Contents

Abstrakt

Habilitační práce je souborem článků [13, 34, 20, 30, 23, 24] publikovaných v mezinárodních časopisech, které jsou všechny evidovány v databázích WoS nebo SCOPUS. Většina těchto článků má spoluautory, jimiž jsou A. Rod Gover, Andreas Čap, Fabian Radoux, Jean-Philippe Michel, Matthias Hammerl, Petr Somberg a Vladimír Souček. Podíl všech autorů na společných článcích je rovnocenný. Reprinty článků jsou v Sekci 6.

Oblast výzkumu těchto matematiků se protíná v konformní geometrii, která je nejznámější strukturou ve třídě tzv. AHS struktur. Význam konformní geometrie spočívá mj. v jejím blízkém vztahu k matematické fyzice a také k analýze. Právě na pomezí těchto oblastí matematiky patří problém, který motivuje výsledky shrnuté v této habilitaci: studium symetrií Laplaceova operátoru Δ. To jsou operátory Σ takové, že $\Delta \circ \Sigma = \Sigma' \circ \Delta$ pro nějaký operátor Σ'. Technická formulace problémů, jejichž vyřešení je nutné pro úplný popis symmetrií Laplaceova operátoru, je v Sekci 1.1.

Symetrie Laplaceova operátoru jsou známé díky článku [15] publikovaném v Annals of Mathematics. Cíl této habilitace je ovšem mnohem obecnější: prezentovat geometrické nástroje a postupy pro studium symetrií invariantních operátorů ve třídě AHS geometrických struktur. Přesnější formulace vyžadují jistý matematický aparát a čtenář je najde v Sekci 1. Hlavní výsledky se týkají *invariantního kvantování* v Sekci 3, symetrií *konformních mocnin* Laplaceova operátoru a diskuze *plochá versus křivá geometrie* v Sekci 4 a prodlužování *prvních BGG operátorů* v Sekci 5.

Preface

The thesis is a collection of articles [13, 34, 20, 30, 23, 24]. All of them have been published in international journals indexed by WoS or SCOPUS. Results and some background theory are summarized in the survey part (Sections 1 – 5) on pages 4 – 27. Reprints of articles are in Section 6.

Our main research interest lies in the theory (bi)linear invariant operators on manifolds with a geometrical structure (known as AHS structure). This provides a suitable geometrical framework for the specific problem which motivates this survey: to understand symmetries of the Laplace operator [15]. Such symmetries play an important role in analysis (in the study of separable solutions of PDE's) and in mathematical physics (where the terminology 'higher' or 'hidden' symmetries is often used). I hope this survey will be useful for researches on the borderline of these fields.

Pronouncement

Almost all papers included in this thesis have co-authors, namely A. Čap, A.R. Gover, J.-P. Michel, F. Radoux, M. Hammerl, P. Somberg and V. Souček. The contributions of all authors were equivalent since the results were based on common discussions. Formally, the author's contribution to the paper [34] was 100%, the author's contribution the papers [13, 20] was 50%, the author's contribution the paper [30] was 33% and the authors contribution to the papers [23, 24] was 25%.

However, the developement of articels [23, 24] was rather specific. Main results of these articels were obtained more or less independently by M. Hammerl on one side and by P. Somberg, V. Souček and the author of the habilition on the other side. After we found out that we work on the same problem, we decided to publish our results together. From this point of view, the author's contribution to [23, 24] can be also considered as 33%.

Acknowledgement

I wish to thank all the co-authors for their friendly and always very helpful collaboration. I would like to express my gratitude to my colleague Prof. Jan Slovák for our numerous interesting discussions.

1 Introduction

1.1 A motivation: symmetries of the Laplacian on $\mathbb{R}^n$

The Laplacian operator is of prominent interest in differential geometry, mathematical physics and analysis and has many analogues in other mathematical fields. We shall start with the simplest version, i.e. the Laplacian on smooth functions on the Euclidean space $\mathbb{R}^n$. This is the operator $\Delta = \nabla^i \nabla_i : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ where $\nabla_i = \partial/\partial x_i$ and we have used the Einstein's summation convention. There is an obvious notion of symmetries of $\Delta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ given by differential operators $\Sigma : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ such that $\Delta\Sigma = \Sigma\Delta$. We shall term such operators *commuting symmetries* of Δ. As a slight reformulation, one can also introduce commuting symmetries as operators preserving eigenspaces of Δ. Another (and weaker) possibility is to study operators Σ which preserve the null space of Δ and this will be the property of our interest. We say Σ is a *conformal symmetry* or just a *symmetry* of Δ if

$$\Delta\Sigma = \Sigma'\Delta \qquad \text{for some } \; \Sigma' : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n). \tag{1}$$

Note this also means that Σ' preserves the range of Δ. A short computation reveals that operators Σ and Σ' have the same symbol. The vector space of symmetries forms obviously an algebra which we denote by $\overline{HS}$. Operators of the form $\Sigma = T\Delta$ are always symmetries which we shall term *trivial* symmetries. Observe the space of trivial symmetries is a left ideal which we denote by $(\Delta) \subseteq \overline{HS}$. The main problem is to describe the quotient $HS := \overline{HS}/(\Delta)$.

What do we need to understand fully symmetries of Δ? First, we need to *classify* symmetries Σ which means to find out which tensor fields can appear as symbols of symmetries in the first place. Another question is to *construct* a preferred symmetry with a prescribed symbol. Then one should understand the *algebra* HS. All these problems are solved in the essential Eastwood's paper [15]. This result also shows the significance of conformal geometry. Although defined using the Euclidean metric, the Laplacian is in a suitable sense conformally invariant, cf. Section 1.2. (Note this follows already from knowledge of first order symmetries.) Thus the basic step in the study of symmetries should be to understand *invariance* of Δ.

These questions motivate the main aim of this thesis: we explore to which extent and directions one can develop a general theory of symmetries of (suitably invariant) differential operators.

1.2 From Euclidean space to conformal geometry

Conformal structure is the pair $(M, [g])$ where M is a smooth manifold of the dimension n, g a (pseudo)Riemannian metric and the class $[g]$ contains metrics obtained by rescaling g by a positive smooth function. This determines the class of corresponding Levi-Civita connections $[\nabla]$. Although the Laplacian $\Delta = g^{ab}\nabla_a\nabla_b$ is not conformally invariant, there is a curvature modification known as the conformal Laplacian (or Yamabe operator) $\Delta_Y := \Delta - \frac{n-2}{4(n-1)}\mathsf{Sc} : \mathcal{E}[-n/2+1] \to \mathcal{E}[-n/2-1]$ where Sc is the scalar curvature of g and $\mathcal{E}[w]$ are density bundles, cf. Section 2.4 for details concerning our convention for conformal weights $w \in \mathbb{R}$. The conformal invariance means $\Delta_Y = \widehat{\Delta}_Y$ for the operator $\widehat{\Delta}_Y$ defined with respect to another choice $\hat{g} \in [g]$ with corresponding $\widehat{\nabla} \in [\nabla]$.

Note the problem of symmetries of Δ_Y is much simpler in the Euclidean case (where $\Delta_Y = \Delta$ because $\mathsf{Sc} = 0$, cf. Section 1.1) than for general conformal structures $(M, [g])$. In fact, the full understanding of symmetries of Δ_Y is available only on locally flat conformal structures $(M, [g])$, i.e. when g is locally isomorphic to the Euclidean metric on $\mathbb{R}^n$. Generally, first order symmetries of Δ_Y come from infinitesimal symmetries of the structure (i.e. conformal Killing fields) but the classification of higher order symmetries is a highly nontrivial problem. Only the case of second order symmetries is solved and we shall discuss this result in Section 4.2.

Above we started with the Laplacian Δ and observed its conformal invariance. Taking a slightly different point of view, we can start with a given conformal structure $(M, [g])$, choose an arbitrary invariant differential operator Φ and consider symmetries of Σ analogously as in (1). We shall do this, in the locally flat setting, for conformal powers of the Laplacian $\Phi : \mathcal{E}[-n/2+\ell] \to \mathcal{E}[-n/2-\ell]$. That is, $\Phi = \Delta^\ell$ on Euclidean spaces. Details are in Section 4. Also symmetries of the Dirac and Maxwell operators are studied [4, 16, 2].

1.3 From conformal geometry to AHS structures

Conformal geometry is the most studied structure among *almost Hermitean symmetric (AHS)* geometrical structures on M. Another important AHS structure is *projective geometry* given as a class of torsion free connections on M which share the same geodesics (as unparametrised curves). This is a general feature: every AHS structure is closely related to a reduction of the structure group of $GL(n)$ to a suitable (simple) subgroup together with a certain class $[\nabla]$ of affine connections on M.

We shall however need a theoretical background of AHS manifolds going far beyond elementary differential geometry used so far. Building on ideas of Cartan and Tanaka, there is nowadays well established notion of so called *Cartan bundle* $\mathcal{G} \to M$ and *Cartan connection* ω of type (G, P) which is a “curved version” of the bundle $G \to G/P$ and the Maurer-Cartan form ω. Here $P \subseteq G$ are Lie groups. Specializing to the case of a parabolic subgroup P of semisimple G, we obtain *parabolic geometry*. A general and widely developed theory (based the Lie representation theory for $P \subseteq G$) is available in [10]. Here AHS structures form an important subclass and we shall extensively use *tractor bundles*, *splitting operators* and other tools in the so called "BGG machinery". A brief summery of this theory is provided in the beginning of Section 2.

Note *invariance* (or *naturality*) of various operations means that such operation is canonically defined using only the geometrical structure (without any additional choices). One can be more specific on locally flat AHS structures M which are locally isomorphic to G/P. Here the G-action gives a realization of the Lie algebra $\mathfrak{g}$ of G as $\mathfrak{g} \subseteq \mathfrak{X}(G/P)$ thus locally $\mathfrak{g} \subseteq \mathfrak{X}(M)$. Invariance of a differential operator Φ then means that Φ intertwines the action of $\mathfrak{g}$. In the curved case (where the G-action is lost) we simply say that Φ is invariant if it can be given by an explicit formula in terms of a chosen $\nabla \in [\nabla]$ and its curvature in such a way that Φ does not depend on this choice.

1.4 Invariant quantization on AHS structures

It turns out that there is indeed a preferred symmetry of the Laplacian for a given symbol. This is provided by a more general notion of the so called invariant quantization. Recall given vector bundles E and F over M and denoting by $\mathrm{Diff}^k(E, F)$ the space of linear differential operators $\Gamma(E) \to \Gamma(F)$ of the order $\leq k$, there is the *(principal) symbol* map $\mathrm{symb} : \mathrm{Diff}^k(E, F) \to \Gamma(\mathrm{Symb}^k(E, F))$. Here $\mathrm{Symb}^k(E, F) = S^k TM \otimes E^* \otimes F$ is the bundle of symbols of $\mathrm{Diff}^k(E, F)$ and S^k denotes the kth symmetric tensor product. By *quantization* we mean a right inverse $\mathcal{Q}^k : \Gamma(\mathrm{Symb}^k(E, F)) \to \mathrm{Diff}^k(E, F)$ of the symbol map symb. Obviously, there is an associated bilinear operator $(\sigma, \varphi) \mapsto \mathcal{Q}^k(\sigma)(\varphi)$ for $\sigma \in \Gamma(\mathrm{Symb}^k(E, F))$ and $\varphi \in \Gamma(E)$. We say $\mathcal{Q}^k$ is *invariant quantization* if this bilinear operator is invariant for a given AHS structure on M.

As an illustration, consider a smooth manifold M and the exterior derivative d. We can view the Lie derivative L_X along the vector field $X \in \mathfrak{X}(M)$ as the invariant quantization for the symbol $X \in \Gamma(TM)$. Then the formula $L_X d = dL_X$ just means L_X is a first order symmetry of d for every vector field X. (Note this formula is the infinitesimal version of the fact that d commutes with local diffeomorphisms.) Also observe this cannot have an analogue for higher order symmetries as there is no higher order analogue of the Lie derivative. The latter follows from the classification of such operators [22]. Of course, if we equip M with an AHS structure (i.e. restrict invariance to this category), there will be many more invariant (bi)linear operators. Thus we can expect existence of invariant quantization on such structures. This is indeed the case and we present a construction of invariant quantization for AHS manifolds in a suitably generic sense in Section 3.1. However, it turns out such construction does not give a complete (non)existence result for the invariant quantization $\mathcal{Q}^k$. The latter problem is much more difficult and will be discussed only for the invariant quantization on conformal densities $\mathcal{Q}^k : \Gamma(\mathrm{Symb}^k(\mathcal{E}[w], \mathcal{E}[w+\delta])) \to \mathrm{Diff}^k(\mathcal{E}[w], \mathcal{E}[w+\delta])$ where $w, \delta \in \mathbb{R}$. More specifically, we shall show how (non)existence of $\mathcal{Q}^k$ depends on δ in Section 3.2.

1.5 Prolongation of first BGG operators

As mentioned in Section 1.1, one of the basic problems in the study of symmetries is to understand which tensor fields can appear as symbols of symmetries of the Laplacian. To formulate the answer, it is useful to employ the abstract index notation in the sense of Penrose [33]. That is, $\mathcal{E}^a = TM$, $\mathcal{E}_a = T^*M$, $\mathcal{E}^{(a_1 \dots a_k)} = S^k TM$ (i.e. symmetrization of indices is denoted by round brackets) and we shall raise and lower indices using the metric $g_{ab} \in \Gamma(\mathcal{E}_{(ab)})$ and its inverse. It turns out that symbols of symmetries (modulo trivial symmetries) of Δ on $\mathbb{R}^n$ are $\sigma^{a_1 \dots a_k} \in \Gamma(\mathcal{E}^{(a_1 \dots a_k)_0})$ characterized by the condition $\nabla^{(a_0}\sigma^{a_1 \dots a_k)_0} = 0$. Here the subscript 0 denotes the projection to trace free part. Sections $\sigma^{a_1 \dots a_k}$ satisfying such system of PDE's are known as *conformal Killing tensors*. Also, we point out that the (conformally invariant) differential operator $\sigma^{a_1 \dots a_k} \mapsto \nabla^{(a_0}\sigma^{a_1 \dots a_k)_0}$ is overdetermined.

The latter operator is actually invariant on any conformal manifold $(M, [g])$. Similar operators control *conformal Killing forms* (which are symbols of first order symmetries of the Dirac), *twistor spinors* etc. They all belong to the class of "first BGG operators" where corresponding systems of PDE's are always overdetermined. They can be naturally constructed using the BGG machinery on all AHS manifolds (see the beginning of Section 5.1). There is a general approach how to deal with overdetermined differential operators known as *prolongation*. Roughly speaking, this means to add new variables to the section σ (on which an overdetermined operator acts) such that differentiating this system, the result can be expressed in terms of variables from the system in an *algebraic* way. (New variables play the role of derivatives of σ.) This leads to another of the main results of this habilitation: we present a canonical (and constructive) way how to design such prolongation for all first BGG operators on AHS structures, see Section 5.1. That is, the prolongation is invariant. Technically, this is formulated in terms of the so called *prolongation connection* $\widetilde{\nabla}$ on certain natural vector bundles (known as tractor bundles) on AHS structures.

In fact, this goes beyond the study of symbols of symmetries as solutions of first BGG operators encode many important geometrical properties of AHS structures (e.g. metrizability of projective classes etc.). We shall comment upon that in Section 5.2 together with an explicit form of $\widetilde{\nabla}$ for some conformal first BGG operators.

1.6 Summary: the author's contribution and further directions

Main results (which summarize the author's contribution to the field) in this habilitation are: a generic construction of invariant quantization on AHS structures (based on [13]), a complete construction of conformal quantization on densities (based on [34]), description of symmetries of powers Δ^ℓ of the Laplacian on conformally flat manifolds (based on [20]) and 2nd order symmetries of Δ_Y on curved conformal structures (based on [30]), a general construction of invariant prolongations for first BGG operators on AHS structures (based on [23]) and conformal examples of explicit form of the prolongation connections (based on [24]).

Reprints of these six articles are appended in Section 6 on page 31 and results from these articles are marked by boxes below. Also note the list of References is at the end of the survey part, i.e. *before* Section 6.

Let as also mention several applications and related research directions of presented results. Understanding of higher (or hidden) symmetries has an interpretation in physics models [35] and can be used in the seek for explicit solutions of physically important systems of PDE's [2, 1]. Symmetries of the Laplacian Δ (or more specifically systems of several symmetries which mutually commute) are closely related to existence of distinguished coordinate systems where Δ has separable solutions [32]. Here [30] is one of few results in the curved case which goes beyond the first order, cf. [2]. In principal, knowing the prolongation connection $\widetilde{\nabla}$ (or rather its curvature $\widetilde{\Omega}$), one can deduce curvature obstruction for existence of solution, see Section 5.1 for details.

Results discussed below are based on geometric considerations (with a substantial use of representation theory) but are of interest also for mathematical physics and analysis. The author believes that the survey will be useful for researches in these fields. Let us also mention at least two current projects the author collaborates on: new properties of Paterson-Walker metrics [25, 26] (where specific first BGG operators play an essential role) and study of "higher supersymmetries" (as symmetries of the (Laplace,Dirac) systems) [31] where a new and physically interesting realization of certain superalgebras of symmetries is obtained.

2 Almost Hermitean symmetric structures

In this section we review the basic theory and invariant calculus of AHS structures. The general theory (cf. Sections 2.1 – 2.3) can be made more explicit for particular AHS geometries and we demonstrate this in the conformal case, cf. Section 2.4. This order is not essential and one can also read Section 2.4 first. This depends on the reader whether he or she prefers to go from 'general' to 'specific' or vise versa. This theory (originating in ideas of Cartan [7]) is nowadays standard and we refer for the recent monograph [10] for details.

AHS structures and invariant calculus

2.1 $|1|$–graded Lie algebras and first order structures

The starting point for defining an AHS–structure is a simple Lie algebra $\mathfrak{g}$ endowed with a so called $|1|$–grading, i.e. a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we agree that $\mathfrak{g}_\ell = 0$ for $\ell \notin \{-1, 0, 1\}$. The classification of such gradings is well known, since it is equivalent to the classification of Hermitean symmetric spaces. We put $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \subset \mathfrak{g}$. By the grading property, $\mathfrak{p}$ is a subalgebra of $\mathfrak{p}$ and $\mathfrak{g}_1$ is a nilpotent ideal in $\mathfrak{p}$.

Given a Lie group G with Lie algebra $\mathfrak{g}$, there are natural subgroups $G_0 \subset P \subset G$ corresponding to the Lie subalgebras $\mathfrak{g}_0 \subset \mathfrak{p} \subset \mathfrak{g}$. For P one may take a subgroup lying between the normalizer $N_G(\mathfrak{p})$ of $\mathfrak{p}$ in G and its connected component of the identity. Then $G_0 \subset P$ is defined as the subgroup of all elements whose adjoint action preserves the grading of $\mathfrak{g}$. In particular, restricting the adjoint action to $\mathfrak{g}_{-1}$, one obtains a representation $G_0 \to GL(\mathfrak{g}_{-1})$. This representation is infinitesimally injective, so it makes sense to talk about first order G–structures with structure group G_0 on smooth manifolds of dimension dim $(\mathfrak{g}_{-1})$.

By definition, such a structure is given by a smooth principal bundle $p : \mathcal{G}_0 \to M$ with structure group G_0, such that the associated bundle $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$ is isomorphic to the tangent bundle TM. It turns out that the Killing form on $\mathfrak{g}$ induces a G_0–equivariant duality between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, so $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_1 \cong T^*M$. Using this, one can realize arbitrary tensor bundles on M as associated bundles to $\mathcal{G}_0$. More generally, any representation of G_0, via forming associated bundles, gives rise to a natural vector bundle on manifolds endowed with such a structure. It turns out that G_0 is always reductive with one–dimensional center. Hence finite dimensional representations of G_0 on which the center acts diagonalizably (which we will always assume in the sequel) are completely reducible, i.e. they split into direct sums of irreducible representations.

The one–dimensional center of G_0 leads to a family of natural line bundles. For $w \in \mathbb{R}$, we can define a homomorphism $G_0 \to \mathbb{R}_+$ by mapping $g \in G_0$ to

$|\det(\mathrm{Ad}_-(g))|^{\frac{w}{n}}$, where $n = \dim\,(\mathfrak{g}_{-1})$ and $\mathrm{Ad}_-(g) : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ is the restriction of the adjoint action of g. This evidently is a smooth homomorphism, thus giving rise to a one–dimensional representation $\mathbb{R}[w]$ of G_0. It is easy to see that $\mathbb{R}[w]$ is non–trivial for $w \neq 0$. (The factor $\frac{1}{n}$ is included to get the usual normalization in the case of conformal structures.) The corresponding associated bundle will be denoted by $\mathcal{E}[w]$, and adding the symbol $[w]$ to the name of a natural bundle will always indicate a tensor product with $\mathcal{E}[w]$. Using the convention that 1–densities are the objects which can be naturally integrated on non–orientable manifolds, $\mathcal{E}[w]$ is by construction the bundle of $(-\frac{w}{n})$–densities. In particular, all the bundles $\mathcal{E}[w]$ are trivial line bundles, but there is no canonical trivialization for $w \neq 0$.

2.2 Canonical Cartan connections and AHS–structures

The exponential mapping restricts to a diffeomorphism from $\mathfrak{g}_1$ onto a closed normal Abelian subgroup $P_+ \subset P$ such that P is the semidirect product of G_0 and P_+. Hence G_0 can also naturally be viewed as a quotient of P. In particular, given a principal P–bundle $\mathcal{G} \to M$, the subgroup P_+ acts freely on $\mathcal{G}$, and the quotient $\mathcal{G}/P_+$ is naturally a principal bundle with structure group G_0. Next, suppose that there is a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ on the principal bundle $\mathcal{G}$. Then the $\mathfrak{g}_{-1}$–component of ω descends to a well defined one–form $\theta \in \Omega^1(\mathcal{G}/P_+, \mathfrak{g}_{-1})$, which is G_0–equivariant and strictly horizontal. This means that $(\mathcal{G}/P_+ \to M, \theta)$ is a first order structure with structure group G_0. In this sense, any Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) has an underlying first order structure with structure group G_0. Conversely, one can talk about extending a first order structure to a Cartan geometry.

It turns out that, for almost all choices of (G, P), for any given first order structure with structure group G_0 there is a unique (up to isomorphism) extension to a Cartan geometry of type (G, P), for which the Cartan connection ω satisfies a certain normalization condition. (The latter is given by the Kostant's codifferential ∂^*, see Section 5.1.) This is usually phrased as saying that such structures admit a canonical *normal* Cartan connection. The main exception is $\mathfrak{g} = \mathfrak{gl}(n+1, \mathbb{R})$ with a $|1|$–grading such that $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g}_{\pm 1} \cong \mathbb{R}^n$. For an appropriate choice of G, the adjoint action identifies G_0 with $GL(\mathfrak{g}_{-1}) = GL(n, \mathbb{R})$. A first order structure for this group on a manifold M is just the full linear frame bundle of M and hence contains no information. In this case, an extension to a normal Cartan geometry of type (G, P) is equivalent to the choice of a projective equivalence class of torsion free connections on the tangent bundle TM, i.e. to a classical projective structure.

Normal Cartan geometries of type (G, P) as well as the equivalent underlying structures (i.e. classical projective structures respectively first order structures with structure group G_0) are often referred to as *AHS–structures*. AHS is short for "almost Hermitian symmetric". To explain this name, recall that the basic example of a Cartan geometry of type (G, P) is provided by the natural projection $G \to G/P$ and the left Maurer–Cartan form as the Cartan connection. This is called the *homogeneous model* of geometries of type (G, P). Now the homogeneous spaces G/P for pairs (G, P) coming from $|1|$–gradings as described above, are exactly the compact irreducible Hermitian symmetric spaces.

2.3 Natural bundles, fundamental derivative and tractor connection

Via forming associated bundles, any representation of the group P gives rise to a natural bundle for Cartan geometries of type (G, P). As we have seen above, P is the semi–direct product of the reductive subgroup G_0 and the normal vector subgroup P_+, so its representation theory is fairly complicated. Via the quotient homomorphism $P \to G_0$, any representation of G_0 gives rise to a representation of P. It turns out that the representations of P obtained in this way are exactly the completely reducible representations, i.e. the direct sums of irreducible representations. Correspondingly, we will talk about completely reducible and irreducible natural bundles on Cartan geometries of type (G, P). If we have a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ with underlying structure $(p_0 : \mathcal{G}_0 \to M, \theta)$ and $\mathbb{E}$ is a representation of G_0, which we also view as a representation of P, then we can naturally identify $\mathcal{G} \times_P \mathbb{E}$ with $\mathcal{G}_0 \times_{G_0} \mathbb{E}$. Hence completely reducible bundles can be easily described in terms of the underlying structure.

There is a second simple source of representations of P, which leads to an important class of natural bundles. Namely, one may restrict any representation of G to the subgroup P and the corresponding natural vector bundles are called *tractor bundles*. The most important tractor bundle is the *adjoint tractor bundle*. For a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ it is defined by $\mathcal{A} := \mathcal{G} \times_P \mathfrak{g}$, so it is the associated bundle with respect to the restriction of the adjoint representation of G to P. Now the P–invariant subspaces $\mathfrak{g}_1 \subset \mathfrak{p} \subset \mathfrak{g}$ give rise to a filtration $\mathcal{A}^1 \subset \mathcal{A}^0 \subset \mathcal{A}$ of the adjoint tractor bundle by smooth subbundles. By construction, $\mathcal{A}^1 \cong T^*M$ and since $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-1}$ we see that $\mathcal{A}/\mathcal{A}^0 \cong TM$. We will write $\Pi : \mathcal{A} \to TM$ for the resulting natural projection. Hence the adjoint tractor bundle has the cotangent bundle as a natural subbundle and the tangent bundle as a natural quotient.

