion for linear Hamiltonian systems*, Appl. Anal. **61** (1996), no. 1-2, 115–119. (Cited on page 12.)

[111] W. Kratz, *Banded matrices and difference equations*, Linear Algebra Appl. **337** (2001), 1–20. (Cited on page 10.)

[112] W. Kratz, *Definiteness of quadratic functionals*, Analysis (Munich) **23** (2003), no. 2, 163–183. (Cited on pages 41 and 54.)

[113] W. Kratz and R. Šimon Hilscher, *A generalized index theorem for monotone matrix-valued functions with applications to discrete oscillation theory*, SIAM J. Matrix Anal. Appl. **34** (2013), no. 1, 228–243. (Cited on page 69.)

[114] M. G. Krein, *Foundations of the theory of lambda-zones of stability of a canonical system of linear differential equations with periodic coefficients*, in "Four Papers on Ordinary Differential Equations", L. J. Leifman (editor), Amer. Math. Soc. Transl. (2), Vol. 120, pp. 1–70, American Mathematical Society, Providence, 1983. (Cited on page 69.)

[115] P. D. Lax, *Linear Algebra and its Applications*, Pure and Applied Mathematics (Hoboken), Wiley-Interscience [John Wiley & Sons], second edition, Hoboken, 2007. ISBN 978-0-471-75156-4. (Cited on page 6.)

[116] M. Lesch and M. M. Malamud, *On the deficiency indices and self-adjointness of symmetric Hamiltonian systems*, J. Differential Equations **189** (2003), no. 2, 556–615. (Cited on pages 7, 90, 93, 94, 98, 100, 106, 107, 113, 120, and 138.)

[117] D. S. Mackey and N. Mackey, *On the determinant of symplectic matrices*, Numerical Analysis Report **422** (2003), 1–12. Manchester Centre for Computational Mathematics, England. (Cited on page 6.)

[118] N. Macrae, *John von Neumann: The Scientific Genius Who Pioneered the Modern Computer, Game Theory, Nuclear Deterrence, and Much More*, reprint of the 1992 original, American Mathematical Society, Providence, 1999. ISBN 0-8218-2064-8. (Cited on page 119.)

[119] M. Marletta and A. Zettl, *The Friedrichs extension of singular differential operators*, J. Differential Equations **160** (2000), no. 2, 404–421. (Cited on page 119.)

[120] J. B. McLeod, *The number of integrable-square solutions of ordinary differential equations*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 285–290. (Cited on pages 27 and 35.)

[121] A. B. Mingarelli, *A limit-point criterion for a three-term recurrence relation*, C. R. Math. Rep. Acad. Sci. Canada **3** (1981), no. 3, 171–175. (Cited on page 13.)

[122] S. J. Monaquel and K. M. Schmidt, *On M-functions and operator theory for non-self-adjoint discrete Hamiltonian systems*, in "Special Issue: 65th birthday of Prof. Desmond Evans", J. Comput. Appl. Math. **208** (2007), no. 1, 82–101. (Cited on page 13.)

[123] M. Muzzulini, *Titchmarsh–Sims–Weyl theory for Complex Hamiltonian Systems of Arbitrary Order*, Ph.D. dissertation, University of Karlsruhe, Karlsruhe, 2007. (Cited on page 12.)

[124] M. A. Naĭmark, *Linear Differential Operators, Part II: Linear Differential Operators in Hilbert Space*, with additional material by the author, and a supplement by V. E. Lyantse, translated from the Russian by E. R. Dawson, English translation edited by W. N. Everitt, George G. Harrap & Company, New York, 1968. (Cited on pages 97 and 119.)

[125] R. Nevanlinna, *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem* (in German), Ann. Acad. Sci. Fenn. A **18** (1922), no. 5, 53 pp. (Cited on page 12.)

[126] H.-D. Niessen and A. Zettl, *The Friedrichs extension of regular ordinary differential operators*, Proc. Roy. Soc. Edinburgh Sect. A **114** (1990), no. 3-4, 229–236. (Cited on page 119.)

[127] H.-D. Niessen and A. Zettl, *Singular Sturm–Liouville problems: The Friedrichs extension and comparison of eigenvalues*, Proc. London Math. Soc. (3) **64** (1992), no. 3, 545–578. (Cited on page 119.)

[128] B. C. Orcutt, *Canonical Differential Equations*, Doctoral dissertation – University of Virginia, ProQuest LLC, Ann Arbor, 1969. (Cited on pages 90 and 93.)

[129] V. Răsvan, *Stability zones for discrete time Hamiltonian systems*, in "CDDE 2000 Proceedings (Brno)", Arch. Math. (Brno) **36** (2000), no. suppl., 563–573. (Cited on page 69.)