The Killing form defines a G–invariant, non–degenerate bilinear form on $\mathfrak{g}$. It turns out that $\mathfrak{g}_1$ is the annihilator of $\mathfrak{p}$ with respect to the Killing form, which leads to duality with $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-1}$ observed above. On the level of associated bundles, we obtain a natural non–degenerate bilinear form on the adjoint tractor bundle

$\mathcal{A}$, which thus can be identified with the dual bundle $\mathcal{A}^*$. Under this pairing, the subbundle $\mathcal{A}^1$ is the annihilator of $\mathcal{A}^0$. The resulting duality between $\mathcal{A}^1$ and $\mathcal{A}/\mathcal{A}^0$ is exactly the duality between T^*M and TM.

The adjoint tractor bundle gives rise to a basic natural differential operator for AHS–structures called *fundamental derivative*. Let us start with an arbitrary representation $\mathbb{E}$ of P and consider the corresponding natural bundle $E := \mathcal{G} \times_P \mathbb{E} \to M$ for a geometry $(p : \mathcal{G} \to M, \omega)$. Then smooth sections of this bundle are in bijective correspondence with smooth maps $f : \mathcal{G} \to \mathbb{E}$, which are P–equivariant. In the special case $\mathbb{E} = \mathfrak{g}$ of the adjoint tractor bundle, we can then use the trivialization of $T\mathcal{G}$ provided by the Cartan connection ω to identify P–equivariant functions $\mathcal{G} \to \mathfrak{g}$ with P–invariant vector fields on $\mathcal{G}$. For a section $s \in \Gamma(\mathcal{A})$, we can form the corresponding vector field $\xi \in \mathfrak{X}(\mathcal{G})$ and use it to differentiate the equivariant function $f : \mathcal{G} \to \mathbb{E}$ corresponding to a section $\varphi \in \Gamma(E)$. The result will again be equivariant, thus defining a smooth section $\mathcal{D}_s\varphi \in \Gamma(E)$. Hence we can view the fundamental derivative as an operator $\mathcal{D} = \mathcal{D}^E : \Gamma(\mathcal{A}) \times \Gamma(E) \to \Gamma(E)$. The basic properties of this operator are the following:

Proposition 2.1 *Let $\mathbb{E}$ be a representation of P and let $E = \mathcal{G} \times_P \mathbb{E}$ be the corresponding natural bundle for an AHS–structure $(p : \mathcal{G} \to M, \omega)$. Then we have:*

(i) $\mathcal{D} : \Gamma(\mathcal{A}) \times \Gamma(E) \to \Gamma(E)$ is a first order differential operator which is natural, i.e. intrisic to the AHS–structure on M.

(ii) $\mathcal{D}$ is linear over smooth functions in the $\mathcal{A}$–entry, so we can also view $\varphi \mapsto \mathcal{D}\varphi$ as an operator $\Gamma(E) \to \Gamma(\mathcal{A}^* \otimes E)$.

(iii) For $s \in \Gamma(\mathcal{A})$, $\varphi \in \Gamma(E)$, and $f \in C^\infty(M)$, we have the Leibniz rule $\mathcal{D}_s(f\varphi) = (\Pi(s) \cdot f)\varphi + f\mathcal{D}_s\varphi$, where $\Pi : \Gamma(\mathcal{A}) \to \Gamma(TM)$ is the natural tensorial projection.

(iv) For a second natural bundle $F = \mathcal{G} \times_P \mathbb{F}$, a P–equivariant map $\mathbb{E} \to \mathbb{F}$, and the corresponding linear bundle map $\Phi : E \to F$, the fundamental derivatives on E and F are related by $\mathcal{D}^F_s(\Phi \circ \varphi) = \Phi \circ \mathcal{D}^E_s\varphi$ for all $s \in \Gamma(\mathcal{A})$ and $\varphi \in \Gamma(E)$.

The naturality statement in (iv) justifies denoting the fundamental derivatives on all natural bundles by the same letter. Since there is no restriction on the bundle E, the fundamental derivative in the form of part (ii) can evidently be iterated. For $\varphi \in \Gamma(E)$ we can form $\mathcal{D}\varphi$, $\mathcal{D}^{(2)}\varphi = \mathcal{D}(\mathcal{D}\varphi)$ and inductively $\mathcal{D}^{(k)}\varphi \in \Gamma(\otimes^k \mathcal{A}^* \otimes E)$.

The fundamental derivative is the basic tool of invariant theory on AHS structures. Further we shall need two natural differential operators which can be easily obtained from $\mathcal{D}$. First, the Killing form on $\mathfrak{g}$ gives rise to a nondegenerate bilinear form B on $\mathcal{A}$ thus also on $\mathcal{A}^*$. Hence composing $\mathcal{D}^{(2)} : \Gamma(E) \to \Gamma(\bigotimes^2 \mathcal{A}^* \otimes E)$ with $B \otimes \mathrm{id} : \Gamma(\bigotimes^2 \mathcal{A}^* \otimes E) \to \Gamma(E)$, we obtain the natural operator $\mathcal{C} : \Gamma(E) \to \Gamma(E)$

called *curved Casimir operator*. It turns out $\mathcal{C}$ is of the first order and acts by a scalar multiplication on the lowest homogeneity component of $\varphi \in \Gamma(E)$. The value of this scalar follows from the representation theory (which motivates the terminology) and we refer to [12] for details.

Next, observe the above theorem indicates $\mathcal{D}$ has some properties of linear connections on E. In fact, there exists such a natural connection if E is a tractor bundle and this will be henceforth our assumption. Then we have the action $\bullet : \mathcal{A} \otimes E \to E$ and it turns out that $\mathcal{D}_s\varphi + s \bullet \varphi \in \Gamma(E)$ depends only on $\Pi(s) \in \Gamma(TM)$. Thus using Proposition 2.1, we obtain the linear (natural) *normal tractor* connection ∇^E on E which, for a vector field $v \in \Gamma(TM)$ and any adjoint tractor $s \in \Gamma(\mathcal{A})$ such that $v = \Pi(s)$, is defined by

$$\nabla^E_v \varphi := \mathcal{D}_s\varphi + s \bullet \varphi \in \Gamma(E), \quad \varphi \in \Gamma(E).$$

2.4 Conformal geometry in the world of AHS structures

Here we present a brief summary, further details may be found in [3, 18]. Let M be a smooth manifold of dimension $n \geq 3$. Recall that a *conformal structure* of signature (p, q) on M is a class $[g]$ of metrics on M such that $\hat{g} \in [g]$ if $\hat{g} = e^{2\Upsilon}g$ for a smooth function Υ on M. Here we use the notation $\mathcal{E}[w]$ for density bundles and our convention for weights $w \in \mathbb{R}$ means $\mathcal{E}[-n] \cong \Lambda^n T^*M$. The conformal class $[g]$ determines the (density valued) *conformal metric* $\boldsymbol{g} \in \Gamma(S^2T^*M[2])$, i.e. the parameter in square brackets indicates tensor product with the corresponding density bundle. This yields the identification $TM \cong T^*M[2]$. For some calculations we shall need abstract indices in an obvious way, i.e. $\mathcal{E}_a = T^*M$, $\mathcal{E}^a = TM$ etc. Given a choice of metric $g \in [g]$, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. Here we are raising, lowering and and contracting inidces using the (inverse) conformal metric. Indices will be raised and lowered in this way without further comment. Note $\mathcal{E}[w]$ is trivialized by a choice of metric g from the conformal class, and we also write ∇ for the connection corresponding to this trivialization. The coupled connection ∇_a preserves the conformal metric.

The curvature $\mathsf{R}_{ab}{}^c{}_d$ of the Levi-Civita connection ∇_a (the Riemannian curvature) is given by $[\nabla_a, \nabla_b]v^c = \mathsf{R}_{ab}{}^c{}_d v^d$ ($[\cdot, \cdot]$ indicates the commutator bracket) for vector field $v^c \in \Gamma(\mathcal{E}^c)$. This can be decomposed into the totally trace-free Weyl curvature W_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to

$$\mathsf{R}_{abcd} = \mathsf{W}_{abcd} + 2\boldsymbol{g}_{c[a}\mathsf{P}_{b]d} + 2\boldsymbol{g}_{d[b}\mathsf{P}_{a]c}, \tag{2}$$

where $[\cdots]$ indicates antisymmetrisation over the enclosed indices. The Schouten tensor is a trace modification of the Ricci tensor $\mathsf{Ric}_{ab} = \mathsf{R}_{ca}{}^c{}_b$ and vice versa:

$\mathsf{Ric}_{ab} = (n-2)\mathsf{P}_{ab} + \mathsf{J}\boldsymbol{g}_{ab}$, where we write $\mathsf{J} = \mathsf{P}_a{}^a$ for the trace of P_{ab}. The *Cotton tensor* is defined by $\mathsf{Y}_{abc} := 2\nabla_{[b}\mathsf{P}_{c]a}$. Via the Bianchi identity this is related to the divergence of the Weyl tensor, $(n-3)\mathsf{Y}_{abc} = \nabla^d \mathsf{W}_{dabc}$.

A *conformal transformation* means to replace $g \in [g]$ by $\hat{g} = e^{2\Upsilon} g \in [g]$. We recall that, in particular, the Weyl curvature is conformally invariant, i.e. $\widehat{\mathsf{W}}_{abcd} = \mathsf{W}_{abcd}$. (Analogously, the Cotton-York tensor is invariant, i.e. $\widehat{\mathsf{Y}}_{abc} = \mathsf{Y}_{abc}$ for $n = 3$). Explicit formulae for the corresponding transformation of the Levi-Civita connection and its remaining curvature components are given in e.g. [3] in terms of the 1-form $\Upsilon_a := \nabla_a \Upsilon$. For example, $\widehat{\nabla}_a \varphi = \nabla_a \varphi + w \Upsilon_a \varphi$ for $\varphi \in \mathcal{E}[w]$ and

$$
\begin{aligned}
\hat{\nabla}_a f_b &= \nabla_a f_b - \Upsilon_a f_b - \Upsilon_b f_a + \boldsymbol{g}_{ab} \Upsilon^r f_r, \quad f_b \in \Gamma(\mathcal{E}_b), \\
\widehat{\mathsf{P}}_{ab} &= \mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \tfrac{1}{2} \Upsilon^c \Upsilon_c \boldsymbol{g}_{ab}.
\end{aligned} \tag{3}
$$

We have introduced tractor bundles in Section 2.3. Here we shall do it directly for the standard tractor bundle over $(M,[g])$ which corresponds to the standard representation of the group $G = SO(p+1, q+1)$. It is a vector bundle of rank $n+2$ defined, for each $g \in [g]$, by $[\mathcal{T}]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\widehat{g} = e^{2\Upsilon} g$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{T}]_g$ with $(\widehat{\alpha}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{T}]_{\widehat{g}}$ by the transformation

$$
\begin{pmatrix} \widehat{\alpha} \\ \widehat{\mu}_a \\ \widehat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_c\Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix}. \tag{4}
$$

It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{T}$ over the conformal manifold. Moreover, $\mathcal{T}$ is equipped by (invariant) normal tractor connection $\nabla^{\mathcal{T}}$ introduced in Section 2.3 and also an invariant metric $h \in \Gamma(S^2\mathcal{T}^*)$ of signature $(p+1, q+1)$ such that $\nabla^{S^2\mathcal{T}^*} h = 0$. That is, $\mathcal{T} \cong \mathcal{T}^*$ and we extend $\nabla^{\mathcal{T}}$ to the normal tractor connection on $\bigotimes \mathcal{T}$ by the Leibnitz rule. In fact, the original (and easier) way is to define h and $\nabla^{\mathcal{T}}$ directly by

$$
h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \nabla^{\mathcal{T}}_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + \mathsf{P}_{ab}\alpha \\ \nabla_a \tau - \mathsf{P}_{ab}\mu^b \end{pmatrix}. \tag{5}
$$

It is readily verified (using (4) and (5)) that both h and $\nabla^{\mathcal{T}}$ are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. It will be sometimes convenient to use also abstract tractor indices and we put $\mathcal{E}^A = \mathcal{T}$. Then h and its inverse allows to raise and lower tractor indices. The curvature $\Omega^{\mathcal{T}}$ of $\nabla^{\mathcal{T}}$ is defined by $[\nabla^{\mathcal{T}}_a, \nabla^{\mathcal{T}}_b] V^C = (\Omega^{\mathcal{T}})_{ab}{}^C{}_E V^E$ for $V^A \in \Gamma(\mathcal{E}^A)$. Note $\Omega^{\mathcal{T}}$ vanishes if and only if $\mathsf{W}_{abcd} = 0$ (for $n \geq 4$) or $\mathsf{Y}_{abc} = 0$ (for $n = 3$), see [3] for details.

Further, given a choice of $g \in [g]$, the *tractor-D operator* or *Thomas-D operator* $D_A : \mathcal{E}[w] \to \mathcal{E}_A[w-1] = \mathcal{T}[w-1]$ is defined by

$$D_A\varphi := \begin{pmatrix} w(n+2w-2)\varphi \\ (n+2w-2)\nabla_a\varphi \\ -(\Delta + w\mathsf{J})\varphi \end{pmatrix} \tag{6}$$

for $\varphi \in \mathcal{E}[w]$. This is conformally invariant, as can be checked directly using the formulae above.

Beside the standard tractor bundle $\mathcal{T} = \mathcal{E}^A$, we shall also need the *adjoint tractor bundle* $\mathcal{A} := \Lambda^2\mathcal{T} = \mathcal{E}^{[AB]}$. It follows from components of $[\mathcal{T}]_g$ that $[\mathcal{A}]_g = \mathcal{E}_b[2] \oplus \big(\mathcal{E}_{[bc]}[2] \oplus \mathcal{E}\big) \oplus \mathcal{E}_b$ and one can deduce from (5) that

$$\nabla^{\mathcal{A}}_a \begin{pmatrix} \alpha_b \\ \mu_{bc} \mid \nu \\ \rho_b \end{pmatrix} = \begin{pmatrix} \nabla_a\alpha_b - 2\mu_{ab} - \boldsymbol{g}_{ab}\nu \\ \nabla_a\mu_{bc} + \mathsf{P}_{a[b}\alpha_{c]} + \boldsymbol{g}_{a[b}\rho_{c]} \mid \nabla_a\nu + \mathsf{P}_{ar}\alpha^r - \rho_a \\ \nabla_a\rho_b - 2\mathsf{P}_a{}^r\mu_{rb} + \mathsf{P}_{ab}\nu \end{pmatrix} \tag{7}$$

where $\alpha_a \in \Gamma(\mathcal{E}_a[2])$, $\mu_{ab} \in \Gamma(\mathcal{E}_{[ab]}[2])$, $\nu \in \Gamma(\mathcal{E})$ and $\rho_a \in \Gamma(\mathcal{E}_a)$. We have $\mathcal{A} \cong \mathcal{A}^*$ because $\mathcal{T} \cong \mathcal{T}^*$. Further we shall need the conformally invariant *fundamental derivative* $\mathcal{D} : \Gamma(V) \to \Gamma(\mathcal{A}^* \otimes V)$ which we introduced in Section 2.3 for any natural bundle V. Alternatively, one can define $\mathcal{D}$ by an explicit formula which we shall do only for $\mathcal{D}$ acting on $\Gamma(\bigotimes TM \otimes \bigotimes \mathcal{T}[w])$, $w \in \mathbb{R}$. First, for $\Phi_A \in \Gamma(\mathcal{E}_A[w]) = \Gamma(\mathcal{T}[w])$ and $\varphi_a \in \Gamma(\mathcal{E}_a) = \Gamma(T^*M) \cong \Gamma(TM[-2])$ we put

$$\mathcal{D}_{BC}\Phi_A = \begin{pmatrix} 0 \\ 0 \mid w\Phi_A \\ \nabla_b\Phi_A \end{pmatrix} + h_{A[B}\Phi_{C]} \in \Gamma(\mathcal{E}_{[BC]A}[w]) = \Gamma(\mathcal{A}^* \otimes \mathcal{T}[w]),$$

$$\mathcal{D}_{BC}\varphi_a = \begin{pmatrix} 0 \\ \boldsymbol{g}_{a[b}\varphi_{c]} \mid -\varphi_a \\ \nabla_b\varphi_a \end{pmatrix} \in \Gamma(\mathcal{E}_{[BC]a}) = \Gamma(\mathcal{A}^* \otimes T^*M).$$

Then we extend $\mathcal{D}$ to sections of tensor products $\bigotimes TM \otimes \bigotimes \mathcal{T}[w]$ using the Leibnitz rule.

Next we discuss so called strong invariance of differential operators in the sense of [17]. This can be defined for any AHS structure but we shall need it only in the conformal case. Let $F : \Gamma(V_1) \to \Gamma(V_2)$ be a differential operator given by an explicit formula in terms of the Levi-Civita connection ∇_a, its curvature R_{abcd} and various algebraic operations. Given any tractor bundle $\mathcal{T}' \subseteq \bigotimes \mathcal{T}$, we define the operator $F^\nabla : \Gamma(V_1 \otimes \mathcal{T}') \to \Gamma(V_2 \otimes \mathcal{T}')$ by replacing every appearance of ∇ by $\nabla^{\mathcal{T}'}$ in the formula for F. Now assume F is conformally invariant; we say F is *strongly invariant* if also the operator F^∇ is conformally invariant for any tractor bundle $\mathcal{T}'$. (That is, strong invariance is rather a property of specified formulae of differential operators.) In particular, the tractor D-operator D_A is strongly invariant which we shall need later. In this case, we shall write D_A instead of $(D_A)^\nabla$ for simplicity. Further examples of strongly invariant operators are the fundamental derivative $\mathcal{D}$ or the conformal Laplacian Δ_Y from Section 1.2.

3 Invariant quantization

Henceforth we assume $(\mathcal{G} \to M, \omega)$ is an AHS structure on M and let E and F be natural irreducible bundles on M. In this section we shall construct an invariant quantization for differential operators $\Gamma(E) \to \Gamma(F)$ of the order $\leq k$ and we denote by $\mathrm{Symb}^k(E, F)$ the corresponding symbol bundle. Our result will be generic (but not complete) and to formulate this precisely, we shall consider rather operators $\Gamma(E) \to \Gamma(F[\delta])$ with the parameter $\delta \in \mathbb{R}$. That is, we shall discuss existence of an invariant quantization

$$\mathcal{Q}^k : \Gamma\big(\mathrm{Symb}^k(E, F[\delta])\big) \to \mathrm{Diff}^k(E, F[\delta]), \qquad \mathrm{symb} \circ \mathcal{Q}^k = \mathrm{id} \tag{8}$$

for kth order operators, cf. Section 1.4, depending on δ. This result, based on [13], is formulated in Section 3.1 below. Then, following [34], we present a different invariant quantization specifically designed for conformal densities (see Section 3.2) to obtain a complete classification in this case.

3.1 A generic construction of invariant quantization

Given $\sigma \in \Gamma\big(\mathrm{Symb}^k(E, F[\delta])\big)$ and $\varphi \in \Gamma(E)$, the required invariant quantization $\mathcal{Q}^k$ from (8) gives rise to the bilinear invariant operator $(\sigma, \varphi) \mapsto \mathcal{Q}^k(\sigma)(\varphi) \in \Gamma(F[\delta])$. We shall construct such operator first (for all δ's) and then discuss for which $\delta \in \mathbb{R}$ this actually yields $\mathcal{Q}^k$ satisfying (8). We shall start with two linear invariant operators (one for σ and one for φ) which we shall combine into a bilinear operator afterwards. Note our construction is motivated by a (much simpler) invariant quantization on Riemannian structures where one can simply combine the symbol σ with iterated Levi-Civita connection applied to φ. Such straightforward construction is not possible on AHS structures (as there is no canonical affine connection) so we replace the Levi-Civita connection by the fundamental derivative. Thus we need to pass to tractors.

First we define a linear invariant operator L acting on σ. As mentioned in Section 2.3, TM is a quotient of $\mathcal{A}$. Thus also $\mathrm{Symb}^k(E, F[\delta]) = S^kTM \otimes E^* \otimes F[\delta]$ is a (completely reducible) quotient of $V := S^k\mathcal{A} \otimes E^* \otimes F[\delta]$. We denote the projection by $\Pi : V \to \mathrm{Symb}^k(E, F[\delta])$. Let us write the decomposition of σ into irreducibles as $\sigma = \bigoplus_{i \in I} \sigma_i \in \Gamma\big(\mathrm{Symb}^k(E, F[\delta])\big)$ where I is an index set. Then there is an invariant differential operator

$$\begin{aligned} &L = P(\mathcal{C}) : \Gamma(S^kTM \otimes E^* \otimes F[\delta]) \to \Gamma(S^k\mathcal{A}M \otimes E^* \otimes F[\delta]) \\ &\text{such that } \ \Pi\big(L(\sigma)\big) = \bigoplus_{i\in I} \gamma_i\sigma_i \end{aligned} \tag{9}$$

where $P(\mathcal{C})$ denotes a polynomial operator in the curved Casimir $\mathcal{C}$, cf. Section 2.3 and $\gamma_i = p_i(\delta, n)$ for some nontrivial polynomials p_i. The upshot of this construction is that for all weights $\delta \in \mathbb{R}$ up to a finite number (when $\gamma_i(\delta, n) = 0$ for some $i \in I$) we have $\Pi\big(L(\bigoplus \gamma_i^{-1}\sigma_i)\big) = \sigma$. In this sense, L is *generically* a splitting operator of the projection Π.

Next we define a linear invariant operator acting on φ. This will be simply the iterated fundamental derivative $\mathcal{D}^{(k)}\varphi = \mathcal{D}\ldots\mathcal{D}\varphi$ for $\varphi \in \Gamma(E)$ where $\mathcal{D}^{(k)} : \Gamma(E) \to \Gamma(\bigotimes^k \mathcal{A}^* \otimes E)$.

These two linear operators are used to build a bilinear operator in the theorem below. We shall use the obvious pairing

$$\langle\, ,\, \rangle : \big(\bigotimes^k \mathcal{A} \otimes E^* \otimes F[\delta]\big) \times \big(\bigotimes^k \mathcal{A}^* \otimes E\big) \to F[\delta].$$

Theorem 1 (Theorem 5 and Corollary 6 from [13]) *Using the notation as above, the following holds:*

(i) The map $(\sigma, \varphi) \mapsto \langle L(\sigma), \mathcal{D}^{(k)}\varphi\rangle$ *defines an invariant bilinear operator* $\Gamma\big(\mathrm{Symb}^k(E, F[\delta])\big) \times \Gamma(E) \to \Gamma(F[\delta])$.

(ii) For $\sigma = \bigoplus_{i\in I} \sigma_i \in \Gamma(S^kTM \otimes E^* \otimes F[\delta])$*, the operator* $\overline{\mathcal{Q}}^k(\sigma) : \Gamma(E) \to \Gamma(F[\delta])$ *defined by* $\overline{\mathcal{Q}}^k(\sigma)(\varphi) := \langle L(\sigma), \mathcal{D}^{(k)}\varphi\rangle$ *is of order at most* k *and has principal symbol* $\bigoplus_{i\in I} \gamma_i\sigma_i$.

(iii) Assume the weight $\delta \in \mathbb{R}$ *is generic in the sense that* $\gamma_i = p_i(n, \delta) \neq 0$ *for all* $i \in I$*, cf. the discussion below (9). Then he map* $\mathcal{Q}^k(\sigma) := \overline{\mathcal{Q}}^k(\bigoplus_{i\in I} \gamma_i^{-1}\sigma_i) \in \mathrm{Diff}^k(E, F[\delta])$ *defines an invariant quantization for operators* $\Gamma(E) \to \Gamma(F[\delta])$ *of the order* k.

In fact, the above construction of $\mathcal{Q}^k$ simplifies the construction in [13] where a bigger set of generic weights was obtained. One can also derive an estimate for the number of nongeneric weights and an upper bound for these. We refer to [13] for details.

Let us emphasize that the previous result does not say anything about possible nonexistence of invariant quantization for nongeneric weights δ. Also, there is no reason to expect uniqueness. Both these questions are of course important and we shall address them (in a special case) in the next section.

3.2 Invariant quantization on conformal densities

The full classification of existence and uniqueness of invariant quantization should be understood primarily in the locally flat case which will be henceforth our assumption. The following crucial result is due to Michel [29] and it is convenient to formulated it for (usually irreducible) subbundle $H \subseteq \mathrm{Symb}^k(E,F) := S^kTM \otimes E^* \otimes F$ for $k \geq 0$.

Theorem 3.1 ([29]) *Let E and F be irreducible natural bundles for the locally flat AHS structure $(\mathcal{G} \to M, \omega)$. Then the invariant quantization $\mathcal{Q}^k_H$ for operators $\mathrm{Diff}^k(E,F)$ restricted to symbols in $H \subseteq \mathrm{Symb}^k(E,F)$ exists and is unique if and only if there is no invariant linear differential operator $\Gamma\big(H\big) \to \Gamma\big(\mathrm{Symb}^{k-i}(E,F)\big)$ for $i = 1, \dots, k$.*

Roughly speaking, this theorem says that unique existence of invariant quantization is obstructed by existence of linear invariant operators between symbols (of different order). This is very useful because linear invariant operator on AHS structures are rare in the sense that, given the bundle E, the space of linear invariant operators with the source bundle E is finite dimensional. Moreover, the space of conformally invariant linear operators is completely classified [5, 6]. Thus, we can expect more specific results in this case.

Assume $(M,[g])$ is a locally flat conformal manifold and put $E := \mathcal{E}[w]$ and $F := \mathcal{E}[w+\delta]$. We say the weight $\delta \in \mathbb{R}$ is *critical* if the unique existence of invariant quantization for operators $\mathrm{Diff}^k(\mathcal{E}[w], \mathcal{E}[w+\delta])$ is lost for some (or equivalently any, cf. Theorem 3.1) weight $w \in \mathbb{R}$. Further we say the weight w is *resonant* for the critical weight δ if invariant quantization for $\mathrm{Diff}^k(\mathcal{E}[w], \mathcal{E}[w+\delta])$ (nonuniquely) exists. Note the set of generic weights from Section 3.1 is significantly smaller than the set of noncritical weights which can be seen already from first order quantization on densities. Here there is at most one linear invariant operator $\Gamma(TM[\delta]) \to \Gamma(\mathcal{E}[\delta])$ (i.e. at most one critical weight δ, cf. Theorem 3.1) whereas there are four nongeneric weights δ. (The latter fact follows from the construction (9) via curved Casimirs as there are four irreducible subquotients of the conformal adjoint tractor bundle $\mathcal{A}$.)

The main aim of this section is to construct invariant quantization on densities for all noncritical weights. Details are in [34] in terms of invariant quantization $Q^k_H : \Gamma(H) \to \mathrm{Diff}^k(\mathcal{E}[w], \mathcal{E}[w+\delta])$ for all irreducible subbundles H of symbols. We put

$$H := S^{k'}_0 TM[\delta'] \subseteq S^k TM[\delta] \quad \text{where} \quad k' = k - 2\ell,\ \delta' = \delta + 2\ell \tag{10}$$

where $0 \leq 2\ell \leq k$ and the subscript 0 indicates the trace free part. First we shall discuss critical weights for H according to Theorem 3.1. For parameters in (10), we put

$$\begin{aligned}&\Sigma_{k'} = \{-(n+k'+j-2) \mid 1\leq j \leq k'\}, \quad \Sigma'_{k',\ell} = \{j-1 \mid 1\leq j \leq \ell\},\\ &\text{and } \Sigma''_{k',\ell} = \{-\frac{1}{2}(n+2k'-2j) \mid 1 \leq j \leq \ell\}\end{aligned} \tag{11}$$

and we put $\Sigma_0 = \Sigma'_{k',0} = \Sigma''_{k',0} = \emptyset$. Using this and the classification in [5, 6], the following display exhausts all linear invariant operators $\Gamma(H) \to \Gamma(S^{k-i}TM[\delta])$:

$$\begin{aligned}&\Gamma(H) \to \Gamma(S_0^{k'-i}TM[\delta']), && \delta' = -(n+k'+i-2) \in \Sigma_{k',0},\\ &\Gamma(H) \to \Gamma(S_0^{k'+2j}TM[\delta'-2j]), && \delta' = j-1 \in \Sigma'_{k',\ell},\\ &\Gamma(H) \to \Gamma(S_0^{k'}TM[\delta'-2j]), && \delta' = -\tfrac{1}{2}(n+2k'-2j) \in \Sigma''_{k',\ell}.\end{aligned} \tag{12}$$

Note the case $\delta' \in \Sigma_{k',0}$ is a divergence type operator of order $k'-i+1$, $\delta' \in \Sigma'_{k',\ell}$ is the generalized conformal Killing operator of order j, cf. Section 4 and $\delta' \in \Sigma''_{k',\ell}$ yields a power of Laplacian type operator (on symmetric tensor fields) of order $2j$. Combining this list with Theorem 3.1, the set of critical weights δ' is $\Sigma_{k',\ell} := \Sigma_{k'} \cup \Sigma'_{k',\ell} \cup \Sigma''_{k',\ell}$.

It remains to construct $\mathcal{Q}^k_H$ for all weights $\delta' \notin \Sigma_{k',\ell}$ which we shall do in the general (curved) setting. We shall start with the special case $k = k'$ and $\delta = \delta'$ and denote the invariant quantization by $\mathcal{Q}^k_\delta : \Gamma(S^k_0TM) \to \mathrm{Diff}^k(\mathcal{E}[w], \mathcal{E}[w+\delta])$ in this case. A detailed analysis of operators in (12) and Theorem 3.1 reveals that the set of critical weights δ for $\mathcal{Q}^k_\delta$ is exactly Σ_k. An existence of $\mathcal{Q}^k_\delta$ is stated in the part (i) of the following theorem.

Theorem 2 (Theorem 3.1 and Theorem 3.3 from [34]) *Let $(M,[g])$ be an arbitrary conformal manifold and consider the symbol $\sigma \in \Gamma(H) \subseteq \Gamma\big(\mathrm{Symb}^k(\mathcal{E}[w], \mathcal{E}[w+\delta])\big)$ and parameters k, k', δ, δ' and ℓ as in (10). Then*

(i) There is an explicit construction of $\mathcal{Q}^{k'}_{\delta'} : \Gamma(H) \to \mathrm{Diff}^{k'}(\mathcal{E}[w], \mathcal{E}[w+\delta'])$ for all noncritical weights $\delta' \notin \Sigma_{k'}$ such that the formula for $\mathcal{Q}^{k'}_{\delta'}$ is strongly invariant.

(ii) Using $\mathcal{Q}^{k'}_{\delta'}$ from (i) and $\sigma \in \Gamma(H)$, the construction

$$\sigma \mapsto D^{A_1}\cdots D^{A_\ell}\mathcal{Q}^{k'}_{\delta'}(\sigma)^\nabla D_{A_\ell}\cdots D_{A_1} : \mathcal{E}[w] \to \mathcal{E}[w+\delta]$$

gives rise to the invariant quantization $\mathcal{Q}^k_H$ for any $w \in \mathbb{R}$ and every noncritical weight $\delta' \notin \Sigma_{k',\ell}$. Here D_A is the tractor D-operator, cf. (6) and the notation $\mathcal{Q}^{k'}_{\delta'}(\sigma)^\nabla$ is explained below.

Recall the notion of strong invariance (together with the notation $\mathcal{Q}^{k'}_{\delta'}(\sigma)^\nabla$) was introduced at the end of Section 2.4.

Idea of the proof. It is worth to emphasize that whereas the generic construction in Section 3.1 was based on basic tools of the invariant calculus on AHS structures, the construction in the theorem is more involved (and it works only for conformal structures). Indeed, the proof of the part (ii) in [34] is based on a rather long computation. But at least the part (i) is related to the generic construction of $\mathcal{Q}^{k'}$. More precisely, $\mathcal{Q}^{k'}_{\delta'}$ is obtained by a careful refinement of $\mathcal{Q}^{k'}$. This is based on the observation that the target spaces of L and $\mathcal{D}^{(k')}$ used in Theorem 1 (i) are actually "too big" which induces too many nongeneric weights. It turns out there is a subbundle $\widetilde{\mathcal{A}} \subseteq \mathcal{A}$ with the subquotient $\widetilde{\mathcal{A}}^*$ of $\mathcal{A}$ and modifications $\widetilde{L} : \Gamma(H) \to \Gamma(\widetilde{\mathcal{A}}^*[\delta'])$ of L and $\widetilde{\mathcal{D}}^{(k')} : \Gamma(\mathcal{E}[w]) \to \Gamma(\widetilde{\mathcal{A}}[w])$ such that $\mathcal{Q}^{k'}_{\delta'}(\sigma)(\varphi) = \langle \widetilde{L}(\sigma), \widetilde{\mathcal{D}}^{(k')}(\varphi)\rangle$ for $\varphi \in \Gamma(\mathcal{E}[w])$. Details are in [34]. $\square$

Finally note that for δ critical, resonant weights w are closely related to existence of linear invariant operators on $\mathcal{E}[w]$. This can be used [34] to construct invariant quantization $\mathrm{Diff}^k(\mathcal{E}[w], \mathcal{E}[w+\delta])$ for such pairs of δ and w. This is, however, limited in the curved case since existence of certain linear invariant operators on $\mathcal{E}[w]$ is lost if we pass to the curved setting [18]. These operators are known as conformal powers of the Laplacian (or GJMS operators [21]). We shall focus on these operators (in the locally flat case) in the next section.

4 Symmetries of conformal powers of the Laplacian

We discussed the conformal version of the Laplacian Δ_Y in Section 1.2. The classification [5, 6] of linear conformal operators tells us that also powers of the Laplacian have conformally invariant analogues

$$P_r = \Delta^r + lot : \Gamma(\mathcal{E}[-n/2+r]) \to \Gamma(\mathcal{E}[-n/2-r]) \tag{13}$$

where lot stands for "lower order terms". It is however a nontrivial question whether such operators P_r exist also in the curved setting and it turns out this is generally not true for n even and $2r > n$ [18].

Similarly as in Section 1.1, we say the operator $\Sigma : \Gamma(\mathcal{E}[-n/2+r]) \to \Gamma(\mathcal{E}[-n/2+r])$ is a *symmetry* of P_r if

$$P_r\Sigma = \Sigma' P_r \quad \text{for some} \quad \Sigma' : \Gamma(\mathcal{E}[-n/2-r]) \to \Gamma(\mathcal{E}[-n/2-r]). \tag{14}$$

We would like to describe the algebra $HS_r := \overline{HS_r}/(P_r)$ where $\overline{HS}_r$ is the algebra of all symmetries of P_r and $(P_r) \subseteq \overline{HS}_r$ denotes the left ideal of *trivial* symmetries, cf. Section 1. The result for the locally flat case is summarized (based on [20]) in Section 4.1 below. However, such question is far too ambitious in general. In fact, it is a complicated problem to understand HS_r (as the vector space) already for Δ_Y and we present (based on [30]) the answer at least for 2nd order symmetries in Section 4.2.

4.1 Symmetries of P_r in the locally flat case

Henceforth we assume $(M, [g])$ is locally flat. Given the symbol $\sigma \in \Gamma(S^kTM)$, the invariant quantization for $\mathrm{Diff}^k(\mathcal{E}[w], \mathcal{E}[w])$, $w = -n/2+r$ is an obvious candidate for a symmetry Σ of the order k of P_r. Assume for simplicity $\sigma \in \Gamma(H) \subseteq \Gamma(S^kTM)$ for an irreducible subbundle H. Then observe the corresponding weight $\delta = 0$ is both generic for $\mathcal{Q}^k$ from Section 3.1 (which follows from properties of curved Casimirs) and also noncritical for $\mathcal{Q}^k_H$ from Section 3.2. Using further the uniqueness in Theorem 3.1, we conclude $\mathcal{Q}^k(\sigma) = \mathcal{Q}^k_H(\sigma)$, i.e. both constructions of invariant quantization coincide.

Of course, not all sections σ can appear as symbols of symmetries so we need to find conditions such symbols must satisfy. As a motivation, consider the commutator $T := P_r\mathcal{Q}^k(\sigma) - \mathcal{Q}^k(\sigma)P_r : \Gamma(\mathcal{E}[-n/2+r]) \to \Gamma(\mathcal{E}[-n/2-r])$. Taking the principal symbol of T, we obtain the operator $\Gamma(H) \ni \sigma \overset{\psi}{\mapsto} \mathrm{symb}(T) \in \Gamma(S^{k+2}TM[-2r])$ which is conformally invariant by construction. Clearly if $P_r\mathcal{Q}^k(\sigma) = \mathcal{Q}^k(\sigma)P_r$ then $\psi(\sigma) = 0$. Now it remains to find out to which of three classes in (12) the operator ψ belongs.

The precise characterization is actually more complicated as the above consideration does not determine ψ uniquely. A direct computation in [20] shows the middle operator in (12) controls symbols of symmetries. More specifically, this is the linear conformally invariant operator

$$\begin{aligned} &\Gamma(S^kTM) \supseteq \Gamma(S_0^{k-2\ell}TM[2\ell]) =: \Gamma(H) \overset{\psi}{\to} \Gamma(S_0^{k+1}[-2\ell-2]), \\ &\sigma^{a_1\dots a_{k-2\ell}} \overset{\psi}{\mapsto} \nabla^{(b_0} \cdots \nabla^{b_{2\ell}}\sigma^{a_1\dots a_{k-2\ell})_0} + lot \end{aligned} \tag{15}$$

where $0 \le 2\ell \le k$ and we have used abstract indices. That is, H is the subbundle of the form (10) with $\delta = 0$, $k' = k - 2\ell$ and $\delta' = 2\ell$. Solutions σ of ψ are known as *generalized conformal Killing tensors*. Also note ψ is overdetermined and conformally invariant also in general curved setting. Such operators will be studied in Section 5 in detail.

The following theorem describes HS_r as the vector space:

Theorem 3 (Theorem 2.4 from [20]) *Let r, k, ℓ, $k' = k - 2\ell$ and H be given as above.*

(i) For each non-zero $\sigma \in \Gamma(H)$ such that $\ell \in \{0, 1, \cdots, r-1\}$, a solution of (15), there is canonically associated a non-

trivial symmetry $\Sigma := \mathcal{Q}^k_H(\sigma)$ *of* P_r *with the leading term*

$$\sigma^{a_1\ldots a_{k'}}(\nabla_{a_1}\cdots\nabla_{a_{k'}})\Delta^\ell.$$

(ii) Modulo trivial symmetries, locally any symmetry of P_r *is a linear combination of such operators* Σ*, for various solutions* σ *of (15), with* ℓ *and* r *in the range assumed here.*

The next step is to describe HS_r as a $\mathfrak{g}$-module, $\mathfrak{g} = \mathfrak{so}_{p+1,q+2}$ where (p,q) is the signature. This is based on the bijective correspondence between the space of solutions of ψ as given by (15), which we denote by $\mathcal{H}^{k'}_\ell$, and ∇^V-parallel sections of the tractor bundle $V := (\boxtimes^{k'}\mathcal{A}) \boxtimes (\boxtimes^{2\ell}\mathcal{T})$ where $\boxtimes$ is the Cartan product, $\mathcal{A}$ is the adjoint tractor bundle and $\mathcal{T}$ is the standard tractor bundle. (More details concerning this relationship can be found in Section 5.) Recall the standard and the adjoint tractor bundle are induced by the standard and the adjoint representation of $\mathfrak{g}$, respectively. Using the symbolism of Young diagrams, the standard and the adjoint representation of $\mathfrak{g}$ are $\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$, respectively. Thus, as a $\mathfrak{g}$-module, we have

$$HS_r \cong \bigoplus_{k'=0}^{\infty}\bigoplus_{\ell=0}^{r-1}\mathcal{H}^{k'}_\ell \quad \text{where} \quad \mathcal{H}^{k'}_\ell \cong \begin{array}{l} \overbrace{\square\square\cdots\square}^{k'}\overbrace{\square\square\cdots\square}^{2\ell} \\ \square\square\cdots\square \end{array}_0 \tag{16}$$

To formulate the algebraic structure of HS_r in the theorem below, we shall further need following projections of the tensor product $U_1 \otimes U_2$ for $U_1, U_2 \in \mathfrak{g} = \begin{smallmatrix}\square\\\square\end{smallmatrix}$:

$$U_1 \boxtimes U_2 \in \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}_0, \quad U_1 \bullet U_2 \in \square\square_0, \quad \langle U_1, U_2\rangle \in \mathbb{R} \quad \text{and} \quad [U_1, U_2] \in \begin{smallmatrix}\square\\\square\end{smallmatrix}.$$

Theorem 4 (Theorem 2.5 from [20]) *The algebra* HS_r *is isomorphic to the tensor algebra* $\bigotimes \mathfrak{g}$ *modulo the two sided ideal generated by*

$$U_1 \otimes U_2 - U_1 \boxtimes U_2 - U_1 \bullet U_2 - \frac{1}{2}[U_1, U_2] + \frac{(n-2r)(n+2r)}{4n(n+1)(n+2)}\langle U_1, U_2\rangle$$

for $U_1, U_2 \in \mathfrak{g}$ *and the image of* $\boxtimes^{2r}\square$ *in* $\otimes^{2r}\mathfrak{g}$*.*

4.2 An inroad to the curved case

Here we briefly discuss 2nd order symmetries of the conformal Laplacian $P_1 = \Delta_Y$ on a general conformal manifold $(M, [g])$. Even such a specific question was understood only recently [30] which illustrates the complexity of the problem for general curved symmetries of operators P_r. Note first order symmetries are given, due to the conformal invariance of Δ_Y, by the Lie derivative L_X along conformal Killing fields X on $(M, [g])$.

Assume Σ is a 2nd order symmetry of Δ_Y; then modulo trivial symmetries, the symbols of Σ is $\sigma \in \Gamma(S^2_0 TM)$. These symbols are controlled by the condition $\psi(\sigma) = 0$ in the locally flat case according to Theorem 3 (i). This is conformal Killing operator $\nabla^{(a}\sigma^{bc)_0} = 0$ in this case and one can show by the direct computation that this condition is necessary also in the curved case. The problem is that $\nabla^{(a}\sigma^{bc)_0} = 0$ is not sufficient, i.e. not all conformal Killing 2-tensors σ^{ab} give rise to symmetries of Δ_Y. There is an additional obstruction for symbols σ^{ab} of symmetries formulated in terms of the 1-form

$$\mathsf{Obs}(\sigma)_a = \tfrac{2(n-2)}{3(n+1)}\big(\mathsf{W}^t{}_{rsa}\nabla_t - 3\mathsf{Y}_{rsa}\big)\sigma^{rs} \in \Gamma(\mathcal{E}_a) = \Gamma(T^*M) \tag{17}$$

where W_{abcd} is the Weyl tensor and Y_{abc} the Cotton-York tensor, cf. Section 2.4. One can show (17) is well defined (i.e. conformally invariant). Also note $\mathcal{Q}^2(\sigma) = \mathcal{Q}^2_H(\sigma)$ for $H = S^2_0 TM \subseteq S^2 TM$ which holds not only in the flat case (cf. the first paragraph of 4.1) but also in the curved case (which can be shown by a direct computation.)

Theorem 5 (Theorem 4.11 from [30]) *Let $\sigma \in \Gamma(S^2_0 TM)$.*

(i) There is a symmetry Σ of Δ_Y with the principal symbol σ if and only if σ is a conformal Killing 2-tensor and $\mathsf{Obs}(\sigma)$ is an exact 1-form.

(ii) Assume σ satisfies conditions in (i). Then modulo trivial symmetries and first and zero order symmetries, Σ has the form $\Sigma = \mathcal{Q}^2_H(\sigma) + f$ for a function $f \in C^\infty(M)$ satisfying $\mathsf{Obs}(\sigma) = -2df$.

This characterization of symbols of symmetries deserves a more detailed discussion, however. There are no conformal Killing tensors (or generally no solutions of overdetermined equations) on generic conformal structures. Thus one might think that the very existence of such solution σ is so restrictive that $\mathsf{Obs}(\sigma)$ is *always* exact. It is proved in [30] that this is not true. That is, we found a conformal manifold $(M, [g])$ with a conformal Killing 2-tensor σ such that $\mathsf{Obs}(\sigma)$ is not exact (actually not closed). Summarizing, exactness of $\mathsf{Obs}(\sigma)$ is a nontrivial condition.

Finally note, above considerations indicate there are curvature obstructions for existence of conformal Killing tensors. The conceptual way in this direction is to study prolongation of the corresponding overdetermined system of PDE's. This is studied, in much more general context, in the next section.

5 Prolongation of first BGG operators

We have already used the bijective correspondence between solutions of ψ on $H \subseteq S^kTM$ from (15) and a parallel sections of the tractor bundle $V := (\boxtimes^{k'}\mathcal{A}) \boxtimes (\boxtimes^{2\ell}\mathcal{T})$ in the locally flat case, cf. the discussion below Theorem 3. A theoretical background of this correspondence is summarized below in Section 5.1, our aim here is to explain that this is indeed a "prolongation" in the usual sense used in the study of PDE's.

The point is that H is a quotient of V hence we have the (algebraic) projection $\Pi_0 : \Gamma(V) \to \Gamma(H)$. The crucial fact is that there is an invariant *differential* splitting $L_0 : \Gamma(H) \to \Gamma(V)$ of Π_0, i.e. $\Pi_0 \circ L_0 = \mathrm{id}_H$. Then we can use the normal tractor connection ∇^V on V and assuming $\psi(\sigma) = 0$, we conclude $\nabla^V L_0(\sigma) = 0$. The latter can be deduced for example from the classification [5, 6] of conformally invariant operators (since nonvanishing $\nabla^V L_0(\sigma)$ would give rise to a conformal operator on $\sigma \in \Gamma(H)$ which cannot exist). Thus $L_0(\sigma)$ is a closed system (composed of σ and new variables) in the sense that $\nabla^V L_0(\sigma)$ is algebraic in components of $L_0(\sigma)$. Thus $L_0(\sigma)$ with ∇^V is the prolonged system for σ and ψ from (15). Also note $L_0 = P(\mathcal{C})$ for some polynomial P in the curved Casimir $\mathcal{C}$ thus operators L_0 and Π_0 are closely related to L and Π used in Section 3.1, cf. the discussion around (9).

In fact, the 1-1 correspondence between solutions of certain invariant operators and parallel sections of suitable tractor bundles holds for all locally flat AHS manifold. This follows from the "BGG machinery" which we review in Section 5.1 below. Thus an obvious question arises: is there such 1-1 correspondence also in the curved case? The answer is positive but not straightforward. The point is that the normal tractor connection ∇^V does not have desired properties in the curved case. Our solution is to find a new normalization and consequently a new tractor connection which we shall denote by $\widetilde{\nabla}^V$ and call *prolongation* connection. Results below are based on articles [23, 24].

5.1 Invariant prolongation connections on AHS manifolds

This construction is parametrised by the choice of the tractor bundle $V = \mathcal{G} \times_P \mathbb{V}$. Then the normal tractor connection ∇^V extends to the exterior covariant derivative $d^\nabla : \mathcal{E}^p(V) \to \mathcal{E}^{p+1}(V)$ where $\mathcal{E}^p(V)$ denotes the space V-valued p-forms on M. (We write d^{∇} instead of d^{∇^V} for simplicity.) Note the lowest homogeneity (i.e. invariant) component of d^{∇} is algebraic, cf. (5) and known as the *Kostant differential* $\partial_p : \mathcal{E}^p(V) \to \mathcal{E}^{p+1}(V)$, [27]. Its adjoint, the *Kostant codifferential* $\partial^*_{p+1} : \mathcal{E}^{p+1}(V) \to \mathcal{E}^p(V)$ is P-invariant (contrary to ∂_p) and satisfies $\partial^*_p \circ \partial^*_{p+1} = 0$. This defines Lie algebra cohomology bundles $H_p := \ker \partial^*_p / \mathrm{im}\, \partial^*_{p+1}$ as subquotients of $\ker \partial^*_p \subseteq \Lambda^p T^* M \otimes V$. (By abuse of notation, we consider ∂^*_p acting both on bundles and sections.) The crucial fact here is that the corresponding projection Π_p has a unique invariant differential *BGG–splitting* L_p with

$$\begin{aligned} &\Pi_p \colon \mathcal{E}^p(V) \supseteq \ker \partial^*_p \to \Gamma(H_p) \quad \text{and} \quad L_p \colon \Gamma(H_p) \to \ker \partial^*_p \subseteq \mathcal{E}^p(V) \\ &\text{such that} \quad \partial^*_{p+1}(d^{\nabla}(L_p(\sigma))) = 0 \quad \text{for every } \sigma \in \Gamma(H_p). \end{aligned} \tag{18}$$

These properties and constructions – usually termed "BGG machinery" – lead to the sequence (actually complex in the locally flat case) of invariant differential operators D_p,

$$D_p : \Gamma(H_p) \to \Gamma(H_{p+1}), \qquad D_p := \Pi_{p+1} \circ d^{\nabla} \circ L_p. \tag{19}$$

Note only the *first BGG-operator* D_0 is overdetermined and we want to find an invariant prolongation of the systems $D_0(\sigma) = 0$ on $\sigma \in \Gamma(H_0)$. (Here $D_0 = \psi$ from (15) for the bundle V as in Section 4.1.) Henceforth we assume $p = 0$. Note the prolongation property from the locally flat case (cf. beginning of Section 5) is lost as generally $L_1 \circ D_0 \neq \nabla^V \circ L_0$. Thus $D_0(\sigma) = 0$ does not generally imply $\nabla^V L_0(\sigma) = 0$. In more detail, the left hand side of the inequality is in the kernel of $\partial^*_2 \circ d^{\nabla}$ according to (18) but this does not generally hold for the right hand side. It gives us a hint, however, as the right hand side leads to $(\partial^*_2 \circ d^{\nabla} \circ \nabla^V \circ L_0)(\sigma) = \partial^*_2 \Omega^V(L_0(\sigma))$ where Ω^V is the curvature of ∇^V. This vanishes in the locally flat case (because $\Omega^V = 0$) and requiring this in general – for a new tractor connection $\widetilde{\nabla}^V$ with the curvature $\widetilde{\Omega}^V$ – one finds the construction below.