[130] V. Răsvan, *On stability zones for discrete-time periodic linear Hamiltonian systems*, Adv. Difference Equ. **2006** (2006), Art. ID 80757, 13 pp. (electronic). (Cited on page 69.)

[131] R. Remmert, *Theory of Complex Functions*, translated from the second German edition by R. B. Burckel, Readings in Mathematics, Graduate Texts in Mathematics, Vol. 122, Springer-Verlag, New York, 1991. ISBN 0-387-97195-5. (Cited on page 145.)

[132] G. Ren, *On the density of the minimal subspaces generated by discrete linear Hamiltonian systems*, Appl. Math. Lett. **27** (2014), 1–5. (Cited on page 89.)

[133] G. Ren and Y. Shi, *The defect index of singular symmetric linear difference equations with real coefficients*, Proc. Amer. Math. Soc. **138** (2010), no. 7, 2463–2475. (Cited on page 13.)

[134] G. Ren and Y. Shi, *Defect indices and definiteness conditions for a class of discrete linear Hamiltonian systems*, Appl. Math. Comput. **218** (2011), no. 7, 3414–3429. (Cited on pages 35, 80, 90, 94, 98, 106, and 120.)

[135] G. Ren and Y. Shi, *Self-adjoint extensions for discrete linear Hamiltonian systems*, Linear Algebra Appl. **454** (2014), 1–48. (Cited on pages 90, 97, 108, 114, 119, 124, 125, and 130.)

[136] N. J. Rose, *Mathematical Maxims and Minims*, Rome Press, Raleigh, 1988. (Cited on page 65.)

[137] V. Růžičková, *Discrete Symplectic Systems and Definiteness of Quadratic Functionals*, Ph.D. dissertation, Masaryk University, Brno, 2006. (Cited on page 6.)

[138] V. Růžičková, *Perturbation of discrete quadratic functionals*, Tatra Mt. Math. Publ. **38** (2007), 229–241. (Cited on page 6.)

[139] K. Setzer, *Definite Sturm–Liouville Matrix Differential Equations and Applications of Moore–Penrose Inverses to Related Problems*, Ph.D. dissertation, Ulm University, Ulm, 2012. (Cited on page 10.)

[140] Y. Shi, *Symplectic structure of discrete Hamiltonian systems*, J. Math. Anal. Appl. **266** (2002), no. 2, 472–478. (Cited on page 6.)

[141] Y. Shi, *On the rank of the matrix radius of the limiting set for a singular linear Hamiltonian system*, Linear Algebra Appl. **376** (2004), 109–123. (Cited on pages 12, 28, and 31.)

[142] Y. Shi, *Weyl–Titchmarsh theory for a class of discrete linear Hamiltonian systems*, Linear Algebra Appl. **416** (2006), no. 2-3, 452–519. (Cited on pages 13, 14, 25, 26, 28, 31, 34, 41, 72, 80, 85, 90, 106, and 109.)

[143] Y. Shi, *The Glazman–Krein–Naimark theory for Hermitian subspaces*, J. Operator Theory **68** (2012), no. 1, 241-256. (Cited on pages 90, 115, 116, 138, and 139.)

[144] Y. Shi, *Stability of essential spectra of self-adjoint subspaces under compact perturbations*, J. Math. Anal. Appl. **433** (2016), no. 2, 832–851. (Cited on page 90.)

[145] Y. Shi and S. Chen, *Spectral theory of second-order vector difference equations*, J. Math. Anal. Appl. **239** (1999), no. 2, 195–212. (Cited on page 43.)

[146] Y. Shi, C. Shao, and G. Ren, *Spectral properties of self-adjoint subspaces*, Linear Algebra Appl. **438** (2013), no. 1, 191–218. (Cited on page 90.)

[147] Y. Shi and H. Sun, *Self-adjoint extensions for second-order symmetric linear difference equations*, Linear Algebra Appl. **434** (2011), no. 4, 903–930. (Cited on pages 13, 90, 108, 109, and 119.)

[148] R. Šimon Hilscher, *Eigenvalue theory for time scale symplectic systems depending nonlinearly on spectral parameter*, Appl. Math. Comput. **219** (2012), no. 6, 2839–2860. (Cited on page 43.)

[149] R. Šimon Hilscher, *Oscillation theorems for discrete symplectic systems with nonlinear dependence in spectral parameter*, Linear Algebra Appl. **437** (2012), no. 12, 2922-2960. (Cited on pages 6 and 69.)

[150] R. Šimon Hilscher and V. Zeidan, *Symmetric three-term recurrence equations and their symplectic structure*, Adv. Difference Equ. (2010), Art. ID 626942, 17 pp. (electronic). (Cited on page 10.)

[151] R. Šimon Hilscher and V. Zeidan, *Symplectic structure of Jacobi systems on time scales*, Int. J. Difference Equ. **5** (2010), no. 1, 55–81. (Cited on page 6.)