Following the approach in [23], we introduce certain class of (nonnormal) tractor connections on V as modifications of ∇^V. There are two conditions on such modification maps $\Phi \in \mathcal{E}^1(\mathrm{End}\, V)$. First, Φ is homogeneous of degree ≥ 1 with respect to the natural filtrations on TM and V, for which we write $\Phi \in \big(\mathcal{E}^1(\mathrm{End}\, V)\big)^1$. (This ensures that if ∇^V is replaced by $\nabla^V + \Phi$ in (19), we recover the original operator D_0.) Second, we require that for any section $s \in \Gamma(V)$ we have that $\Phi(s) \in \mathrm{im}\ \partial^*_2 \subseteq \mathcal{E}^1(V)$. (This guarantees that if ∇^V is replaced by $\nabla^V + \Phi$ in (18), the BGG-splitting L_0 is unchanged.) The latter condition can be rewritten as $\Phi \in \mathrm{im}\ (\partial^*_V \otimes \mathrm{Id}_{V^*}) \subseteq \mathcal{E}^1(V \otimes V^*)$ where ∂^*_V denotes ∂^*_2 acting on $\mathcal{E}^2(V)$ only. Thus we arrive at a class of *admissible* tractor connections

$$\mathcal{C} = \left\{ \nabla^V + \Phi \mid \Phi \in \mathrm{im}\ (\partial^*_V \otimes \mathrm{Id}_{V^*}) \cap \big(\mathcal{E}^1(\mathrm{End}\, V)\big)^1 \right\}.$$

The main theorem of [23] is then

Theorem 6 (Theorem 1.2 from [23]) *There exists a unique tractor connection* $\widetilde{\nabla}^V \in \mathcal{C}$ *on* V *characterized by the property* $(\partial_V^* \otimes \mathrm{Id}_{V^*})(\widetilde{\Omega}^V) = 0$ *where* $\widetilde{\Omega}^V$ *is the curvature of* $\widetilde{\nabla}^V$*.*

This implies $\widetilde{\nabla}^V \circ L_0 = L_1 \circ D_0$ because now both sides are in the kernel of both ∂_1^* and $\partial_2^* \circ d^{\widetilde{\nabla}}$ (thus both sides are equal according to (18) and uniqueness of L_0). Therefore $\widetilde{\nabla}^V$ gives a prolongation of the first BGG operator D_0 in the sense that Π_0 and L_0 from (18) provide the bijective correspondence

$$\{\sigma \in \Gamma(H_0) \mid D_0(\sigma) = 0\} \xleftrightarrow{1-1} \{s \in \Gamma(V) \mid \widetilde{\nabla}^V s = 0\}.$$

We say that $\widetilde{\nabla}^V$ is the tractor *prolongation connection* on V. Finally note there is an analogue of the equality $\widetilde{\nabla}^V \circ L_0 = L_1 \circ D_0$ for higher operators D_p in the BGG sequence, see [23] for detail.

Above we stated only *existence* of the prolongation connection $\widetilde{\nabla}^V$. In fact, there is also an *iterative construction* [23] of the form

$$\begin{aligned} &\overset{0}{\nabla}{}^V := \nabla^V, \quad \widetilde{\nabla}^V := \overset{s}{\nabla}{}^V \quad \text{and} \quad \overset{i}{\nabla}{}^V = \overset{i-1}{\nabla}{}^V + \Phi_i \in \mathcal{C}, \quad 1 \le i \le s \\ &\text{where} \quad \Phi_i = a_i \left(\partial_V^* \otimes \mathrm{Id}_{V^*}\right) \overset{i-1}{\Omega}{}^V \in \mathcal{C}, \quad a_i \in \mathbb{R} \end{aligned} \tag{20}$$

i.e. Φ_i is computed from the curvature $\overset{i-1}{\Omega}{}^V$ of $\overset{i-1}{\nabla}{}^V$. Moreover, the upper bound for the number of iteration steps s is the number of irreducible subquotients of V. That is, complexity of the construction of $\widetilde{\nabla}^V$ grows with the dimension of V. We shall illustrate the difference between ∇^V and $\widetilde{\nabla}^V$ on two simple conformal examples in the next section.

Finally note the curvature $\widetilde{\Omega}^V$ essentially captures information about (non)existence of solutions of D_0. The point is that if $D_0(\sigma) = 0$ then $\widetilde{\Omega}^V(L_0(\sigma)) = 0$ hence a suitably defined 'determinant' of $\widetilde{\Omega}^V$ must vanish. Overdetermined operators generically do not have any solutions hence the curvature $\widetilde{\Omega}^V$ provides an algebraic test for (otherwise difficult) problem of existence of solutions of D_0.

5.2 Prolongation connection in conformal geometry

Henceforth we shall consider the conformal manifold $(M, [g])$. First note that it easily follows from (20) that $\nabla^{\mathcal{T}} = \widetilde{\nabla}^{\mathcal{T}}$ where $\mathcal{T}$ is the standard conformal tractor bundle and the same is true for the spinor tractor bundle. Moreover, $\widetilde{\nabla}^V$ differs from ∇^V for all remaining tractor bundles V.

Next we put $V := \mathcal{A} = \Lambda^2\mathcal{T}$. In this case, we obtain $\widetilde{\nabla}^V = \nabla^V + \Phi$ where $\Phi(s) = -\frac{1}{2}\iota_{\Pi(s)}\Omega^V \in \mathcal{E}^1(\mathcal{A})$ for $s \in \Gamma(\mathcal{A})$. Here $\iota_{\Pi_0(s)}$ inserts the vector field $\Pi_0(s) \in \Gamma(TM)$ into the curvature Ω^V. This case is known also by [8]. Note solutions of D_0 are conformal Killing fields in this case.

Our next example is the bundle $V = \Lambda^3\mathcal{T}$. The projecting slot is the quotient $H_0 = \Lambda^2T^*M[3]$ of V and solutions of D_0 in this case are conformal Killing 2-forms $\sigma \in \Gamma(H_0)$. An invariant prolongation in this case was solved already in [19] in a slightly different way (for a different splitting operator). The form $\widetilde{\nabla}^V$ in our setting is presented in [24]. This is already rather complicated so we shall restrict to the dimension $n = 4$ and the tractor subbundle $V_+ \subseteq V$ of self-adjoint tractor 3-forms. (Note dim $\mathcal{T} = 6$ for $n = 4$.) Then $\sigma \in \Gamma(H_0) = \Gamma(\Lambda^2_+T^*M[3])$ is a self-adjoint 2-form. Employing abstract index notation, we have $\widetilde{\nabla}^V_c = \nabla^V_c + \Phi_c$ where, for $s \in \Gamma(V_+)$, we have

$$s=\begin{pmatrix}\sigma_{a^1a^2}\\ \mu_{a^0a^1a^2}\,|\,\nu_a\\ \rho_{a^1a^2}\end{pmatrix}\in\begin{pmatrix}\mathcal{E}_{[a^1a^2]}[3]\\ \mathcal{E}_{[a^0a^1a^2]}[3]\mid\mathcal{E}_a[1]\\ \mathcal{E}_{[a^1a^2]}[1]\end{pmatrix},\quad \Phi_c(s)=\begin{pmatrix}0\\ -\frac{1}{4}\mathsf{W}_c{}^r{}_{[a^1a^2}\sigma_{a^0]r}\,|\,-\frac{1}{4}\mathsf{W}_{ca}{}^{rs}\sigma_{rs}\\ \beta_{a^1a^2}\end{pmatrix}$$

where $\beta_{a^1a^2} = \frac{1}{4}\big(4\mathsf{Y}_c{}^r{}_{[a^1}\sigma_{a^2]r} + \boldsymbol{g}_{c[a^1}\mathsf{Y}^{rs}{}_{a^2]}\sigma_{rs} + 2\mathsf{Y}_{c[a^1}{}^r\sigma_{a^2]r} - \mathsf{Y}_{a^1a^2}{}^r\sigma_{cr} + 2\mathsf{W}_{c[a^1}{}^{rs}\mu_{a^2]rs} + \mathsf{W}_c{}^r{}_{a^1a^2}\nu_r\big)$.

The modification Φ_c simplifies on half-flat conformal four manifolds. In particular, assuming $(M, [g])$ is anti-self-adjoint, the result is

$$\widetilde{\nabla}^V_c s = \nabla^V_c s + \begin{pmatrix}0\\ 0\mid 0\\ -\mathsf{Y}_{c(pa^1)}\sigma^p{}_{a^2} + \mathsf{Y}_{c(pa^2)}\sigma^p{}_{a^1} + \frac{1}{2}\mathsf{W}_c{}^p{}_{a^1a^2}\nu_p\end{pmatrix}.$$

Finally note solutions of first BGG operators often encode additional geometrical structure. For example, self-adjoint conformal Killing 2-forms $\sigma_{a_1a_2}$ discussed above correspond to Kähler metrics in the conformal class [14]. Another important case is $V = \mathcal{T}$ where nonvanishing solutions of D_0 yield Einstein metrics in the conformal class [3]. Finally, the problem of metrizability of AHS manifolds (i.e. existence of a metrizable compatible affine connection on M) is also closely related to solutions of suitable first BGG operators, cf. [28] and the related parts of [24] for the case of projective structures.

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Equivariant quantizations for AHS-structures

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Abstract

We construct an explicit scheme to associate to any potential symbol an operator acting between sections of natural bundles (associated to irreducible representations) for a so-called AHS-structure. Outside of a finite set of critical (or resonant) weights, this procedure gives rise to a quantization, which is intrinsic to this geometric structure. In particular, this provides projectively and conformally equivariant quantizations for arbitrary symbols on general (curved) projective and conformal structures.

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1. Introduction

Consider a smooth manifold M, two vector bundles E and F over M and a linear differential operator $D : \Gamma(E) \to \Gamma(F)$, where $\Gamma(\,)$ indicates the space of smooth sections. If D is of order at most k, then it has a well-defined (kth order) *principal symbol* σ_D, which can be viewed as a vector bundle map $S^k T^*M \otimes E \to F$ or as a smooth section of the vector bundle $S^k TM \otimes$

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$E^* \otimes F$. Here TM and T^*M are the tangent respectively cotangent bundle of M, E^* is the bundle dual to E, and S^k denotes the kth symmetric power.

A *quantization* on M is a right inverse to the principal symbol map. This means that to each smooth section τ of the bundle $S^kTM \otimes E^* \otimes F$, one has to associate a differential operator $A_\tau : \Gamma(E) \to \Gamma(F)$ of order k with principal symbol τ. Note that operators of order 0 coincide with their principal symbols, so there is a unique possible quantization in order 0. Given any kth order operator D with principal symbol τ, the difference $D - A_\tau$ is of order $k-1$. Iterating this, we conclude that, having a quantization in each order $\leqslant k$, one actually obtains an isomorphism between the space $\mathit{Diff}^k(E, F)$ of differential operators $\Gamma(E) \to \Gamma(F)$ of order at most k and the space of smooth sections of the bundle $\bigoplus_{i=0}^k S^iTM \otimes E^* \otimes F$.

A classical example of a quantization is provided by the Fourier transform for smooth functions on $\mathbb{R}^n$. However, it is well known that (even for $E = F = M \times \mathbb{R}$) there is no canonical quantization on a general manifold M, but one has to make additional choices. For our purposes, the most relevant example is to choose linear connections on the vector bundles E and TM. Having done this, one obtains induced linear connections on duals and tensor products of these bundles, and we will denote all these connections by ∇. For a smooth section s of E, one can then form the k-fold covariant derivative $\nabla^k s$, which is a section of $\bigotimes^k T^*M \otimes E$. Symmetrizing in the T^*M entries, we obtain a section $\nabla^{(k)}s$ of $S^kT^*M \otimes E$. Viewing a symbol τ as a bundle map $S^kT^*M \otimes E \to F$, we can simply put $A_\tau(s) := \tau(\nabla^{(k)}s)$. Clearly this defines a differential operator A_τ of order k and it is well known that its principal symbol is τ, so we have obtained a quantization in this way.

This provides a link to geometry. Suppose that M is endowed with some geometric structure which admits a canonical connection. Then one obtains quantizations for all natural bundles associated to this structure. The classical example of this situation is the case when (M, g) is a Riemannian manifold. Then the natural bundles are tensor and spinor bundles, and on each such bundle one has the Levi-Civita connection. Hence the above procedure leads to a natural quantization (in the sense that it is intrinsic to the Riemannian structure) for any pair E and F of natural vector bundles.

At this point there arises the question whether weaker geometric structures, which do not admit canonical connections, still do admit natural quantizations. This problem has been originally posed in [15] and has been intensively studied since then. The examples above naturally lead to the two geometric structures for which this problem has been mainly considered. On the one hand, one may replace a single linear connection on TM by a projective equivalence class of such connections. Here two connections are considered as equivalent if they have the same geodesics up to parametrization. On the other hand, the most natural weakening of Riemannian metrics is provided by conformal structures. Here one takes an equivalence class of (pseudo-)Riemannian metrics which are obtained from each other by multiplication by positive smooth functions.

Projective and conformal structures fit into the general scheme of so-called AHS-structures. These are geometric structures which admit an equivalent description by a canonical Cartan connection modelled on a compact Hermitian symmetric space G/P, where G is semisimple and $P \subset G$ is an appropriate parabolic subgroup. These geometries and the more general class of parabolic geometries have been studied intensively during the last years, and several striking results have been obtained, see e.g. [8]. In particular, an efficient differential calculus for these structures based on so-called tractor bundles has been worked out in [4].

This general point of view has shown up in the theory of equivariant quantizations already. Namely, it turns out that the homogeneous space G/P always contains a dense open subset (the big Schubert-cell) which is naturally diffeomorphic to $\mathbb{R}^n$. While the G-action on G/P cannot be restricted to this subspace, one obtains a realization of the Lie algebra $\mathfrak{g}$ of G as a Lie algebra of vector fields on $\mathbb{R}^n$. For the homogeneous model G/P and geometries locally isomorphic to it, naturality of a quantization is then equivalent to equivariancy for the action of this Lie algebra of vector fields. In many articles, the question of quantizations naturally associated to a projective and/or conformal structure is posed in this setting. Also, the algebras corresponding to general AHS-structures have been studied in this setting under the name "IFFT-equivariant quantizations", see [1]. It should be pointed out however, that these methods only apply to geometries locally isomorphic to G/P (e.g. to locally conformally flat conformal structures). As it is well known from the theory of linear invariant differential operators, passing from the locally flat category to general structures is a very difficult problem.

Most of the work on natural quantizations only applies to operators on sections of line bundles (density bundles). It was only recently that the methods for projective structures have been extended to general natural vector bundles in [12]. The construction there uses the Thomas–Whitehead (or ambient) description of projective structures, which is an equivalent encoding of the canonical Cartan connection for projective structures. This approach is only available in the projective case, though. As mentioned in [12], there is hope to use the Fefferman–Graham ambient metric for conformal structures to find conformally invariant quantizations, but there are several immediate problems with this approach. For the other AHS-structures, there is no clear analog of the ambient description.

It should be mentioned that the results for projective structures have also been obtained using the canonical Cartan connection, see [16]. After this article was essentially completed, we learned about the recent preprint [17], in which the Cartan approach is extended to prove existence of a natural quantization for conformal structures and it is claimed that the method further extends to all AHS-structures.

In this article, we use the recent advances on invariant calculi for parabolic geometries to develop a scheme for constructing equivariant quantizations. This scheme is explicit and uniform, it applies to all AHS-structures and to all (irreducible) natural bundles for such structures. As it is known from the special cases studied so far, equivariant quantizations do not always exist, so our scheme does not always lead to an equivariant quantization.

To formulate the result more precisely, we need a bit more background. It turns out that for any AHS-structure there is a family of natural line bundles $\mathcal{E}[w]$ parametrized by a real number w, the so-called density bundles. Any natural bundle E can be twisted by forming tensor products with density bundles to obtain bundles $E[w] := E \otimes \mathcal{E}[w]$. (For conformal structures, this free parameter is known as "conformal weight".) Doing this to the target bundle of differential operators, we can view a section $\tau \in \Gamma(S^k TM \otimes E^* \otimes F \otimes \mathcal{E}[\delta])$ as the potential symbol of an operator $\Gamma(E) \to \Gamma(F[\delta])$. We first universally decompose the bundle of symbols into a finite direct sum of subbundles. On the level of sections, we write this decomposition as $\tau = \sum_i \tau_i$. Given such a section, our scheme constructs a differential operator $A_\tau : \Gamma(E) \to \Gamma(F[\delta])$ for any choice of weight δ. The principal symbol of A_τ is $\sum_i \gamma_i \tau_i$ for real numbers γ_i which only depend on i, and δ (and not on τ or on the manifold in question). We prove that each γ_i is non-zero except for finitely many values of δ. Whenever all γ_i are non-zero, we obtain a natural quantization by mapping τ to $A_{\sum_i \gamma_i^{-1} \tau_i}$.

Our method does not only lead to an abstract proof that the set of critical weights (i.e. of weights δ for which some γ_i vanishes) is finite. We also get general information on the number and size of critical weights. In each concrete example, one can determine the set of critical weights explicitly, and this needs only finite dimensional representation theory.

We should mention that the developments in this article are closely related to the results in the recent thesis [14] of J. Kroeske, in which the author systematically constructs bilinear natural differential operators for AHS-structures and, more generally, for parabolic geometries.

2. AHS-structures and invariant calculus

In this section we review basic facts on AHS-structures and invariant differential calculus for these geometries. Our basic references are [18,6,7].

2.1. $|1|$-graded Lie algebras and first order structures

The starting point for defining an AHS-structure is a real simple Lie algebra $\mathfrak{g}$ endowed with a so-called $|1|$-grading, i.e. a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we agree that $\mathfrak{g}_\ell = 0$ for $\ell \notin \{-1, 0, 1\}$. The classification of such gradings is well known, since it is equivalent to the classification of Hermitian symmetric spaces. We put $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \subset \mathfrak{g}$. By the grading property, $\mathfrak{p}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_1$ is a nilpotent ideal in $\mathfrak{p}$.

Given a Lie group G with Lie algebra $\mathfrak{g}$, there are natural subgroups $G_0 \subset P \subset G$ corresponding to the Lie subalgebras $\mathfrak{g}_0 \subset \mathfrak{p} \subset \mathfrak{g}$. For P one may take a subgroup lying between the normalizer $N_G(\mathfrak{p})$ of $\mathfrak{p}$ in G and its connected component of the identity. Then $G_0 \subset P$ is defined as the subgroup of all elements whose adjoint action preserves the grading of $\mathfrak{g}$. In particular, restricting the adjoint action to $\mathfrak{g}_{-1}$, one obtains a representation $G_0 \to GL(\mathfrak{g}_{-1})$. This representation is infinitesimally injective, so it makes sense to talk about first order G-structures with structure group G_0 on smooth manifolds of dimension $\dim(\mathfrak{g}_{-1})$.

By definition, such a structure is given by a smooth principal bundle $p : \mathcal{G}_0 \to M$ with structure group G_0, such that the associated bundle $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$ is isomorphic to the tangent bundle TM. It turns out that the Killing form on $\mathfrak{g}$ induces a G_0-equivariant duality between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, so $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_1 \cong T^*M$. Using this, one can realize arbitrary tensor bundles on M as associated bundles to $\mathcal{G}_0$. More generally, any representation of G_0, via forming associated bundles, gives rise to a natural vector bundle on manifolds endowed with such a structure. It turns out that G_0 is always reductive with one-dimensional center. Hence finite dimensional representations of G_0 on which the center acts diagonalizably (which we will always assume in the sequel) are completely reducible, i.e. they split into direct sums of irreducible representations.

The one-dimensional center of G_0 leads to a family of natural line bundles. For $w \in \mathbb{R}$, we can define a homomorphism $G_0 \to \mathbb{R}_+$ by mapping $g \in G_0$ to $|\det(\mathrm{Ad}_-(g))|^{\frac{w}{n}}$, where $n = \dim(\mathfrak{g}_{-1})$ and $\mathrm{Ad}_-(g) : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ is the restriction of the adjoint action of g. This evidently is a smooth homomorphism, thus giving rise to a one-dimensional representation $\mathbb{R}[w]$ of G_0. It is easy to see that $\mathbb{R}[w]$ is non-trivial for $w \neq 0$. (The factor $\frac{1}{n}$ is included to get the usual normalization in the case of conformal structures.) The corresponding associated bundle will be denoted by $\mathcal{E}[w]$, and adding the symbol $[w]$ to the name of a natural bundle will always indicate a tensor product with $\mathcal{E}[w]$. Using the convention that 1-densities are the objects which can be naturally integrated on non-orientable manifolds, $\mathcal{E}[w]$ is by construction the bundle of $(-\frac{w}{n})$-densities. In particular, all the bundles $\mathcal{E}[w]$ are trivial line bundles, but there is no canonical trivialization for $w \neq 0$.

2.2. Canonical Cartan connections and AHS-structures

The exponential mapping restricts to a diffeomorphism from $\mathfrak{g}_1$ onto a closed normal Abelian subgroup $P_+ \subset P$ such that P is the semidirect product of G_0 and P_+. Hence G_0 can also naturally be viewed as a quotient of P. In particular, given a principal P-bundle $\mathcal{G} \to M$, the subgroup P_+ acts freely on $\mathcal{G}$, and the quotient $\mathcal{G}/P_+$ is naturally a principal bundle with structure group G_0. Next, suppose that there is a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ on the principal bundle $\mathcal{G}$. Then the $\mathfrak{g}_{-1}$-component of ω descends to a well-defined one-form $\theta \in \Omega^1(\mathcal{G}/P_+, \mathfrak{g}_{-1})$, which is G_0-equivariant and strictly horizontal. This means that $(\mathcal{G}/P_+ \to M, \theta)$ is a first order structure with structure group G_0. In this sense, any Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) has an underlying first order structure with structure group G_0. Conversely, one can talk about extending a first order structure to a Cartan geometry.

It turns out (see e.g. [7]) that, for almost all choices of (G, P), for any given first order structure with structure group G_0 there is a unique (up to isomorphism) extension to a Cartan geometry of type (G, P), for which the Cartan connection ω satisfies a certain normalization condition. This is usually phrased as saying that such structures admit a canonical Cartan connection. The main exception is $\mathfrak{g} = \mathfrak{gl}(n+1, \mathbb{R})$ with a $|1|$-grading such that $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g}_{\pm 1} \cong \mathbb{R}^n$. For an appropriate choice of G, the adjoint action identifies G_0 with $GL(\mathfrak{g}_{-1}) = GL(n, \mathbb{R})$. A first order structure for this group on a manifold M is just the full linear frame bundle of M and hence contains no information. In this case, an extension to a normal Cartan geometry of type (G, P) is equivalent to the choice of a projective equivalence class of torsion-free connections on the tangent bundle TM, i.e. to a classical projective structure.

Normal Cartan geometries of type (G, P) as well as the equivalent underlying structures (i.e. classical projective structures respectively first order structures with structure group G_0) are often referred to as *AHS-structures*. AHS is short for "almost Hermitian symmetric". To explain this name, recall that the basic example of a Cartan geometry of type (G, P) is provided by the natural projection $G \to G/P$ and the left Maurer–Cartan form as the Cartan connection. This is called the *homogeneous model* of geometries of type (G, P). Now the homogeneous spaces G/P for pairs (G, P) coming from $|1|$-gradings as described above, are exactly the compact irreducible Hermitian symmetric spaces.

2.3. Natural bundles and the fundamental derivative

Via forming associated bundles, any representation of the group P gives rise to a natural bundle for Cartan geometries of type (G, P). As we have seen above, P is the semidirect product of the reductive subgroup G_0 and the normal vector subgroup P_+, so its representation theory is fairly complicated. Via the quotient homomorphism $P \to G_0$, any representation of G_0 gives rise to a representation of P. It turns out that the representations of P obtained in this way are exactly the completely reducible representations, i.e. the direct sums of irreducible representations. Correspondingly, we will talk about completely reducible and irreducible natural bundles on Cartan geometries of type (G, P). Consider a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ with underlying structure $(p_0 : \mathcal{G}_0 \to M, \theta)$ and let V be a representation of G_0, which we view as a representation of P via the quotient homomorphism. Then by definition, the subgroup $P_+ \subset P$ acts trivially on V and since $\mathcal{G}_0 = \mathcal{G}/P_+$, we can naturally identify $\mathcal{G} \times_P V$ with $\mathcal{G}_0 \times_{G_0} V$. Hence completely reducible bundles can be easily described in terms of the underlying structure.

There is a second simple source of representations of P, which leads to an important class of natural bundles. Namely, one may restrict any representation of G to the subgroup P. The corresponding natural vector bundles are called *tractor bundles*, their general theory is developed in [4]. The most important tractor bundle is the *adjoint tractor bundle*. For a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ it is defined by $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$, so it is the associated bundle with respect to the restriction of the adjoint representation of G to P. Now the P-invariant subspaces $\mathfrak{g}_1 \subset \mathfrak{p} \subset \mathfrak{g}$

give rise to a filtration $\mathcal{A}^1M \subset \mathcal{A}^0M \subset \mathcal{A}M$ of the adjoint tractor bundle by smooth subbundles. By construction, $\mathcal{A}^1M \cong T^*M$ and since $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-1}$ we see that $\mathcal{A}M/\mathcal{A}^0M \cong TM$. We will write $\Pi : \mathcal{A}M \to TM$ for the resulting natural projection. Hence the adjoint tractor bundle has the cotangent bundle as a natural subbundle and the tangent bundle as a natural quotient.

The Killing form defines a G-invariant, non-degenerate bilinear form on $\mathfrak{g}$. It turns out that $\mathfrak{g}_1$ is the annihilator of $\mathfrak{p}$ with respect to the Killing form, which leads to duality with $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-1}$ as observed above. On the level of associated bundles, we obtain a natural non-degenerate bilinear form on the adjoint tractor bundle $\mathcal{A}M$, which thus can be identified with the dual bundle $\mathcal{A}^*M$. Under this pairing, the subbundle $\mathcal{A}^1M$ is the annihilator of $\mathcal{A}^0M$. The resulting duality between $\mathcal{A}^1M$ and $\mathcal{A}M/\mathcal{A}^0M$ is exactly the duality between T^*M and TM.

The adjoint tractor bundle gives rise to a basic family of natural differential operators for AHS-structures (and more generally for parabolic geometries). These have been introduced in [4] under the name "fundamental D-operators", more recently, the name *fundamental derivative* is commonly used. Let us start with an arbitrary representation V of P and consider the corresponding natural bundle $E := \mathcal{G} \times_P V \to M$ for a geometry $(p : \mathcal{G} \to M, \omega)$. Then smooth sections of this bundle are in bijective correspondence with smooth maps $f : \mathcal{G} \to V$, which are P-equivariant. In the special case $V = \mathfrak{g}$ of the adjoint tractor bundle, we can then use the trivialization of $T\mathcal{G}$ provided by the Cartan connection ω to identify P-equivariant functions $\mathcal{G} \to \mathfrak{g}$ with P-invariant vector fields on $\mathcal{G}$. For a section $s \in \Gamma(\mathcal{A}M)$, we can form the corresponding vector field $\xi \in \mathfrak{X}(\mathcal{G})$ and use it to differentiate the equivariant function $f : \mathcal{G} \to V$ corresponding to a section $\sigma \in \Gamma(E)$. The result will again be equivariant, thus defining a smooth section $D_s\sigma \in \Gamma(E)$. Hence we can view the fundamental derivative as an operator $D = D^E : \Gamma(\mathcal{A}M) \times \Gamma(E) \to \Gamma(E)$. The basic properties of this operator as proved in Section 3 of [4] are:

Proposition 1. *Let V be a representation of P and let $E = \mathcal{G} \times_P V$ be the corresponding natural bundle for an AHS-structure $(p : \mathcal{G} \to M, \omega)$. Then we have:*

1. *$D : \Gamma(\mathcal{A}M) \times \Gamma(E) \to \Gamma(E)$ is a first order differential operator which is natural, i.e. intrinsic to the AHS-structure on M.*
2. *D is linear over smooth functions in the $\mathcal{A}M$-entry, so we can also view $\sigma \mapsto D\sigma$ as an operator $\Gamma(E) \to \Gamma(\mathcal{A}^*M \otimes E)$.*
3. *For $s \in \Gamma(\mathcal{A}M)$, $\sigma \in \Gamma(E)$, and $f \in C^\infty(M, \mathbb{R})$, we have the Leibniz rule $D_s(f\sigma) = (\Pi(s) \cdot f)\sigma + f D_s\sigma$, where $\Pi : \Gamma(\mathcal{A}M) \to \Gamma(TM)$ is the natural tensorial projection.*
4. *For a second natural bundle $F = \mathcal{G} \times_P W$, a P-equivariant map $V \to W$, and the corresponding bundle map $\Phi : E \to F$, the fundamental derivatives on E and F are related by $D_s^F(\Phi \circ \sigma) = \Phi \circ D_s^E\sigma$ for all $s \in \Gamma(\mathcal{A}M)$ and $\sigma \in \Gamma(E)$.*

The naturality statement in (4) justifies denoting the fundamental derivatives on all natural bundles by the same letter. Since there is no restriction on the bundle E, the fundamental derivative in the form of part (2) can evidently be iterated. For $\sigma \in \Gamma(E)$ we can form $D\sigma$, $D^2\sigma = D(D\sigma)$ and inductively $D^k\sigma \in \Gamma(\bigotimes^k \mathcal{A}^*M \otimes E)$.

2.4. Curved Casimir operators

Curved Casimir operators form another basic set of natural differential operators defined on AHS-structures. They have been introduced in [9] in the general context of parabolic geometries.

That article contains all the facts about curved Casimir operators we will need, as well as the general construction for splitting operators that we will use below.

As above, we start with a representation V of P and consider the corresponding natural vector bundle $E = \mathcal{G} \times_P V$ for an AHS-structure $(p : \mathcal{G} \to M, \omega)$. As noticed above, the composition of two fundamental derivatives defines an operator $D^2 : \Gamma(E) \to \Gamma(\bigotimes^2 \mathcal{A}^*M \otimes E)$. From 2.3 we know that the Killing form on $\mathfrak{g}$ induces a non-degenerate bilinear form on $\mathcal{A}M$. Using this to identify $\mathcal{A}M$ with $\mathcal{A}^*M$, we also get a natural bilinear form B on $\mathcal{A}^*M$. This can be used to define a bundle map $B \otimes \mathrm{id} : \bigotimes^2 \mathcal{A}^*M \otimes E \to E$. Now one defines the *curved Casimir operator* $\mathcal{C} = \mathcal{C}^E : \Gamma(E) \to \Gamma(E)$ by $\mathcal{C}(\sigma) := (B \otimes \mathrm{id}) \circ D^2\sigma$.

Part (4) of Proposition 1 easily implies (compare with Proposition 2 of [9]) that for another natural vector bundle F and a bundle map $\Phi : E \to F$ coming from a P-equivariant map between the inducing representations, one gets $\mathcal{C}^F(\Phi \circ \sigma) = \Phi \circ \mathcal{C}^E(\sigma)$. This is the justification for denoting all curved Casimir operators by the same symbol.

From the construction it is clear that $\mathcal{C}$ is a natural differential operator of order at most 2. However, it turns out that $\mathcal{C}$ actually always is of order at most one. Moreover, on sections of bundles induced by irreducible representations, the operator $\mathcal{C}$ acts by a scalar which can be computed from representation theory data. One can associate to any irreducible representation of $\mathfrak{g}_0$ a highest and a lowest weight by passing to complexifications, see Section 3.4 of [9]. The weights are functionals on the Cartan subalgebra $\mathfrak{h}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, which at the same time is a Cartan subalgebra for $(\mathfrak{g}_0)_{\mathbb{C}}$. Recall that the Killing form of $\mathfrak{g}$ induces a positive definite inner product on the real space of functionals on $\mathfrak{h}$ spanned by possible weights for finite dimensional representations. Denoting this inner product by $\langle\, , \rangle$ and the corresponding norm by $\|\ \|$, the following result is proved as Theorem 1 in [9].

Proposition 2. *Let V be a representation of P and let $E = \mathcal{G} \times_P V$ be the corresponding natural vector bundle for an AHS-structure $(p : \mathcal{G} \to M, \omega)$. Then:*

(1) *$\mathcal{C} : \Gamma(E) \to \Gamma(E)$ is a natural differential operator of order at most one.*
(2) *If the representation V is irreducible of lowest weight $-\lambda$, then $\mathcal{C}$ acts on $\Gamma(E)$ by multiplication by $\|\lambda\|^2 + 2\langle\lambda, \rho\rangle$, where ρ is half the sum of all positive roots of $\mathfrak{g}_{\mathbb{C}}$.*

3. The quantization scheme

Throughout this section, we fix a pair (G, P), two irreducible representations V and W of G_0 with corresponding natural bundles E and F, as well as an order $k > 0$. Given these data, we try to construct a quantization for kth order symbols of operators mapping sections of E to sections of $F[\delta]$ for $\delta \in \mathbb{R}$.

The basic idea for the construction is very simple. The bundle of symbols in this situation is $S^kTM \otimes E^* \otimes F[\delta]$. We know from 2.3 that TM naturally is a quotient of the adjoint tractor bundle $\mathcal{A}M$, so the bundle of symbols is a quotient of $S^k\mathcal{A}M \otimes E^* \otimes F[\delta]$. Using the general machinery of splitting operators, we can associate to a symbol a section of the latter bundle. But such a section can be interpreted as a bundle map $S^k\mathcal{A}^*M \otimes E \to F[\delta]$, so we can apply it to the values of the symmetrized k-fold fundamental derivative of sections of E.

3.1. Some properties of the fundamental derivative

To carry out this idea, we first have to derive, for some fixed k, some properties of the iterated fundamental derivative D^k and its symmetrization $D^{(k)} : \Gamma(E) \to \Gamma(\mathcal{W}M)$, where

$\mathcal{W}M := S^k\mathcal{A}^*M \otimes E$. Recall from 2.3 that $\mathcal{A}M$ admits a natural filtration of the form $\mathcal{A}^1M \subset \mathcal{A}^0M \subset \mathcal{A}^{-1}M := \mathcal{A}M$. Since elements of $\mathcal{W}M$ can be interpreted as k-linear, symmetric maps $(\mathcal{A}M)^k \to E$, we get an induced filtration of the bundle $\mathcal{W}M$. We first take the natural filtration of $S^k\mathcal{A}M$, with components indexed from $-k$ to k, and then define $\mathcal{W}^\ell M$ to be the annihilator of the filtration component with index $-\ell + 1$. Explicitly, this means that $\mathcal{W}^\ell M$ consists of all maps $\Psi \in \mathcal{W}M$ such that $\Psi(s_1, \dots, s_k) = 0$ for arbitrary elements $s_j \in \mathcal{A}^{i_j}M$, provided that $i_1 + \dots + i_k > -\ell$. Then by definition, we get $\mathcal{W}^{\ell+1}M \subset \mathcal{W}^\ell M$ for each ℓ, $\mathcal{W}^{k+1}M = 0$, and $\mathcal{W}^{-k}M = \mathcal{W}M$. Moreover, a map $\Phi \in \mathcal{W}^kM$ by definition vanishes if at least one of its entries is from $\mathcal{A}^0M \subset \mathcal{A}M$. Hence this factors to a k-linear symmetric map on copies of $\mathcal{A}M/\mathcal{A}^0M \cong TM$, and we get an isomorphism $\mathcal{W}^kM \cong S^kT^*M \otimes E$. We will denote by $\iota : S^kT^*M \otimes E \to \mathcal{W}M$ the corresponding natural inclusion.

Proposition 3. (1) *The symmetrized k-fold fundamental derivative $D^{(k)} : \Gamma(E) \to \Gamma(\mathcal{W}M)$ has values in the space of sections of the subbundle $\mathcal{W}^0M$.*

(2) *Consider any principal connection on the bundle $\mathcal{G}_0 \to M$, denote by ∇ all the induced connections on associated vector bundles, by ∇^k the k-fold covariant derivative, and by $\nabla^{(k)}$ its symmetrization.*

Then the operator $\Gamma(E) \to \Gamma(\mathcal{W}M)$ given by $\varphi \mapsto D^k\varphi - \iota(\nabla^k\varphi)$ has order at most $k - 1$. In particular, $D^{(k)}\varphi$ is the sum of $\iota(\nabla^{(k)}\varphi)$ and terms of order at most $k - 1$ in φ.

Proof. We will proceed by induction on k. Recall that there is a family of preferred connections on the bundle $\mathcal{G}_0$ which is intrinsic to the AHS-structure, see [6,4]. Any such connection also determines a splitting of the filtration of the adjoint tractor bundle, i.e. an isomorphism $\mathcal{A}M \to T^*M \oplus \mathrm{End}_0(TM) \oplus TM$, where $\mathrm{End}_0(TM) = \mathcal{G}_0 \times_{G_0} \mathfrak{g}_0$, which behaves well with respect to the filtration. In particular, the last component is given by the natural projection $\Pi : \mathcal{A}M \to TM$, while the first component restricts to the natural isomorphism $\mathcal{A}^1M \to T^*M$. Fixing one preferred connection, the difference to any other principal connection on $\mathcal{G}_0$ is given by a tensorial operator, so it suffices to prove part (2) for the chosen preferred connection.

A formula for the action of the fundamental derivative on tensor bundles in terms of ∇ and this splitting is derived in Section 4.14 of [4]. The argument used there applies to all bundles constructed from completely reducible subquotients of tractor bundles, and hence to all bundles associated to $\mathcal{G}_0$. If $s \in \Gamma(\mathcal{A}M)$ corresponds to (ψ, Φ, ξ) in the splitting determined by ∇ (so in particular $\xi = \Pi(s)$), then $D_s\varphi = \nabla_\xi\varphi - \Phi \bullet \varphi$, where $\bullet : \mathrm{End}_0(TM) \times E \to E$ is the tensorial operation induced by the infinitesimal action $\mathfrak{g}_0 \times V \to V$. Now $s \in \Gamma(\mathcal{A}^1M)$ if and only if $\xi = 0$ and $\Phi = 0$, so $D_s\varphi = 0$ in this case. On the other hand, $\xi = \Pi(s)$ so $(D\varphi - \iota(\nabla\varphi))(s) = D_s\varphi - \nabla_{\Pi(s)}\varphi = \Phi \bullet \varphi$ is tensorial. Hence we have proved (1) and (2) for $k = 1$.

Next observe that naturality of the fundamental derivative implies that for $s_0, \dots, s_k \in \Gamma(\mathcal{A}M)$ we obtain the Leibniz rule

$$\big(D^{k+1}\varphi\big)(s_0, \dots, s_k) = D_{s_0}\big(D^k\varphi(s_1, \dots, s_k)\big) - \sum_{i=1}^{k}\big(D^k\varphi\big)(s_1, \dots, D_{s_0}s_i, \dots, s_k), \qquad (*)$$

compare with Proposition 3.1 of [4]. Assuming inductively that part (2) holds for k, the second summand is evidently of order at most k in φ. Moreover, the first summand is given by $\nabla_{\Pi(s_0)}(\nabla^k\varphi(\Pi(s_1), \dots, \Pi(s_k)))$ plus terms of order at most $k - 1$ in φ which immediately implies (2).

To prove (1), observe $D^k\varphi \in \Gamma(\mathcal{W}^0M)$ if and only if $D^k\varphi(s_1,\dots,s_k) = 0$ provided that at least r of the sections s_i have values in $\mathcal{A}^0M$ and at least $k-r+1$ of them even have values in $\mathcal{A}^1M$. We assume this inductively and prove the corresponding property of $D^{k+1}\varphi$. Hence we take sections $s_0,\dots,s_k$, and assume that r' of them have values in $\mathcal{A}^0M$ and $k-r'+2$ even have values in $\mathcal{A}^1M$.

If s_0 has values in $\mathcal{A}^1M$, then D_{s_0} acts trivially on $\Gamma(E)$ as well as on sections of $\mathcal{A}^1M$, it maps sections of $\mathcal{A}M$ to sections of $\mathcal{A}^0M$ and sections of $\mathcal{A}^0M$ to sections of $\mathcal{A}^1M$. Hence the first summand of the right-hand side of $(*)$ vanishes. In the second term of this right-hand side, only summands in which s_i does not have values in $\mathcal{A}^1M$ can provide a non-zero contribution. If $s_i \in \Gamma(\mathcal{A}^0M)$, then in the corresponding summand we have $r'-1$ sections of $\mathcal{A}^0M$, and $k-r'+2 = k-(r'-1)+1$ of them have values in $\mathcal{A}^1M$, so the corresponding summand vanishes by inductive hypothesis. If s_i is not a section of $\mathcal{A}^0M$, then in the corresponding summand we have r' sections of $\mathcal{A}^0M$, and $k-r'+1$ of them have values in $\mathcal{A}^1M$, so again vanishing follows by induction.

If s_0 has values in $\mathcal{A}^0M$ but not in $\mathcal{A}^1M$, then we only need to take into account that, acting on sections of $\mathcal{A}M$, D_{s_0} preserves sections of each filtration component. This shows that in each of the summands in the right-hand side of $(*)$, there are $r'-1$ sections of $\mathcal{A}^0M$ inserted into $D^k\varphi$, and $k-r'+2 = k-(r'-1)+1$ of them have values in $\mathcal{A}^1M$. Hence again vanishing of each summand follows by induction.

Finally, if s_0 does not have values in $\mathcal{A}^0M$, then we again need only that D_{s_0} preserves sections of each of the filtration components of $\mathcal{A}M$. This shows that in each summand of the right-hand side of $(*)$, we have r' sections of $\mathcal{A}^0M$ and $k-r'+2$ of them have values in $\mathcal{A}^1M$. Thus vanishing of each summand again follows by induction, and the proof of (1) follows by symmetrization. □

3.2. The splitting operators

According to the idea described in the beginning of Section 3, we should next, for fixed k, consider the bundle $S^kTM \otimes E^* \otimes F[\delta]$ of symbols as a quotient of the bundle $\tilde{\mathcal{V}}M := S^k\mathcal{A}M \otimes E^* \otimes F[\delta]$. However, in view of Proposition 3, we can already improve the basic idea. As we have noted in 3.1, the bundle $S^k\mathcal{A}M$ carries a natural filtration. Taking the tensor product with E^* and $F[\delta]$, we obtain a filtration of the bundle $\tilde{\mathcal{V}}M$ of the form

$$\tilde{\mathcal{V}}^kM \subset \dots \subset \tilde{\mathcal{V}}^0M \subset \dots \subset \tilde{\mathcal{V}}^{-k}M = \tilde{\mathcal{V}}M.$$

As we have observed in the beginning of Section 3, there is a well-defined bilinear pairing $\tilde{\mathcal{V}}M \times \mathcal{W}M \to F[\delta]$. By definition of the filtration on $\mathcal{W}M$, this factorizes to a bilinear pairing of $\mathcal{V}M \times \mathcal{W}^0M \to F[\delta]$, where $\mathcal{V}M := \tilde{\mathcal{V}}M/\tilde{\mathcal{V}}^1M$. We denote all these pairings by $\langle\,,\rangle$. As we shall see below, replacing the bundle $\tilde{\mathcal{V}}M$ by its quotient $\mathcal{V}M$ leads to a smaller set of critical weights δ.

For the same reason, it is preferable to take a further decomposition according to irreducible components of the bundle of symbols as follows. By construction, the filtration on $S^k\mathcal{A}M$ is induced by P-invariant subspaces of the representation $S^k\mathfrak{g}$, so the filtration of $\tilde{\mathcal{V}}M$ comes from a P-invariant filtration of $S^k\mathfrak{g} \otimes V^* \otimes W[\delta]$. The quotient of this space by the largest proper filtration component by construction is $S^k(\mathfrak{g}/\mathfrak{p}) \otimes V^* \otimes W[\delta]$, which induces the bundle of symbols. Now if we restrict to the subgroup $G_0 \subset P$, then $\mathfrak{g}$ decomposes into the direct sum

$\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and the filtration components are just $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}_1$. Correspondingly, the filtrations on $S^k\mathfrak{g}$ and $S^k\mathfrak{g} \otimes V^* \otimes W[\delta]$, viewed as G_0-representations, are induced from direct sum decompositions.

Since we have assumed that V and W are irreducible representations of P (and hence of G_0), the tensor product $S^k(\mathfrak{g}/\mathfrak{p}) \otimes V^* \otimes W[\delta]$ splits into a direct sum $\bigoplus_i R_i$ of irreducible representations of G_0. Identifying $\mathfrak{g}/\mathfrak{p}$ with $\mathfrak{g}_{-1}$, we can view each R_i as a subspace in the quotient of $S^k\mathfrak{g} \otimes V^* \otimes W$ by the P-invariant filtration component with index 1. Then for each i, we can look at the P-module S_i generated by R_i. Each S_i has a P-invariant filtration with completely reducible subquotients, and the quotient of S_i by the largest proper filtration component is R_i.

Passing to associated bundles, we see that for each i, we can consider $\mathcal{G} \times_P R_i$ as a subbundle of the bundle $S^kTM \otimes E^* \otimes F[\delta]$ of symbols, and these subbundles form a decomposition into a direct sum. In particular, any section τ of the bundle of symbols can be uniquely written as $\tau = \sum_i \tau_i$ of sections $\tau_i \in \Gamma(\mathcal{G} \times_P R_i)$. Likewise, for each i, we can view $\mathcal{G} \times_P S_i$ as a subbundle of $\mathcal{V}M$, so in particular, sections of $\mathcal{G} \times_P S_i$ can be viewed as sections of $\mathcal{V}M$.

Now for each i, we denote by β_i^0 the eigenvalue by which the curved Casimir operator acts on sections of the irreducible bundle $\mathcal{G} \times_P R_i$, see Proposition 2. Further, by $\beta_i^1, \dots, \beta_i^{n_i}$ we denote the different Casimir eigenvalues occurring for irreducible components in the other quotients of consecutive filtration components of S_i. Using this, we can now formulate:

Proposition 4. *Let $\Pi : \mathcal{V}M \to S^kTM \otimes E^* \otimes F[\delta]$ be the natural projection and denote the induced tensorial operator on sections by the same symbol. For each i define $\gamma_i := \prod_{j=1}^{n_i}(\beta_i^0 - \beta_i^j)$.*

Then there is a natural differential operator

$$L : \Gamma\big(S^kTM \otimes E^* \otimes F[\delta]\big) \to \Gamma(\mathcal{V}M)$$

such that $\Pi(L(\tau)) = \sum_i \gamma_i \tau_i$ for any section $\tau = \sum_i \tau_i$ of the bundle of symbols.

Proof. Of course for each i, mapping τ to $\tau_i \in \Gamma(\mathcal{G} \times_P R_i)$ defines a tensorial natural operator. The construction of splitting operators in Theorem 2 of [9] gives us, for each i, a natural differential operator $L_i : \Gamma(\mathcal{G} \times_P R_i) \to \Gamma(\mathcal{G} \times_P S_i)$. This has the property that denoting by Π_i the tensorial projection in the other direction, we obtain $\Pi_i(L(\tau_i)) = \gamma_i \tau_i$ for the number γ_i defined in the proposition. As we have noted above, we can naturally view sections of $\mathcal{G} \times_P S_i$ as sections of $\mathcal{V}M$, so we can simply define $L(\tau) := \sum_i L_i(\tau_i)$. □

It is easy to give an explicit description of L, since the construction of splitting operators in [9] is explicit. Given τ, we have to choose sections $s_i \in \Gamma(\mathcal{G} \times_P S_i) \subset \Gamma(\mathcal{V}M)$ such that $\Pi(s_i) = \tau_i$ for all i. Then we claim that

$$L(\tau) = \sum_i \prod_{j=1}^{n_i} \big(\mathcal{C} - \beta_i^j\big)(s_i).$$

The product for fixed i exactly corresponds to the definition of the splitting operator from [9]. Naturality of the curved Casimir operator thus implies that each of the summands equals $L_i(\tau_i)$, viewed as a section of $\mathcal{V}M$, and the claim follows.

3.3. The quantization scheme

We are now ready to formulate our first main result.

Theorem 5. *The map $(\tau, \varphi) \mapsto \langle L(\tau), D^{(k)}\varphi\rangle$ defines a natural bilinear operator $\Gamma(S^kTM \otimes E^* \otimes F[\delta]) \times \Gamma(E) \to \Gamma(F[\delta])$.*

For $\tau = \sum_i \tau_i \in \Gamma(S^kTM \otimes E^* \otimes F[\delta])$, the operator $A_\tau : \Gamma(E) \to \Gamma(F[\delta])$ defined by $A_\tau(\varphi) := \langle L(\tau), D^{(k)}\varphi\rangle$ is of order at most k and has principal symbol $\sum_i \gamma_i \tau_i$.

Proof. Naturality of L, $D^{(k)}$, and the pairing $\langle\, ,\rangle$ implies naturality of the bilinear operator. Now fix τ and consider the operator A_τ. Choose any principal connection on $\mathcal{G}_0$ and denote by ∇ all the induced linear connections on associated vector bundles. Using Proposition 3 we see that $A_\tau(\varphi) = \langle L(\tau), i(\nabla^{(k)}\varphi)\rangle$ up to terms of order at mots $k-1$ in φ. Hence A_τ is of order at most k and by the properties of the pairing $\langle\, ,\rangle$, the principal symbol is obtained as the result of pairing $\Pi(L(\tau)) \in \Gamma(S^kTM \otimes E^* \otimes F[\delta])$ with $\nabla^{(k)}\varphi \in \Gamma(S^kT^*M \otimes E)$. Thus the result follows from Proposition 4. □

Now we define a weight $\delta \in \mathbb{R}$ to be *critical* if at least one of the γ_i is zero for the chosen value of δ. For non-critical weights, our theorem immediately leads to a natural quantization:

Corollary 6. *If the weight δ is not critical, then the map $\tau \mapsto A_{\sum_i \gamma_i^{-1}\tau_i}$ defines a natural quantization for the bundles E and $F[\delta]$.*

We want to emphasize that the naturality result in the corollary in particular implies that in the case of the homogeneous model G/P of the AHS-structure in question the quantization is equivariant (as a bilinear map) under the natural G-action on the spaces of sections of the bundles in question (which are homogeneous vector bundles in this case). We can restrict the quantization to the big Schubert cell in G/P, which is diffeomorphic to $\mathbb{R}^n$, $n = \dim(G/P)$. The G-equivariancy on G/P immediately implies that the result is equivariant for the Lie subalgebra of vector fields on $\mathbb{R}^n$ formed by the fundamental vector fields for this G-action. Hence our quantization will specialize to an equivariant quantization in the usual sense.