[152] R. Šimon Hilscher and V. Zeidan, *Rayleigh principle for time scale symplectic systems and applications*, Electron. J. Qual. Theory Differ. Equ. (2011), no. 83, 16 pp. (electronic). (Cited on page 43.)

[153] R. Šimon Hilscher and V. Zeidan, *Oscillation theorems and Rayleigh principle for linear Hamiltonian and symplectic systems with general boundary conditions*, Appl. Math. Comput. **218** (2012), no. 17, 8309–8328. (Cited on pages 41, 42, 43, 46, and 54.)

[154] H. Sun, *On the limit-point case of singular linear Hamiltonian systems*, Appl. Anal. **89** (2010), no. 5, 663–675. (Cited on pages 34 and 41.)

[155] H. Sun, Q. Kong, and Y. Shi, *Essential spectrum of singular discrete linear Hamiltonian systems*, Math. Nachr. **289** (2016), no. 2-3, 343–359. (Cited on page 90.)

[156] H. Sun and Y. Shi, *Eigenvalues of second-order difference equations with coupled boundary conditions*, Linear Algebra Appl. **414** (2006), no. 1, 361–372. (Cited on page 13.)

[157] H. Sun and Y. Shi, *Limit-point and limit-circle criteria for singular second-order linear difference equations with complex coefficients*, Comput. Math. Appl. **52** (2006), no. 3-4, 539–554. (Cited on pages 13 and 123.)

[158] H. Sun and Y. Shi, *Strong limit point criteria for a class of singular discrete linear Hamiltonian systems*, J. Math. Anal. Appl. **336** (2007), no. 1, 224–242. (Cited on page 13.)

[159] H. Sun and Y. Shi, *Self-adjoint extensions for linear Hamiltonian systems with two singular endpoints*, J. Funct. Anal. **259** (2010), no. 8, 2003–2027. (Cited on page 119.)

[160] H. Sun and Y. Shi, *Self-adjoint extensions for singular linear Hamiltonian systems*, Math. Nachr. **284** (2011), no. 5-6, 797–814. (Cited on page 119.)

[161] H. Sun and Y. Shi, *Spectral properties of singular discrete linear Hamiltonian systems*, J. Difference Equ. Appl. **20** (2014), no. 3, 379–405. (Cited on page 90.)

[162] J. Sun, *On the selfadjoint extensions of symmetric ordinary differential operators with middle deficiency indices*, Acta Math. Sinica (N.S.) **2** (1986), no. 2, 152–167. (Cited on pages 119 and 130.)

[163] S. Sun, Y. Shi, and S. Chen, *The Glazman–Krein–Naimark theory for a class of discrete Hamiltonian systems*, J. Math. Anal. Appl. **327** (2007), no. 2, 1360–1380. (Cited on page 90.)

[164] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, Mathematical Surveys and Monographs, Vol. 72, American Mathematical Society, Providence, 2000. ISBN 0-8218-1940-2. (Cited on page 13.)

[165] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations, Part II*, Clarendon Press, Oxford, 1958. (Cited on page 12.)

[166] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations, Part I*, second edition, Clarendon Press, Oxford, 1962. (Cited on page 12.)

[167] J. von Neumann, *Über adjungierte Funktionaloperatoren* (in German), Ann. of Math. (2) **33** (1932), no. 2, 294–310. (Cited on page 90.)

[168] P. W. Walker, *A note on differential equations with all solutions of integrable-square*, Pacific J. Math. **56** (1975), no. 1, 285–289. (Cited on pages 2, 55, 56, and 63.)

[169] A. Wang, J. Sun, and A. Zettl, *Characterization of domains of self-adjoint ordinary differential operators*, J. Differential Equations **246** (2009), no. 4, 1600–1622. (Cited on page 119.)

[170] Y. Wang and Y. Shi, *Eigenvalues of second-order difference equations with periodic and antiperiodic boundary conditions*, J. Math. Anal. Appl. **309** (2005), no. 1, 56–69. (Cited on pages 13, 42, and 43.)

[171] J. Weidmann, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, Vol. 68, Springer-Verlag, New York, 1980. ISBN 0-387-90427-1. (Cited on page 34.)

[172] H. Weyl, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen* (in German), Math. Ann. **68** (1910), no. 2, 220–269. (Cited on pages 12 and 22.)

[173] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, Princeton, 1946. (Cited on page 6.)

[174] H. Weyl, *Topology and abstract algebra as two roads of mathematical comprehension I*, Amer. Math. Monthly **102** (1995), no. 5, 453–460. Translated from: Unterrichtsblatter fur Mathematik und Naturwissenschaften **38** (1932), 177-188 (German). (Cited on page 11.)

[175] D. L. Wilcox, *Essential spectra of linear relations*, Linear Algebra Appl. **462** (2014), 110–125. (Cited on page 90.)