3.4. The set of critical weights

To complete our results, we have to prove that for any choice of bundles E and F and any order k, the set of critical weights is finite. Verifying this is a question of finite dimensional representation theory. In fact, we not only get an abstract proof of finiteness of the set of critical weights, but a method to determine the set of critical weights for any given example.

In view of Proposition 4 and Theorem 5, it is clear that we have to understand the dependence of the Casimir eigenvalues, or more precisely of the differences $\beta_i^0 - \beta_i^j$, on δ. To get a complete understanding of the set of critical weights, one has to determine the composition series (i.e. the structure of the quotients of iterated filtration components), of the P-modules S_i. Recall from 3.2 that, as a representation of G_0, S_i is simply the direct sum of all the composition factors, so essentially we have to determine the decomposition of S_i into irreducible components as a G_0-module. From Proposition 2 we know how to determine the numbers β from the lowest weights of these irreducible components. Notice that changing the weight δ corresponds to taking a tensor product with a one-dimensional representation. In particular, this does not influence the basic decompositions into irreducible components, apart from the fact that each of these components is tensorized with that one-dimensional representation. As we shall see, we can get quite a bit of information without detailed knowledge of the decomposition into irreducibles, using only structural information on the possible irreducible components. We start by proving a basic finiteness result.

Theorem 7. *Fix an irreducible component $R_i \subset S^k\mathfrak{g}_{-1} \otimes V^* \otimes W[\delta]$, consider the corresponding Casimir eigenvalue β_i^0, and one of the other Casimir eigenvalues β_i^j. Then there is exactly one value of δ for which $\beta_i^0 = \beta_i^j$. Hence there are at most n_i many values for δ for which $\gamma_i = 0$, and at most $\sum_i n_i$ critical weights.*

Proof. Let us first make a few comments. The Casimir eigenvalues can be computed from lowest weights, which are defined via complexification of non-complex representations and of the Lie algebra in question. Since these complexifications do not change the decomposition into irreducible components, we may work in the setting of complex $|1|$-graded Lie algebras throughout the proof. Second, recall that for an irreducible representation of a complex semisimple Lie algebra, the negative of the lowest weight coincides with the highest weight of the dual representation. In this way, standard results on highest weights have analogs for the negatives of lowest weights.

As we have noted in Proposition 2, for a representation with lowest weight $-\lambda$, the Casimir eigenvalue on sections of the corresponding induced bundle is given by $\|\lambda\|^2 + 2\langle\lambda, \rho\rangle = \langle\lambda, \lambda + 2\rho\rangle$. Writing c_λ for this number, the last expression immediately shows that for two weights λ and λ', we have

$$c_{\lambda'} - c_\lambda = 2\langle\lambda' - \lambda, \lambda + \rho\rangle + \|\lambda' - \lambda\|^2. \tag{1}$$

We have to understand, how this is influenced by changing δ. Denoting by μ the highest weight associated to the representation $\mathbb{R}[1]$, which induces the bundle $\mathcal{E}[1]$, the bundle $\mathcal{E}[w]$ corresponds to the weight $w\mu$. Moving from δ to $\delta + w$ corresponds to forming a tensor product with $\mathcal{E}[w]$, and hence replacing λ by $\lambda + w\mu$ and λ' by $\lambda' + w\mu$. This means that the difference of the two weights remains unchanged, and Eq. (1) shows that

$$c_{\lambda' + w\mu} - c_{\lambda + w\mu} = c_{\lambda'} - c_\lambda + 2w\langle\lambda' - \lambda, \mu\rangle. \tag{2}$$

Now by definition, the weights of the representation $\mathfrak{g}$ are exactly the roots of $\mathfrak{g}$. Consequently, any weight of $S^k\mathfrak{g}$ is a sum of k roots. Further, it is well known that the highest weight of any irreducible component in a tensor product of two irreducible representations can be written as a sum of the highest weight of one of the two factors and some weight of the other factor. Passing to duals, we see that the same statement holds for the negatives of lowest weights. Thus, the negative of the lowest weight of any irreducible component of $S^k\mathfrak{g} \otimes V^* \otimes W$ can be written as a linear combination of the negative of the lowest weight of an irreducible component of $V^* \otimes W$ and at most k roots.

Now recall (see [18]) that for a complex $|1|$-graded Lie algebra, one can choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and positive roots in such a way that there is a unique simple root α_0 for which the corresponding root space is contained in $\mathfrak{g}_1$. More precisely, for a root α, the corresponding root space sits in $\mathfrak{g}_i$ for $i = -1, 0, 1$, where i is the coefficient of α_0 in the expansion of α as a linear combination of simple roots. Then the center of $\mathfrak{g}_0$ is generated by the unique element $H_0 \in \mathfrak{h}$ for which $\alpha_0(H_0) = 1$ while all other simple roots vanish on H_0. The orthocomplement of H_0 in $\mathfrak{h}$ is a Cartan subalgebra of the semisimple part of $\mathfrak{g}_0$.

Since the semisimple part of $\mathfrak{g}_0$ acts trivially on $\mathbb{R}[1]$, we conclude that $\mu(H) = aB(H, H_0)$ for some non-zero number a and all $H \in \mathfrak{h}$, where B denotes the Killing form of $\mathfrak{g}$. Going through the conventions, it is easy to see that actually $a < 0$. By definition of the inner product, this means that for any weight ν, we have $\langle \mu, \nu \rangle = a\nu(H_0)$. Since H_0 acts by a scalar on any irreducible representation, it also acts by a scalar on all of $S^k\mathfrak{g}_{-1} \otimes V^* \otimes W$. But this implies that if $-\nu$ is the lowest weight of an irreducible component of $V^* \otimes W$, then $\nu(H_0) = a_0$ for a fixed number a_0. Consequently, if $-\nu$ is the lowest weight of an irreducible component of the quotient of two consecutive filtrations components in $\mathcal{V}M$, say the one with index ℓ by the one with index $\ell + 1$, $\nu(H_0) = a_0 + \ell$. In particular, if $-\lambda$ is the lowest weight of R_i, then $\lambda(H_0) = a_0 - k$. Likewise if $-\lambda'$ is the lowest weight giving rise to β_i^j then $\lambda'(H_0) = a_0 + \ell$ for some $\ell > -k$. Thus we conclude that $\langle \lambda' - \lambda, \mu \rangle = a(k + \ell) < 0$, and formula (2) shows that λ and λ' give rise to exactly one critical weight. □

Note that the proof actually leads to an explicit formulae for the critical weights. Suppose that $-\lambda$ and $-\lambda'$ are the lowest weights of irreducible components giving rise to β_i^0 and β_i^j, and that the irreducible component corresponding to $-\lambda'$ sits in the quotient of the ℓth by the $(\ell + 1)$st filtration component. Then formulae (1) and (2) from the proof show that the critical weight caused by these two components is given by

$$\delta = \frac{2\langle \lambda' - \lambda, \lambda + \rho \rangle + \|\lambda' - \lambda\|^2}{2\langle \lambda' - \lambda, \mu \rangle} \tag{3}$$

where μ is the highest weight of the representation $\mathbb{R}[1]$. In particular, we can use this formula to completely determine the set of all critical weights if we know all the P-representations S_i together with their composition structure.

3.5. Restrictions on critical weights

We can also get some information on the set of critical weights without this detailed knowledge. For any P-module, we can look at the restriction of the P-action to G_0 and the restriction of the infinitesimal action of $\mathfrak{p}$ to the abelian subalgebra $\mathfrak{g}_1$. Since P is the semidirect product of G_0 and $\exp(\mathfrak{g}_1)$, one immediately concludes that any subspace in a representation of P, which is G_0-invariant and closed under the infinitesimal action of $\mathfrak{g}_1$ is actually P-invariant. By construction, the actions of elements of $\mathfrak{g}_1$ on any P-module commute. Hence the iterated action of elements of $\mathfrak{g}_1$ (in the P-module $S^k\mathfrak{g} \otimes V^* \otimes W[\delta]$) on R_i defines maps $S^\ell\mathfrak{g}_1 \otimes R_i \to S^k\mathfrak{g} \otimes V^* \otimes W[\delta]$. By construction, the image sits in the filtration component with index $\ell - k$ as well as in S_i. Hence we actually obtain a map $\bigoplus_{\ell=0}^k S^\ell\mathfrak{g}_1 \otimes R_i \to S_i$, which is evidently G_0-equivariant. In particular, the image is a G_0-invariant subspace of S_i and from the construction it follows immediately that it is also closed under the infinitesimal action of $\mathfrak{g}_1$.

The upshot of this is that any G_0-irreducible component of S_i also occurs in $\bigoplus_{\ell=0}^k S^\ell\mathfrak{g}_1 \otimes R_i$. If we determine the set of all weights δ for which an irreducible component of $\bigoplus_{\ell=1}^k S^\ell\mathfrak{g}_1 \otimes R_i$ corresponds to the same Casimir eigenvalue as R_i, then the union of these sets for all i contains the set of all critical weights.

We next work out more details on the set of critical weights for some examples in the case of even-dimensional conformal structures of arbitrary signature (p,q). (This is significantly more complicated than the case of projective structures, which is mainly considered in the literature.) Hence G_0 is the conformal group $CO(p,q)$ and $\mathfrak{g}_{-1}$ is the standard representation $\mathbb{R}^n$, $n = p+q$ of this group, and we assume that n is even. As above, we may work in the complexified setting, and we will use the notation, conventions and results from [5] for weights. We will fix representations V and W and determine critical weights starting from $S^k\mathfrak{g}_{-1} \otimes V^* \otimes W$ (i.e. with $\delta = 0$).

Let us assume that $S^k\mathfrak{g}_{-1} \otimes V^* \otimes W$ contains an irreducible component $R_i \cong \mathbb{R}[w]$ for some $w \in \mathbb{R}$. The decomposition of $S^\ell\mathbb{R}^{n*}$ into irreducible components is given by $S_0^\ell\mathbb{R}^{n*} \oplus S_0^{\ell-2}\mathbb{R}^{n*}[-2] \oplus S_0^{\ell-4}\mathbb{R}^{n*}[-4] \oplus \cdots$, where the subscript 0 indicates the totally trace-free part. From 3.5 we thus conclude that in any case all the irreducible components of P-module S_i generated by R_i must be of the form $S^\ell\mathbb{R}^{n*}[w-2m]$ for non-negative integers ℓ and m such that $\ell + 2m \leqslant k$.

In particular, for $k = 1$, the only possibility is $\mathbb{R}^n[w]$. In the notation from Section 2.4 of [5], $\mathbb{R}[w]$ corresponds to the weight $(w|0,\dots)$ while $\mathbb{R}^{n*}[w]$ corresponds to $(w-1|1,0,\dots)$, which immediately shows that the corresponding critical weight is $\delta = -w$. For $k = 2$, we get $S_0^2\mathbb{R}^{n*}[w]$ and $\mathbb{R}[w-2]$, which correspond to $(w-2|2,0,\dots)$ and $(w-2|0,\dots)$ and the critical weights $1-w$ and $1-w-\frac{n}{2}$.

For a general order k, the possible representations are $(w-\ell|\ell-2m,0,\dots)$ for $\ell \leqslant k$ and $\ell - 2m \geqslant 0$ and one easily verifies directly:

Proposition 8. *The possible critical weights caused by an irreducible component $\mathbb{R}[w] \subset S^k\mathfrak{g}_{-1} \otimes V^* \otimes W$ are contained in the set*

$$\left\{-w-1+\ell-2m+\frac{m(2+2m-n)}{\ell}\colon\ 0<\ell\leqslant k,\ 0\leqslant 2m\leqslant \ell\right\}.$$

We can derive an effective upper bound, above which there are no critical weights for quantization in *any* order. This can be viewed as a vast generalization of the results in Section 3.1 of [11] on quantization of operators on functions. Observe first that it may happen that for the representations V and W inducing E and F, the tensor product $V^* \otimes W$ itself splits into several irreducible components. For example, if $V = W$, then one always has the one-dimensional invariant subspace spanned by the identity. Given an irreducible component $U \subset V^* \otimes W$ and $\delta \in \mathbb{R}$, we have $S^k\mathfrak{g}_{-1} \otimes U[\delta] \subset S^k\mathfrak{g}_{-1} \otimes V^* \otimes W[\delta]$, so one may talk about symbols of type U of any order and any weight. Of course, one may apply the constructions from 3.1–3.3 directly to this subspace. As an irreducible representation of $\mathfrak{g}_0$, $U[\delta]$ has an associated lowest weight. Using this, we can now formulate

Theorem 9. *Let $-\lambda$ be the lowest weight of $U[\delta]$ and assume that δ is chosen in such a way that λ is $\mathfrak{g}$-dominant. Then for any order k, the weight δ is non-critical for symbols of type U. In particular, this always holds for sufficiently large values of δ.*

Proof. Let us first assume that λ is $\mathfrak{g}$-dominant and integral. Then there is a finite dimensional irreducible representation $\tilde{U}$ of $\mathfrak{g}$ with lowest weight $-\lambda$. We can pass to the dual $\tilde{U}^*$, and look at the $\mathfrak{p}$-submodule generated by a highest weight vector. It is well known that this realizes the irreducible representation of $\mathfrak{p}$ with highest weight λ. Passing back, we see that $U[\delta]$ can be naturally viewed as a quotient of $\tilde{U}$. Consequently, for any $k \geqslant 0$, we can naturally view $S^k\mathfrak{g}_{-1} \otimes U[\delta]$ as quotient of the representation $S^k\mathfrak{g} \otimes \tilde{U}$ of $\mathfrak{g}$. In particular, for any irreducible component $R_i \subset S^k\mathfrak{g}_{-1} \otimes U[\delta]$ we obtain a corresponding $\mathfrak{g}$-invariant subset $\tilde{S}_i \subset S^k\mathfrak{g} \otimes \tilde{U}$ (which can be taken to be $\mathfrak{g}$-irreducible) with $\mathfrak{p}$-irreducible quotient R_i. It is also evident that applying the natural map $S^k\mathfrak{g} \otimes \tilde{U} \to S^k\mathfrak{g} \otimes U[\delta]$ to $\tilde{S}_i$ and then factoring by the filtration component of degree zero, the image has to contain the $\mathfrak{p}$-submodule S_i generated by R_i. In particular, any $\mathfrak{g}_0$-irreducible component of S_i also has to occur in $\tilde{S}_i$.

But for the bundles corresponding to irreducible representations of $\mathfrak{g}$, the critical weights are described in Lemma 2 of [9] in terms of the Kostant Laplacian $\Box$ and the value c_0 by which the (algebraic) Casimir operator of $\mathfrak{g}$ acts on the irreducible representation $\tilde{S}_i$. Now c_0 coincides with the Casimir eigenvalue β_i^0 in our sense and hence Lemma 2 of [9] shows that $\beta_i^j - \beta_i^0$ can be computed as twice the eigenvalue of $\Box$ on the irreducible component giving rise to β_j^i. Now Kostant's theorem from [13] in particular implies that the kernel of $\Box$ on $\tilde{S}_i$ consists of R_i (viewed as a $\mathfrak{g}_0$-invariant subspace) only. This implies the result if λ is $\mathfrak{g}$-dominant and integral.

More is known about the eigenvalues of $\Box$, however. The lemma in Cartier's remarks [10] to Kostant's article shows that all eigenvalues of $\Box$ are non-positive. In the terminology of the proof of Theorem 7 this means that $c_{\lambda'} - c_\lambda < 0$. There we have also seen that $\langle \lambda' - \lambda, \mu \rangle < 0$, so formula (2) from that proof shows that $c_{\lambda'+w\mu} - c_{\lambda+w\mu} < 0$ for $w \geqslant 0$. Now if $-\lambda$ is the lowest weight of a finite dimensional irreducible representation of $\mathfrak{p}$, then λ is $\mathfrak{p}$-dominant and $\mathfrak{p}$-integral. But this means that $\lambda + w\mu$ is $\mathfrak{g}$-dominant for sufficiently large values of w and $\mathfrak{g}$-integral for all integral values of w, which implies all the remaining claims. $\Box$

3.6. Low order quantizations for even-dimensional conformal structures

Let us move to more complete examples in the setting from above. We will restrict to the cases that $V^* \otimes W \cong \mathbb{R}$ and $V^* \otimes W \cong \mathbb{R}^n$, and to orders at most three in the first case and at most two in the second case. For $V^* \otimes W \cong \mathbb{R}$, we get quantizations on density bundles, which can be compared to available results in the literature. The case $V^* \otimes W \cong \mathbb{R}^n$ can be used to understand operators mapping weighted one-forms to densities and, vice versa, mapping densities to weighted one-forms.

We have already noted in 3.5 that the decomposition of $S^k\mathfrak{g}_{-1}$ is given by $\bigoplus_{\ell \leqslant k/2} S^\ell\mathfrak{g}_{-1}[2\ell]$.

First order operators on densities. Here the symbol representation is $\mathfrak{g}_{-1} \cong \mathbb{R}^n$, so this is irreducible and corresponds to the weight $(1|1, 0, \ldots)$. Likewise, $\mathfrak{g}$ is an irreducible representation of $\mathfrak{g}$, and there is only one relevant level which may produce critical weights, namely $\mathfrak{g}_0 \cong \Lambda^2\mathbb{R}^n[2] \oplus \mathbb{R}$, which is the quotient of the filtration components of degrees 0 and 1. The summands correspond to the weights $(0|1, 1, 0, \ldots) \oplus (0|0, \ldots)$ and we obtain the critical weights $-n$ and -2.

Second order operators on densities. The symbol representation splits into two irreducible components R_1 and R_2 corresponding to the weights $(2|2, 0, \ldots)$ (trace-free symbols) and $(2|0, \ldots)$ (symbols which are pure trace, i.e. of Laplace type). Also, the representation $S^2\mathfrak{g}$ of $\mathfrak{g}$ is not irreducible any more, but splits into four irreducible components. One of them is a trivial representation (corresponding to the Killing form) and one is isomorphic to $\Lambda^4\mathbb{R}^{n+2}$. These two components are entirely contained in the filtration component of degree -1, so they do not contribute to the quotient by the largest filtration component. One of the remaining two irreducible components is isomorphic to $S_0^2\mathbb{R}^{n+2}$. The quotient of this component by its intersection with the largest filtration component is exactly R_2, so all of S_2 must be contained in this part. Finally, there is the highest weight component $\circledcirc^2\mathfrak{g} \subset S^2\mathfrak{g}$ (the Cartan product of two copies of $\mathfrak{g}$), whose quotient by the largest filtration component is R_1. Hence S_1 is contained in this component.

To determine the possible critical weights it thus suffices to analyze the composition structure of the representations $\circledcirc^2\mathfrak{g}$ and $S_0^2\mathbb{R}^{n+2}$. This can be done fairly easily using the description of representations of $\mathfrak{g}$ in terms of their $\mathfrak{p}$-irreducible quotients from Section 3 of [3], in particular the result in Lemma 3.1 of this article. One has to use the fact that the Lie algebra cohomology groups that occur are algorithmically computable using Kostant's version of the Bott–Borel–Weil theorem.

This shows that in the language of weights, the two relevant levels of $\circledcirc^2\mathfrak{g}$ decompose as

$$(1|2,1,0,\dots) \oplus (1|1,0,\dots),$$
$$(0|2,2,0,\dots) \oplus (0|2,0,\dots) \oplus (0|1,1,0,\dots) \oplus (0|0,\dots),$$

and consequently, one obtains the critical weights -3, -2, $-2-n$, $-1-n$, $(-2-n)/2$, and $(-4-n)/2$.

For the case of symbols which are pure trace, the decompositions of the level for the index -1 is irreducible corresponding to the weight $(1|1,0,\dots)$, while the level for index zero decomposes as $(0|2,0,\dots) \oplus (0|0,\dots)$. This gives rise to the critical weights -2, -1 and $(-2-n)/2$.

Third order operators on densities. The analysis is closely analogous to the second order case, we mainly include the results for comparison to [2]. The symbol representation splits into two irreducible components and again these two components correspond to two of the seven irreducible components in $S^3\mathfrak{g}$. Namely, trace-free symbols ($S_0^3\mathbb{R}^n$) correspond to the highest weight component $\circledcirc^3\mathfrak{g}$, while trace symbols ($\mathbb{R}^n[2]$) correspond to the Cartan product $\mathfrak{g} \circledcirc S_0^2\mathbb{R}^{n+2}$. The relevant parts of the composition series for these two representations of $\mathfrak{g}$ can be determined as in the second order case. From these, one computes the critical weights. In the trace-free case, one obtains -4, -3, -2, $-4-n$, $-3-n$, $-2-n$, $(-7-n)/2$, $(-4-n)/2$, $(-8-n)/3$, $(-8-2n)/3$, $(-6-n)/3$, and $(-6-2n)/3$. For trace-type symbols, we get the critical weights -1, -2, -4, $-5/2$, $-4/3$, $(-4-n)/2$, $(-4-n)/3$, $(-6-n)/3$, and $(-4-2n)/3$. These are the critical weights from [2], plus quite a few additional ones. We'll comment on that in 3.7 below.

First order operators for $V^* \otimes W \cong \mathbb{R}^n$. Here the symbol representation decomposes as

$$\mathbb{R}^n \otimes \mathbb{R}^n = R_1 \oplus R_2 \oplus R_3 = S_0^2\mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n \oplus \mathbb{R}[2],$$

or in weights $(2|2,0,\dots) \oplus (2|1,1,0,\dots) \oplus (2|0,\dots)$. There is only one relevant level in the composition series of $\mathfrak{g} \otimes \mathbb{R}^n$, which can be determined by decomposing the tensor product $\mathfrak{g}_0 \otimes \mathbb{R}^n$ into irreducibles. In terms of weights, the result is $(1|2,1,0,\dots) \oplus (1|1,1,1,0,\dots) \oplus 2(1|1,0,\dots)$, so the last irreducible component occurs with multiplicity two. Decomposing the tensor products $R_i \otimes \mathbb{R}^n$, one concludes that S_1 can only contain the first and a copy of the last irreducible components, while S_3 can only contain one copy of the last irreducible component. Consequently, there are three critical weights for skew symmetric symbols (which turn out to be -1, -4, and $-n$) but only two (namely -3 and $-2-n$) for symmetric symbols. For trace-type symbols we obtain only one critical weight, namely -2, which agrees with the result from 3.5.

Second order operators for $V^* \otimes W \cong \mathbb{R}^n$. Here the symbol representation $S^2\mathbb{R}^n \otimes \mathbb{R}^n$ decomposes into four irreducible components, in weight notation, it is given by

$$(3|3,0,\ldots) \oplus (3|2,1,0,\ldots) \oplus 2(3|1,0,\ldots).$$

Here one of the two copies of $\mathbb{R}^n[2]$ is contained in $S_0^2\mathbb{R}^n \otimes \mathbb{R}^n$, while the other comes from the trace part. Let us write this decomposition as $R_1 \oplus \cdots \oplus R_4$, with R_4 coming from the trace part. From above, we know that $S^2\mathfrak{g}$ contains the irreducible components $\circledcirc^2\mathfrak{g}$ and $S_0^2\mathbb{R}^{n+2}$, which correspond to $S_0^2\mathbb{R}^n$ and $\mathbb{R}[2] \subset S^2\mathbb{R}^n$, respectively. Consequently, we can determine the relevant composition factors for S_1, S_2, and S_3 by decomposing the tensor products of the composition factors of $\circledcirc^2\mathfrak{g}$ as listed above with $\mathbb{R}^n$, and then checking with of the components may be contained in each S_i. For S_4, we proceed similarly with $S_0^2\mathbb{R}^{n+2}$ replacing $\circledcirc^2\mathfrak{g}$.

For the first relevant level (corresponding to filtration index -1), we first have to decompose $(1|2,1,0,\ldots) \otimes (1|1,0,\ldots)$ which gives

$$(2|3,1,0,\ldots) \oplus (2|2,2,0,\ldots) \oplus (2|2,1,1,0,\ldots) \oplus (2|2,0,\ldots) \oplus (2|1,1,0,\ldots).$$

Second, $(1|1,0,\ldots) \otimes (1|1,0,\ldots) \cong (2|2,0,\ldots) \oplus (2|1,1,0,\ldots) \oplus (2|0,\ldots)$.

Looking at the tensor products $R_i \otimes \mathbb{R}^n$, we conclude that S_1 can only contain $(2|3,1,0,\ldots)$ and $(2|2,0,\ldots)$, S_3 can only contain $(2|2,0,\ldots)$ and $(2|1,1,0,\ldots)$, while all components of the first sum may occur in S_2. Hence from this level, we get the critical weights -4 and $-4-n$ for R_1. For R_2, we obtain the critical weights -1, -3, -5, $-1-n$, and $-3-n$, while for R_3, the critical weights are -2, -4, and $-2-n$.

The second relevant level is dealt with in an analogous way. The result is that for R_1, we get the additional critical weights -3, $-3-n$, $(-4-n)/2$, and $(-7-n)/2$. For R_2, we obtain $-3/2$, $-7/2$, $(-1-n)/2$, $(-4-n)/2$, $(-7-n)/2$, $(-3-2n)/2$. Finally, for R_3, we get the additional critical weights -1, $-5/2$, and $(-4-n)/2$. A direct evaluation shows that for R_4 we get exactly the same critical weights as for R_3 (although the bundle involved is different).

3.7. Discussion and remarks

(1) Note that the results in the examples from 3.6 are consistent with Theorem 9, which implies that in all the cases discussed in 3.6 all critical weights have to be negative.

(2) From the examples of operators on densities discussed in 3.6 it is evident that the sets of critical weights we obtain with our general procedure are far from being optimal. It is actually easy to see why this happens, and even to partly improve the procedure, to get smaller sets of critical weights. The point here is that part (1) of Proposition 3 can be heavily improved in special cases, and in particular for the fundamental derivative on densities. In the case of densities, already the values of a single fundamental derivative do not exhaust $\mathcal{A}^0M[w]$. On the contrary, projecting to $(\mathcal{A}^0M/\mathcal{A}^1M)[w] \cong \Lambda^2TM[w-2] \oplus \mathcal{E}[w]$, the values always lie in the density summand only. By naturality of the fundamental derivative, this implies that higher order fundamental derivatives always will lie in subbundles which are much smaller than the bundle $\mathcal{W}^0M$ from Proposition 3.

Knowing this, one can run the analog of the procedure from 3.2 and 3.3 on the quotient by the annihilator of this subbundle, which will be significantly smaller than the bundle $\mathcal{V}M$ we have used. For this smaller quotient, there will be less irreducible components in the individual subquotients and hence less critical weights. In fact, it is easy to see directly that in the examples discussed in 3.6 most (but not all) of the superfluous critical weights will disappear.

(3) In the case $V \otimes W \cong \mathbb{R}^n$ the set of critical weights we have obtained in 3.6 will be closer to the optimum than in the case of densities. As we have noted, this case can be used to study both quantizations for operators mapping sections of $\mathcal{E}[w]$ to sections of $TM[w+\delta]$ and for operators mapping sections of $T^*M[w] \cong TM[w-2]$ to sections of $\mathcal{E}[w+\delta]$. While these two cases are completely symmetric from our point of view, this is no more true if one looks at the best possible sets of critical weights. The point is that in the first case, the value of the splitting operator will be paired with $D^{(k)} f \in \Gamma(S^k \mathcal{A}^* M[w])$ for $f \in \Gamma(\mathcal{E}[w])$, and as discussed above, this has values in a much smaller subbundle than just the filtration component of degree zero. In the second interpretation, we will have to pair it with $D^{(k)}\alpha \in \Gamma(S^k \mathcal{A}^* M \otimes T^*M[w])$ for $\alpha \in \Gamma(T^*M[w])$, and the values of this operator fill a more substantial part of the filtration component of degree zero. Hence in the first case, we can remove more superfluous critical weights than in the second one.

(4) There is a systematic way to derive explicit formulae for the procedures we have developed in terms of distinguished connections (e.g. the Levi-Civita connections of the metrics in a conformal class), but this becomes quickly rather tedious. In view of the construction, the main point is to obtain an explicit formula for the curved Casimir operator on irreducible components of $S^k \mathcal{A}M$. This can be done along the lines of Proposition 2.2 of [5] which holds (with obvious modifications) for general AHS-structures.

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Conformally invariant quantization – towards the complete classification ☆

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A R T I C L E   I N F O

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A B S T R A C T

Let M be a smooth manifold equipped with a conformal structure, $\mathcal{E}[w]$ the space of densities with the conformal weight w and $\mathcal{D}_{w,w+\delta}$ the space of differential operators from $\mathcal{E}[w]$ to $\mathcal{E}[w+\delta]$. Conformal quantization Q is a right inverse of the principle symbol map on $\mathcal{D}_{w,w+\delta}$ such that Q is conformally invariant and exists for all w. There are partial results concerning the dependence on δ: the existence of Q is known for generic values of $\delta \in \mathbb{R}$ and the nonexistence is known for a set of critical weights $\delta \in \Sigma \subseteq \mathbb{R}$. We complete these results, i.e. we show that Q exists for *all* noncritical weights $\delta \notin \Sigma$. We give explicit formulae for Q for all such δ in terms of the conformal tractor calculus.

1. Introduction

The notion of quantization originates in physics. Here we view it as quest for a correspondence between a space of differential operators and the corresponding space of symbols. More specifically, consider the space $\mathcal{D}_0$ of differential operators acting on smooth functions on a smooth manifold M and the space of symbols $\mathcal{S}_0$. Quantization is a map $Q_0 : \mathcal{S}_0 \to \mathcal{D}_0$ such that $\mathrm{Symb} \circ Q_0 = \mathrm{id}\,|_{\mathcal{S}_0}$ where $\mathrm{Symb} : \mathcal{D}_0 \to \mathcal{S}_0$ is the principal symbol map. If $\Phi \in \mathcal{D}_0$ of the order k has the principal symbol σ then $\Phi - Q_0(\sigma) \in \mathcal{D}_0$ has the order $k-1$. Iterating this we obtain the isomorphism of vector spaces $\bigoplus_{i=0}^{k} \mathcal{S}_0^i \cong \mathcal{D}_0^k$ where $\mathcal{S}_0^i = \Gamma(\bigodot^i TM) \subseteq \mathcal{S}_0$ and $\mathcal{D}_0^k \subseteq \mathcal{D}_0$ is the space of operators of order at most k. Here $\bigodot^k$ is the kth symmetric tensor product. We shall use the notation $Q_0^\sigma := Q_0(\sigma)$.

There is no natural quantization on a smooth manifold M. On the other hand, e.g. a choice of a linear connection ∇ on M yields a preferred quantization in an obvious way: if $\sigma \in \mathcal{S}_0^k$ and $f \in C^\infty(M)$, we put $Q_0^\sigma(f) = \sigma(\nabla^{(k)} f)$ where $\nabla^{(k)} f$ is the symmetrized k-fold covariant derivative. Therefore there is a

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canonical quantization on every pseudo-Riemannian manifold M. Motivated by this observation one can ask whether there is a natural quantization for less rigid geometrical structures on M.

In this article we study the case when the manifold M is equipped with a conformal structure. This was initiated by Lecomte et al. [24,13] and we refer to [25] for details about further development and references. On one hand, existence of conformally invariant quantization is known in many cases, see e.g. [13,4,26,11]. On the other hand, [25] shows the nonexistence for so-called *critical weights* of symbols. (The terminology will be explained below.) Here we prove the complementary result, i.e. we provide an explicit construction of the conformally invariant quantization for all noncritical weights.

The conformal structure on manifold M is a class of pseudo-Riemannian metrics $[g] = \{fg \mid f \in C^\infty(M),\ f > 0\}$ on a manifold M. The homogeneous model is the pseudosphere $M = S^{p,q} := S^p \times S^q$, where (p,q) is the signature of g, the product of the standard metrics on S^p and S^q where one of them has the opposite sign. This is a homogeneous space for $G = SO_0(p+1, q+1)$ acting on $S^{p,q}$ by conformal motions of $[g]$ and we have the isomorphism $S^{p,q} \cong G/P$ where $P \subseteq G$ is the Poincaré group of conformal motions fixing a point, see [8] for details. Then both $\mathcal{S}_0$ and $\mathcal{D}_0$ are G-modules and the question of conformally invariant quantization means to construct $Q_0 : \mathcal{S}_0 \to \mathcal{D}_0$ which intertwines these G-actions. If we pass from $S^{p,q}$ to $\mathbb{R}^{p,q}$ (where only the rational action of G is defined) via the stereographic projection, we replace the G-action by the infinitesimal $\mathfrak{g}$-action. The Lie algebra $\mathfrak{g}$ of G can be realized as a Lie algebra of (polynomial) vector fields on $\mathbb{R}^{p,q}$ and they act by the Lie derivative as infinitesimal conformal symmetries. The same can be done for every locally conformally flat manifold and the invariance of Q_0 is given by this $\mathfrak{g}$-action. This setting is often taken as the starting point in the study of invariant (or equivariant) quantization [13]. It is natural to consider more generally bundles of conformal densities $E[w]$, $w \in \mathbb{R}$ (instead just functions) and the space of differential operators $\Gamma(E[w_1]) \to \Gamma(E[w_2])$ denoted by $\mathcal{D}_{w_1,w_2}$. Denoting by $\mathcal{D}^k_{w_1,w_2}$ the space of operators of degree $\leqslant k$, the corresponding bundle of kth degree symbols is then $S^k_\delta = (\bigodot^k TM) \otimes E[\delta] \cong \mathcal{D}^k_{w_1,w_2}/\mathcal{D}^{k-1}_{w_1,w_2}$ where $\delta = w_2 - w_1$. Note this is the notation used in the conformally invariant calculus; the space of densities can be also defined as $\mathcal{F}_\lambda = \Gamma(\bigotimes^\lambda(|\bigwedge^n T^*M|))$ where $\bigwedge^n T^*M \to M$ is the determinant line bundle, $n = \dim(M)$. Then one has the relation $\Gamma(E[-nw]) = \mathcal{F}_w$. Finally note one can also consider complex densities but this would not change results obtained in this article. (That is, all weights w, which are significant from the point of view of invariant quantization, are real.)

Summarizing, the question in the conformally flat case is whether for a given $\delta \in \mathbb{R}$ there is an isomorphism of $\mathfrak{so}_{p+1,q+1}$-modules

$$Q_\delta : \mathcal{S}_\delta \to \mathcal{D}_{w,w+\delta} \tag{1}$$

for all $w \in \mathbb{R}$ where $S_\delta = (\bigodot TM) \otimes E[\delta]$. That is, the corresponding bilinear differential operator $Q_\delta : \mathcal{S}_\delta \times \mathcal{E}[w] \to \mathcal{E}[w+\delta]$ is conformally invariant. It turns out the answer is positive for generic weights δ, i.e. for all weights up to a certain number (which is finite for symbols of a fixed degree) [13,4,11]. On the other hand, [25] identifies the set of *critical weights* δ for which the nonexistence is proved. Here we extend known existence results to all noncritical weights δ. Note to get such complete answer one needs to study critical weights for particular irreducible components of $\mathcal{S}_\delta$ (similarly as in [25]). Note also the requirement that (1) holds for all $w \in \mathbb{R}$ is essential for our notion of critical weights δ. If we drop this requirement, (1) could hold also for critical weights δ, and certain weights w, see below.

Analogous problem can be formulated for all manifolds M with the given conformal class $[g]$. Then there are generically no infinitesimal symmetries on $(M,[g])$ and by invariance of the quantization $Q_\delta : \mathcal{S}_\delta \to \mathcal{D}_{w,w+\delta}$ we mean the corresponding bilinear operator $Q_\delta : \mathcal{S}_\delta \times \mathcal{E}[w] \to \mathcal{E}[w+\delta]$ is given in terms of a Levi-Civita connection ∇ from the conformal class, its curvature R and algebraic operations in such a way that Q_δ does not depend on the choice of ∇. Using the terminology of conformal geometry, the bilinear invariant operator Q_δ has a *curved analogue*. Note there is generally no hope for uniqueness of Q_δ as the curvature can modify conformal operators in various ways. Note the general problem of construction of invariant bilinear operators $V_1 \times V_2 \to W$ for conformal (and more generally parabolic) geometries for irreducible bundles V_1, V_2 and W is considered in Kroeske's thesis [23]. It is however not clear how to obtain quantization from the machinery developed there.

Our main result is Theorem 3.3 which provides an explicit (and inductive) formula for Q_δ on all curved conformal manifolds and all noncritical (in the sense of [25]) weights δ. This is achieved using the conformal tractor calculus, see [1] for a discussion on its origin and [7,6] for the relation to the Cartan connection. In Section 4 we comment upon critical weights. First we recall their relation to existence of linear invariant operators on symbols (cf. [25]) and then we discuss weights w such that $Q_\delta : \mathcal{S}_\delta \to \mathcal{D}_{w,w+\delta}$ exists even when δ is critical, i.e. *resonant weights* $w \in \mathbb{R}$. Details are available in the flat case [25], we shall indicate existence of curved analogues for some resonant weights in Theorem 4.4.

2. Conformal geometry and tractor calculus

2.1. Notation and background

We present here a brief summary, further details may be found in [5,20]. Let M be a smooth manifold of dimension $n \geqslant 3$. Recall that a *conformal structure* of signature (p,q) on M is a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$ whose fibre over x consists of conformally related signature (p,q) metrics at the point x. Sections of $\mathcal{Q}$ are metrics g on M. So we may equivalently view the conformal structure as the equivalence class $[g]$ of these conformally related metrics. The principal bundle $\pi : \mathcal{Q} \to M$ has structure group $\mathbb{R}_+$, and so each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$ induces a natural line bundle on $(M,[g])$ that we term the conformal density bundle $E[w]$. We shall write $\mathcal{E}[w]$ for the space of sections of this bundle. We write $\mathcal{E}^a$ for the space of sections of the tangent bundle TM and $\mathcal{E}_a$ for the space of sections of T^*M. The indices here are abstract in the sense of [27] and we follow the usual conventions from that source. So for example $\mathcal{E}_{ab}$ is the space of sections of $\bigotimes^2 T^*M$. Here and throughout, sections, tensors, and functions are always smooth. When no confusion is likely to arise, we will use the same notation for a bundle and its section space.

We write $\boldsymbol{g}$ for the *conformal metric*, that is the tautological section of $S^2T^*M \otimes E[2]$ determined by the conformal structure. This is used to identify TM with $T^*M[2]$. For many calculations we employ abstract indices in an obvious way. Given a choice of metric g from $[g]$, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. Here we are raising indices and contracting using the (inverse) conformal metric. Indices will be raised and lowered in this way without further comment. Note $E[w]$ is trivialized by a choice of metric g from the conformal class, and we also write ∇ for the connection corresponding to this trivialization. The coupled connection ∇_a preserves the conformal metric.

The curvature $R_{ab}{}^c{}_d$ of the Levi-Civita connection (the Riemannian curvature) is given by $[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d$ ($[\cdot,\cdot]$ indicates the commutator bracket). This can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to

$$R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}P_{b]d} + 2\boldsymbol{g}_{d[b}P_{a]c}, \tag{2}$$

where $[\cdots]$ indicates antisymmetrization over the enclosed indices. The Schouten tensor is a trace modification of the Ricci tensor $\mathrm{Ric}_{ab} = R_{ca}{}^c{}_b$ and vice versa: $\mathrm{Ric}_{ab} = (n-2)P_{ab} + J\boldsymbol{g}_{ab}$, where we write J for the trace $P_a{}^a$ of P. The *Cotton tensor* is defined by $A_{abc} := 2\nabla_{[b}P_{c]a}$. Via the Bianchi identity this is related to the divergence of the Weyl tensor as follows:

$$(n-3)A_{abc} = \nabla^d C_{dabc}. \tag{3}$$

Under a *conformal transformation* we replace a choice of metric g by the metric $\hat{g} = e^{2\Upsilon} g$, where Υ is a smooth function. We recall that, in particular, the Weyl curvature is conformally invariant $\hat{C}_{abcd} = C_{abcd}$. With $\Upsilon_a := \nabla_a \Upsilon$, the Schouten tensor transforms according to

$$\hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2}\Upsilon^c \Upsilon_c \boldsymbol{g}_{ab}. \tag{4}$$

Explicit formulae for the corresponding transformation of the Levi-Civita connection and its curvatures are given in e.g. [1,20]. From these, one can easily compute the transformation for a general valence (i.e. rank) s section $f_{bc\cdots d} \in \mathcal{E}_{bc\cdots d}[w]$ using the Leibniz rule:

$$\begin{aligned} \hat{\nabla}_{\bar{a}} f_{bc\cdots d} = {} & \nabla_{\bar{a}} f_{bc\cdots d} + (w-s)\Upsilon_{\bar{a}} f_{bc\cdots d} - \Upsilon_b f_{\bar{a}c\cdots d} - \cdots - \Upsilon_d f_{bc\cdots \bar{a}} \\ & + \Upsilon^p f_{pc\cdots d} \boldsymbol{g}_{b\bar{a}} + \cdots + \Upsilon^p f_{bc\cdots p} \boldsymbol{g}_{d\bar{a}}. \end{aligned} \tag{5}$$

We next define the standard tractor bundle over $(M,[g])$. It is a vector bundle of rank $n+2$ defined, for each $g \in [g]$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\hat{g} = e^{2\Upsilon} g$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\hat{\alpha}, \hat{\mu}_a, \hat{\tau}) \in [\mathcal{E}^A]_{\hat{g}}$ by the transformation

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\mu}_a \\ \hat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_c \Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix}. \tag{6}$$

It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{E}^A$ over the conformal manifold. (Alternatively the standard tractor bundle may be constructed as a canonical quotient of a certain 2-jet bundle or as an associated bundle to the normal conformal Cartan bundle [6].) On a conformal structure of signature (p,q), the bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(p+1, q+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. Up to an isomorphism, this is the unique *normal conformal tractor connection* [7] and it induces a normal connection on $\bigotimes \mathcal{E}^A$ that we will also denote by ∇_a and term the (normal) tractor connection. In a conformal scale g, the metric h_{AB} and ∇_a on $\mathcal{E}^A$ are given by

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \nabla_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + P_{ab}\alpha \\ \nabla_a \tau - P_{ab}\mu^b \end{pmatrix}. \tag{7}$$

It is readily verified that both of these are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A$, $\mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we see that

$$Y_A X^A = 1, \qquad Z_{Ab} Z^A{}_c = \boldsymbol{g}_{bc}, \tag{8}$$

and all other quadratic combinations that contract the tractor index vanish. In (6) note that $\hat{\alpha} = \alpha$ and hence X^A is conformally invariant.

The curvature Ω of the tractor connection is defined on $\mathcal{E}^C$ by $[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E$. Using (7) and the decomposition (2) for the Riemannian curvature yields

$$\Omega_{abCE} = Z_C{}^c Z_E{}^e C_{abce} - 2X_{[C} Z_{E]}{}^e A_{eab}. \tag{9}$$

Given a choice of $g \in [g]$, the *tractor-D operator* $D_A : \mathcal{E}[w] \to \mathcal{E}_A[w-1]$ is defined by

$$D_A V := (n+2w-2)wY_A V + (n+2w-2)Z_{Aa}\nabla^a V - X_A \Box V, \tag{10}$$

where $\Box V := \Delta V + wJV$. This is conformally invariant, as can be checked directly using the formulae above (or alternatively there are conformally invariant constructions of D, see e.g. [17]).

The operator D_A is strongly invariant. That is, it is invariant as an operator

$$D_A : \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1]$$

where now we interpret ∇ in (10) as the coupled Levi-Civita–tractor connection. Note the strong invariance is a property of a *formula*, see [18, p. 21] for a more detailed discussion and [14, (2)] for an illustrative example. We shall say an operator is strongly invariant if it is clear which formula we mean. Note composition of two strongly invariant operators is strongly invariant.

Beside the standard tractor bundle $\mathcal{E}^A \cong \mathcal{E}_A$, we shall also need more details about the structure of the *adjoint tractor bundle* $\mathcal{E}_{[A^0A^1]}$. To simplify the notation, we shall use form indices $\mathbf{A} = [A^0A^1]$ and $\mathbf{a} = [a^0a^1]$. For a chosen metric g, we have $[\mathcal{E}_\mathbf{A}]_g = \mathcal{E}^a \oplus \mathcal{E}_{[a^0a^1]}[2] \oplus \mathcal{E} \oplus \mathcal{E}_a$. We can write sections $F_\mathbf{A} \in \mathcal{E}_\mathbf{A}$ as

$$[F_\mathbf{A}]_g = \mathbb{Y}^a_\mathbf{A}\sigma_a + \mathbb{Z}^\mathbf{a}_\mathbf{A}\mu_\mathbf{a} + \mathbb{W}_\mathbf{A}\nu + \mathbb{X}^a_\mathbf{A}\rho_a$$

where $\sigma_a \in \mathcal{E}_a[2] \cong \mathcal{E}^a$, $\mu_\mathbf{a} \in \mathcal{E}_\mathbf{a}[2]$, $\nu \in \mathcal{E}$ and $\rho_a \in \mathcal{E}_a$ and we use the notation $\mathbb{Y}^a_\mathbf{A} = Y_{[A^0}Z^a_{A^1]}$, $\mathbb{Z}^\mathbf{a}_\mathbf{A} = Z^{a^0}_{[A^0}Z^{a^1}_{A^1]}$, $\mathbb{W}_\mathbf{A} = X_{[A^0}Y_{A^1]}$ and $\mathbb{X}^a_\mathbf{A} = X_{[A^0}Z^a_{A^1]}$.

2.2. Tractor connection and standard tractors

Using the standard tractors X_B, Z^b_B and Y_B, the tractor connection takes the form

$$\begin{aligned}
&\nabla_a Y_B\sigma = Y_B\nabla_a\sigma + Z^b_B P_{ab}\sigma, \quad \sigma \in \mathcal{E}[w]\\
&\nabla_a Z^b_B\mu_b = -Y_B\mu_a + Z^b_B\nabla_a\mu_b - X_B P_a{}^b\mu_b, \quad \mu_b \in \mathcal{E}_b[w]\\
&\nabla_a X_B\rho = Z^b_B \boldsymbol{g}_{ab}\rho + X_B\nabla_a\rho, \quad \rho \in \mathcal{E}[w]
\end{aligned} \tag{11}$$

which follows from (7) (or see e.g. [20]). More accurately, ∇ denotes the coupled tractor–Levi-Civita connection in expressions like in the previous display.

We shall need, more generally, to know how the composition of several applications of the tractor connection acts on standard tractors. In fact, we shall need this only on $\mathbb{R}^{p,q}$. It follows from (11) (and can be verified easily by induction wrt. $k \geqslant 1$) that

$$\begin{aligned}
&\nabla_{(a_1}\cdots\nabla_{a_k)}Y_B\sigma = Y_B\nabla_{(a_1}\cdots\nabla_{a_k)}\sigma + ct,\\
&\nabla_{(a_1}\cdots\nabla_{a_k)}Z^b_B\mu_b = -kY_B\delta^b_{(a_1}\nabla_{a_2}\cdots\nabla_{a_k)}\mu_b + Z^b_B\nabla_{(a_1}\cdots\nabla_{a_k)}\mu_b + ct,\\
&\nabla_{(a_1}\cdots\nabla_{a_k)}X_B\rho = -\frac{1}{2}k(k-1)Y_B\boldsymbol{g}_{(a_1a_2}\nabla_{a_3}\cdots\nabla_{a_k)}\rho + kZ^b_B\boldsymbol{g}_{b(a_1}\nabla_{a_2}\cdots\nabla_{a_k)}\rho + X_B\nabla_{(a_1}\cdots\nabla_{a_k)}\rho + ct
\end{aligned}$$

where $\sigma \in \mathcal{E}[w]$, $\mu_b \in \mathcal{E}_b[w]$, $\rho \in \mathcal{E}[w]$ and "ct" denotes terms which involve curvature and at most $k-2$ derivatives. (That is, "ct" vanishes on $\mathbb{R}^{p,q}$.) Here and below, $(\cdots)$ denotes symmetrization of the enclosed indices and the notation $(\cdots)_0$ will denote the projection to the symmetric trace-free part. In fact, the previous display holds also for $k = 0$ if we consider expressions with k free indices $a_1 \cdots a_k$ simply being absent for $k = 0$. Henceforth we shall use this convention. It follows from the previous display (or can be verified by induction directly) that for $k \geqslant 0$ we obtain

$$\begin{aligned}
&\nabla_{(a_1} \cdots \nabla_{a_k)_0} Y_B \sigma = Y_B \nabla_{(a_1} \cdots \nabla_{a_k)_0} \sigma + ct, \\
&\nabla_{(a_1} \cdots \nabla_{a_k)_0} Z_B^b \mu_b = -k Y_B \delta^b_{(a_1} \nabla_{a_2} \cdots \nabla_{a_k)_0} \mu_b + Z_B^b \nabla_{(a_1} \cdots \nabla_{a_k)_0} \mu_b + ct, \\
&\nabla_{(a_1} \cdots \nabla_{a_k)_0} X_B \rho = k Z_B^b \boldsymbol{g}_{b(a_1} \nabla_{a_2} \cdots \nabla_{a_k)_0} \rho + X_B \nabla_{(a_1} \cdots \nabla_{a_k)_0} \rho + ct
\end{aligned} \tag{12}$$

and for $\ell \geqslant 0$ we have

$$\begin{aligned}
&\Delta^\ell Y_B \sigma = Y_B \Delta^\ell \sigma + ct, \\
&\Delta^\ell Z_B^b \mu_b = -2\ell Y_B \nabla^b \Delta^{\ell-1} \mu_b + Z_B^b \Delta^\ell \mu_b + ct, \\
&\Delta^\ell X_B \rho = -\ell(n + 2\ell - 2) Y_B \Delta^{\ell-1} \rho + 2\ell Z_B^b \nabla_b \Delta^{\ell-1} \rho + X_B \Delta^\ell \rho + ct
\end{aligned} \tag{13}$$

where $\sigma \in \mathcal{E}[w]$, $\mu_b \in \mathcal{E}_b[w]$, $\rho \in \mathcal{E}[w]$.

3. Tractor construction of conformal quantization and critical weights

We assume $\sigma^{a_1 \cdots a_k} \in \mathcal{E}^{(a_1 \cdots a_k)}[\delta] =: \mathcal{S}_{\delta,k}$ and $f \in \mathcal{E}[w]$. Our aim is to construct a quantization, i.e. a differential operator $Q^\sigma_\delta : \mathcal{E}[w] \to \mathcal{E}[w + \delta]$ with the leading term $\sigma^{a_1 \cdots a_k} \nabla_{a_1} \cdots \nabla_{a_k}$. The bundle of symbols $\mathcal{E}^{(a_1 \cdots a_k)}[\delta]$ decomposes into irreducibles as

$$\mathcal{E}^{(a_1 \cdots a_k)}[\delta] = \bigoplus_{i=0}^{\lfloor \frac{k}{2} \rfloor} \mathcal{E}^{(a_1 \cdots a_{k-2i})_0}[\delta + 2i],$$

where $\lfloor a \rfloor$ denotes the lower integer part of $a \in \mathbb{R}$. (Recall that the notation $\mathcal{E}^{(a \cdots b)_0}$ denotes the symmetric trace-free part.) We can assume σ is irreducible (as Q^σ_δ is linear in σ) so

$$\begin{aligned}
&\sigma^{a_1 \cdots a_k} = \sigma'^{\,(a_1 \cdots a_{k'}} \boldsymbol{g}^{a_{k'+1} a_{k'+2}} \cdots \boldsymbol{g}^{a_{k'+2\ell-1} a_{k'+2\ell})}, \quad k' + 2\ell = k \\
&\quad \text{where } (\sigma')^{a_1 \cdots a_{k'}} \in \mathcal{E}^{(a_1 \cdots a_{k'})_0}[\delta'], \ \delta' = \delta + 2\ell
\end{aligned}$$

since $\boldsymbol{g}^{ab} \in \mathcal{E}^{ab}[-2]$.

Henceforth we consider the irreducible symbol σ' as in the previous display. Our aim is to construct a differential operator

$$Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2l], \qquad Q^{\sigma'}_{k',\ell}(f) = (\sigma')^{a_1 \cdots a_{k'}} \nabla_{(a_1} \cdots \nabla_{a_{k'})_0} \Delta^\ell f + lot \tag{14}$$

for some scalar $y \neq 0$ which is conformally invariant as the bilinear operator $Q_{k',\ell} : \mathcal{E}^{(a_1 \cdots a_{k'})_0}[\delta'] \times \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2l]$. Here "*lot*" denotes lower order terms and we have suppressed the parameter δ' in the notation for Q. The reason is that we will define the operator $Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2l]$ by a universal tractor formula for all $\delta' \in \mathbb{R}$ up to a finite number of critical weights. Then we shall discuss these critical weights δ' in detail.

The construction of $Q_{k',\ell}$ is divided into two steps – the cases $\ell = 0$ and $\ell > 0$.

3.1. The quantization $Q_{k',0}$

This case is known, the explicit formula is obtained in [28] or can be deduced from [23]. To keep our presentation self-contained and also to verify the strong invariance, we present a construction (of a tractor formula) here.

Theorem 3.1. *Let $(\sigma')^{a_1\cdots a_{k'}} \in \mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta']$. There is a tractor formula for the quantization $Q^{\sigma'}_{k',0} : \mathcal{E}[w] \to \mathcal{E}[w+\delta']$ with the leading term $(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}$ for every weight $\delta' \in \mathbb{R}$ satisfying*

$$\delta' \notin \Sigma_{k',0} \quad \text{where } \Sigma_{k',0} = \begin{cases} \{-(n+k'+i-2) \mid i = 1,\dots,k'\} & k' \geqslant 1, \\ \emptyset & k' = 0. \end{cases} \tag{15}$$

Moreover, $Q_{k',0}$ is strongly conformally invariant in the following sense: if we replace $f \in \mathcal{E}[w]$ by $f \in \mathcal{T} \otimes \mathcal{E}[w]$ for any tractor bundle $\mathcal{T}$ and, in the formula for $Q_{k',0}$, we replace the Levi-Civita connection acting on f by the coupled Levi-Civita–tractor connection then $Q_{k',0}$ is a conformally invariant bilinear operator $\mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta'] \times (\mathcal{T} \otimes \mathcal{E}[w]) \to \mathcal{T} \otimes \mathcal{E}[w+\delta']$.

Proof. We shall use certain splitting operators from $\mathcal{E}[w]$ and $\mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta']$ into symmetric tensor products of the adjoint tractor bundle $\mathcal{E}_{\mathbf{A}}$ and their subquotients where we use the form index notation $\mathbf{A} = [A^0A^1]$, see Section 2.1. These are just abstract indices of the adjoint tractor bundle. We shall use the notation $f_{(\mathbf{AB})} = \frac{1}{2}(f_{\mathbf{AB}} + f_{\mathbf{BA}})$, $f_{\mathbf{AB}} \in \mathcal{E}_{\mathbf{AB}}$ for the symmetrization, the symmetric tensor products of the adjoint tractor bundle will be denoted by $\mathcal{E}_{(\mathbf{A}_1\cdots\mathbf{A}_k)}$. This notation means symmetrization over adjoint indices (and *not* over standard tractor indices), i.e. generally $f_{(\mathbf{AB})} \neq 0$. The completely trace-free component with respect to h^{AB} will be denoted by $\mathcal{E}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$. Note the latter bundles are generally not irreducible tractor bundles.

The skew symmetrization with the tractor $X_{A_i^0}$ defines bundle maps $\mathcal{E}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0} \to \mathcal{E}_{\mathbf{A}_1\cdots[\mathbf{A}_i^0\mathbf{A}_i]\cdots\mathbf{A}_k}$. The joint kernel of all these maps for $i = 1,\dots,k$ will be denoted by $\bar{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$. Using the complement $\bar{\mathcal{E}}^{\perp}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0} \subseteq \mathcal{E}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$ (via the tractor metric h), we obtain the quotient bundle $\tilde{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0} := \mathcal{E}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}/\bar{\mathcal{E}}^{\perp}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$. That is, we have the duality $(\tilde{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0})^* \cong \bar{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$. One can easily see that choosing a metric from the conformal class, sections of these have the form

$$\bar{F}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0} = \sum_{i=0}^{k} \mathbb{X}^{a_1}_{\mathbf{A}_1}\cdots\mathbb{X}^{a_i}_{\mathbf{A}_i}\mathbb{W}_{\mathbf{A}_{i+1}}\cdots\mathbb{W}_{\mathbf{A}_k}\bar{f}^i_{(a_1\cdots a_i)_0} \quad \text{for } \bar{F}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0} \in \bar{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0},$$

$$\tilde{F}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0} = \sum_{i=0}^{k} \mathbb{Y}^{a_1}_{\mathbf{A}_1}\cdots\mathbb{Y}^{a_i}_{\mathbf{A}_i}\mathbb{W}_{\mathbf{A}_{i+1}}\cdots\mathbb{W}_{\mathbf{A}_k}\tilde{f}^i_{(a_1\cdots a_i)_0} \quad \text{for } \tilde{F}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0} \in \tilde{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$$

for some sections $\bar{f}^i_{(a_1\cdots a_i)_0}$ and $\tilde{f}^i_{(a_1\cdots a_i)_0}$. Note i is *not* an abstract index here. This describes the composition series for $\bar{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$ and $\tilde{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$. (In particular, choosing a metric in the conformal class, both these bundles decompose to exactly $k+1$ irreducible components, e.g. $\bar{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0} = \mathcal{E} \oplus \mathcal{E}_{a_1} \oplus \cdots \oplus \mathcal{E}_{(a_1\cdots a_k)_0}$.) Finally, taking the tensor product with density bundles, we obtain $\bar{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}[w]$ and $\tilde{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}[w]$ for any $w \in \mathbb{R}$.

Further we shall need the *curved Casimir operator* $\mathcal{C}$ introduced in [10]. We refer for the definition of $\mathcal{C}$ to [10]; here we recall crucial properties of $\mathcal{C}$ in special cases. (Note also that, more explicitly, one has $\mathcal{C} = h^{AB}\mathcal{D}^2_{AB}$ where the operator $\mathcal{D}^2_{AB}$ is given by [21, (21)].) Consider a subquotient $\mathcal{T}_k$ of $\mathcal{E}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}[w]$ with the filtration $\{0\} = \mathcal{T}_{-1} \subseteq \mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \cdots \subseteq \mathcal{T}_k$ such that $\mathcal{T}_\ell/\mathcal{T}_{\ell-1}$ are irreducible for $\ell = 0,\dots,k$. This in particular covers cases of our interest, i.e. $\mathcal{T}_k = \tilde{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}[w]$ and $\mathcal{T}_k = \bar{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}[w]$. Then $\mathcal{C} : \mathcal{T}_k \to \mathcal{T}_k$ has the following properties: it is conformally invariant and of the first order, it preserves subquotients and the induced operator on $\mathcal{T}_\ell/\mathcal{T}_{\ell-1}$ is multiplication by a scalar which is polynomial in w. We shall denote the latter scalar by α_ℓ. These properties guarantee that the operator

$$\mathcal{T}_k/\mathcal{T}_\ell \to \mathcal{T}_k/\mathcal{T}_{\ell-1} \xrightarrow{\mathcal{C}-\alpha_{\ell-1}} \mathcal{T}_k/\mathcal{T}_{\ell-1},$$

where the first arrow is an arbitrary (not necessarily conformally invariant) extension, is overall conformally invariant for $\ell = 1, \dots, k$. Iterating this k times we obtain the conformally invariant extension $P(\mathcal{C}) := (\mathcal{C} - \alpha_{k-1}) \circ \dots \circ (\mathcal{C} - \alpha_0) : \mathcal{T}_k/\mathcal{T}_{k-1} \to \mathcal{T}_k$. Moreover, $P(\mathcal{C})$ restricted to $\mathcal{T}_k/\mathcal{T}_{k-1}$ acts by multiplication by $(\alpha_k - \alpha_{k-1}) \circ \dots \circ (\alpha_k - \alpha_0) \in \mathbb{R}$. This is a polynomial in w which we denote by $p(w)$. In particular, degrees of both polynomials P and p are k. (Note that although $\alpha_k, \dots, \alpha_0$ are quadratic in w, differences $\alpha_k - \alpha_{\ell-1}$ are affine in w).

Henceforth assume $k' \geqslant 1$ and put $k := k'$. We shall start with the splitting $\mathcal{E}[w] \to \bar{\mathcal{E}}_{(\mathbf{A}_1 \cdots \mathbf{A}_{k'})_0}[w] =: \mathcal{T}_{k'}$. Corresponding subquotients are $\mathcal{T}_\ell/\mathcal{T}_{\ell-1} \cong \mathcal{E}_{(a_1 \cdots a_{k'-\ell})_0}[w]$, $\ell = 0, \dots, k'-1$ and $\mathcal{T}_{k'}/\mathcal{T}_{k'-1} \cong \mathcal{E}[w]$. Assume $f \in \mathcal{E}[w]$. For an arbitrarily chosen metric from the conformal class, we consider the inclusion

$$\bar{\iota} : \mathcal{E}[w] \hookrightarrow \bar{\mathcal{E}}_{(\mathbf{A}_1 \cdots \mathbf{A}_{k'})_0}[w], \qquad f \overset{\bar{\iota}}{\mapsto} \mathbb{W}_{\mathbf{A}_1} \cdots \mathbb{W}_{\mathbf{A}_{k'}} f.$$

Note the induced map to the subquotient $\mathcal{E}[w] \overset{\bar{\iota}}{\hookrightarrow} \mathcal{T}_{k'} \twoheadrightarrow \mathcal{T}_{k'}/\mathcal{T}_{k'-1}$ is just the isomorphism mentioned above. Now $\bar{\iota}(f)$ can be extended to a conformally invariant section $\bar{F}_{\mathbf{A}_1 \cdots \mathbf{A}_{k'}} := \bar{P}(\mathcal{C})(\bar{\iota}(f)_{A_1 \cdots A_{k'}})$ for a suitable polynomial $\bar{P}(\mathcal{C})$ in $\mathcal{C}$ such that the degree of $\bar{P}$ is k'. Let us compute the highest order term of $\bar{P}(\mathcal{C})(\mathbb{W}_{\mathbf{A}_1} \cdots \mathbb{W}_{\mathbf{A}_{k'}} f)$. For this it is sufficient to work on $\mathbb{R}^{p,q}$ with the standard metric. Then if P' is a suitable polynomial of degree r, $0 \leqslant r \leqslant k'$ such that $P'(\mathcal{C})(\mathbb{W}_{\mathbf{A}_1} \cdots \mathbb{W}_{\mathbf{A}_{k'}} f)$ is conformally invariant as in the construction above, then there is a (degree r) polynomial p' such that $P'(\mathcal{C})(\mathbb{W}_{\mathbf{A}_1} \cdots \mathbb{W}_{\mathbf{A}_{k'}} f) = \mathbb{W}_{\mathbf{A}_1} \cdots \mathbb{W}_{\mathbf{A}_{k'}} p'(w) f + \dots + \mathbb{X}^{a_1}_{(\mathbf{A}_1} \cdots \mathbb{X}^{a_r}_{\mathbf{A}_r} \mathbb{W}_{\mathbf{A}_{r+1}} \cdots \mathbb{W}_{\mathbf{A}_{k'})} \nabla_{(a_1} \cdots \nabla_{a_r)_0} f$ up to a (nonzero) scalar multiple. This can be easily verified by the induction using the fact that $\mathcal{C}$ is of the first order. Putting $r := k'$, there is a k'-order polynomial $\bar{p}(w)$ such that

$$\bar{F}_{\mathbf{A}_1 \cdots \mathbf{A}_{k'}} = \mathbb{W}_{\mathbf{A}_1} \cdots \mathbb{W}_{\mathbf{A}_{k'}} \bar{p}(w) f + \dots + \mathbb{X}^{a_1}_{(\mathbf{A}_1} \cdots \mathbb{X}^{a_{k'}}_{\mathbf{A}_{k'})} \nabla_{(a_1} \cdots \nabla_{a_{k'})_0} f \tag{16}$$

up to a nonzero scalar multiple.

The splitting $\mathcal{E}^{(a_1 \cdots a_{k'})_0}[\delta'] \to \tilde{\mathcal{E}}_{(\mathbf{A}_1 \cdots \mathbf{A}_{k'})_0}[\delta'] =: \mathcal{T}_{k'}$ is analogous. Corresponding subquotients are now $\mathcal{T}_\ell/\mathcal{T}_{\ell-1} \cong \mathcal{E}^{(a_1 \cdots a_\ell)_0}[\delta']$, $\ell = 1, \dots, k'$ and $\mathcal{T}_0/\mathcal{T}_{-1} \cong \mathcal{E}[\delta']$. Assume $(\sigma')^{a_1 \cdots a_{k'}} \in \mathcal{E}^{(a_1 \cdots a_{k'})_0}[\delta']$. As above, we shall start with the inclusion

$$\tilde{\iota} : \mathcal{E}^{(a_1 \cdots a_{k'})_0}[\delta'] \hookrightarrow \tilde{\mathcal{E}}_{(\mathbf{A}_1 \cdots \mathbf{A}_{k'})_0}[\delta'], \qquad (\sigma')^{a_1 \cdots a_{k'}} \overset{\tilde{\iota}}{\mapsto} \mathbb{Y}^{a_1}_{\mathbf{A}_1} \cdots \mathbb{Y}^{a_{k'}}_{\mathbf{A}_{k'}} (\sigma')_{a_1 \cdots a_{k'}}$$

for a chosen metric in the conformal class. Then we apply a suitable polynomial operator in the curved Casimir to obtain a conformally invariant extension $\tilde{F}_{\mathbf{A}_1 \cdots \mathbf{A}_{k'}} := \tilde{P}(\mathcal{C})(\tilde{\iota}(\sigma')_{\mathbf{A}_1 \cdots \mathbf{A}_{k'}}) \in \tilde{\mathcal{E}}_{(\mathbf{A}_1 \cdots \mathbf{A}_{k'})_0}[\delta']$. A similar reasoning as above shows that $\tilde{P}$ has order k' and

$$\tilde{F}_{\mathbf{A}_1 \cdots \mathbf{A}_{k'}} = \mathbb{Y}_{\mathbf{A}_1 a_1} \cdots \mathbb{Y}_{\mathbf{A}_{k'} a_{k'}} \tilde{p}(\delta') (\sigma')^{a_1 \cdots a_{k'}} + \dots + \mathbb{W}_{(\mathbf{A}_1} \cdots \mathbb{W}_{\mathbf{A}_{k'})} \nabla_{(a_1} \cdots \nabla_{a_{k'})_0} (\sigma')^{a_1 \cdots a_{k'}} \tag{17}$$

on $\mathbb{R}^{p,q}$ for a polynomial $\tilde{p}$ of the order k'. In this case we need to know $\tilde{p}(\delta')$ explicitly; following [10] one computes

$$\tilde{p}(\delta') = \prod_{i=1}^{k'} (\delta' + n + k' + i - 2)$$

which is non-vanishing by the hypothesis, see (15). In fact, analogues of this splitting are well-known, see e.g. [23, 6.2.3] or [29, 2.1.4].

In the last step we use the duality between $\bar{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$ and $\tilde{\mathcal{E}}_{(\mathbf{A}_1\cdots\mathbf{A}_k)_0}$. From this it follows that $Q^{\sigma'}_{k',0}(f) := \frac{1}{\tilde{p}(\delta')}\tilde{F}^{\mathbf{A}_1\cdots\mathbf{A}_{k'}}\bar{F}_{\mathbf{A}_1\cdots\mathbf{A}_{k'}}$ is a conformally invariant bilinear operator. Considering $Q^{\sigma'}_{k',0}$ as a linear operator $\mathcal{E}[w]\to\mathcal{E}[w+\delta']$, it follows from (16) and (17) that

$$\tilde{F}^{\mathbf{A}_1\cdots\mathbf{A}_{k'}}\bar{F}_{\mathbf{A}_1\cdots\mathbf{A}_{k'}} = \tilde{p}\big(\delta'\big)\big(\sigma'\big)^{a_1\cdots a_{k'}}\nabla_{(a_1}\cdots\nabla_{a_{k'})_0}f + lot$$

where "*lot*" denotes the lower order terms.

It remains to verify the strong invariance of $Q^{\sigma'}_{k',0}(f)$. But this follows from the fact that the curved Casimir is a strongly invariant linear differential operator. □

3.2. The general case $Q_{k',\ell}$

Recall $k',\ell\geqslant 0$, $(\sigma')^{a_1\cdots a_{k'}}\in\mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta']$ and $f\in\mathcal{E}[w]$, $\delta',w\in\mathbb{R}$. We shall construct $Q_{k',\ell}$ by an inductive procedure. The main step is the construction of $Q^{\sigma'}_{k',\ell+1}$ from $Q^{\sigma'}_{k',\ell}$. This construction (in the proposition below) is a specific implementation of the idea known as *curved translation principle* (see [27,16] for its origins) which can be formulated as follows. Consider $f\in\mathcal{E}[w]$. Then first extend f invariantly into a tractor field (in our case D_Bf), then apply the known invariant operator (in our case $Q^{\sigma'}_{k',\ell}$) and finally go back to densities (by applying D^B in our case) to obtain the new operator (in our case $Q^{\sigma'}_{k',\ell+1}$). Overall this yields the following construction:

Proposition 3.2. *Fix $\delta'\in\mathbb{R}$ and assume there is an explicit construction of the quantization $Q^{\sigma'}_{k',\ell}:\mathcal{E}[w]\to\mathcal{E}[w+\delta'-2\ell]$, $k',\ell\geqslant 0$ with the leading term $\sigma^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell$ for every $w\in\mathbb{R}$. Also assume $Q_{k',\ell}$ is strongly invariant in the sense of Theorem 3.1. Then the operator*

$$\tilde{Q}^{\sigma'}_{k',\ell+1} := D^BQ^{\sigma'}_{k',\ell}D_B:\mathcal{E}[w]\to\mathcal{E}\big[w+\delta'-2(\ell+1)\big]$$

has the form

$$\tilde{Q}^{\sigma'}_{k',\ell+1}(f) = -\big(\delta'-\ell\big)\big(n+2\delta'+2\big(k'-\ell\big)-2\big)\sigma^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}f + lot$$

for every $w\in\mathbb{R}$. Here "lot" denotes lower order terms.

The operator $\tilde{Q}_{k',\ell+1}:\mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta']\times\mathcal{E}[w]\to\mathcal{E}[w+\delta'-2(\ell+1)]$ is a conformally invariant bilinear operator. Moreover, it is strongly invariant in the sense of Theorem 3.1. When well-defined, we put $Q^{\sigma'}_{k',\ell+1} := -\frac{1}{(\delta'-\ell)(n+2\delta'+2(k'-\ell)-2)}\tilde{Q}^{\sigma'}_{k',\ell+1}$.

Proof. We shall start with the discussion on the invariance. Since $Q_{k',\ell}:\mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta']\times\mathcal{E}[w]\to\mathcal{E}[w+\delta'-2\ell]$ is assumed to be strongly invariant (in the sense of Theorem 3.1), it is also invariant as $Q_{k',l}:\mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta']\times\mathcal{E}_B[w]\to\mathcal{E}_B[w+\delta'-2\ell]$. Therefore the composition

$$\mathcal{E}^{(a_1\cdots a_{k'})_0}\big[\delta'\big]\times\mathcal{E}[w]\xrightarrow{\mathrm{id}\times D_B}\mathcal{E}^{(a_1\cdots a_{k'})_0}\big[\delta'\big]\times\mathcal{E}_B[w-1]\xrightarrow{Q_{k',\ell}}\mathcal{E}_B\big[(w-1)+\delta'-2\ell\big]\xrightarrow{D^B}\mathcal{E}\big[w+\delta'-2\ell-2\big]$$

is a conformally invariant bilinear operator. The strong invariance of $\tilde{Q}_{k',l+1}$ follows from the strong invariance of $\mathrm{id}\times D^B$, $\tilde{Q}_{k',l}$ and D^B.

It remains to compute the symbol of $\tilde{Q}^{\sigma'}_{k',\ell+1}$, we shall do it by a direct computation. The tractor D-operator is explicitly given by the sum of three terms on the right hand side of (10). Decomposing both D_B and D^B in the formula for $\tilde{Q}^{\sigma'}_{k',l+1}$ accordingly, we obtain overall 9 leading terms. Note $Q^{\sigma'}_{k',\ell}D_Bf = [(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}} + lot]\Delta^\ell D_Bf\in\mathcal{E}_B[w']$ where $f\in\mathcal{E}[w]$ and $w' = w+\delta'-2\ell-1$.

Since the tractor D is of the second order and $Q^{\sigma'}_{k',\ell}$ is of the order $k'+2\ell$, the leading term of $\tilde{Q}^{\sigma'}_{k',\ell+1}$ has the order at most $k'+2\ell+4$. In fact, it turns out this order is $k'+2\ell+2$ in the generic case (or less for certain values of δ'). To show this we will collect all terms of the order at least $k'+2\ell+2$. In fact, we shall do this in details only for the leading term $(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell$ of $Q^{\sigma'}_{k',\ell}$. But it will be obvious from the form of all 9 summands this is sufficient. Below we shall use $lot_{\leqslant o}$ to denote terms of the order at most o and $lot_{<o}$ will denotes terms of order smaller than o. To simplify the notation we will henceforth work with the Euclidean metric. (Terms involving the curvature have necessarily lower degree than $2\ell+k'$.) Then all terms on the right hand side of (12) and (13) denoted by "ct" vanish.

I. We shall start with $w'(n+2w'-2)Y^B Q^{\sigma'}_{k',\ell} D_B f$; decomposing D_B here according to (10) yields first three summands.

I.a The first one is

$$\begin{aligned} &w'\big(n+2w'-2\big)Y^B Q^{\sigma'}_{k',\ell}\big[w(n+2w-2)Y_B f\big] \\ &\quad = w'\big(n+2w'-2\big)w(n+2w-2)Y^B\big(\sigma'\big)^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell Y_B f = 0. \end{aligned} \tag{18}$$

The reason is that the tractor Y^B contracts nontrivially only with X_B according to (8) and if we compute $\nabla_{(a_1}\cdots\nabla_{a_{k'})_0}\Delta^\ell Y_B f$ according to (12) and (13), X_B always involves curvature.

I.b Analogously we obtain

$$w'\big(n+2w'-2\big)Y^B Q^{\sigma'}_{k',\ell}\big[(n+2w-2)Z^b_B\nabla_b f\big] = 0. \tag{19}$$

I.c Looking at the X_B-terms of $Q^{\sigma'}_{k',\ell}(-X_B\Delta f)$, we see from (12) and (13) that

$$\begin{aligned} &w'\big(n+2w'-2\big)Y^B Q^{\sigma'}_{k',\ell}[-X_B\Delta f] \\ &\quad = -w'\big(n+2w'-2\big)\big(\sigma'\big)^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1} f + lot_{\leqslant k'+2\ell+1}. \end{aligned} \tag{20}$$

II. Next we shall compute $(n+2w'-2)Z^{Bb}\nabla_b Q^{\sigma'}_{k',\ell} D_B f$, we obtain again three summands. They are produced by contraction of $(n+2w'-2)Z^b_B$ with

$$\nabla_b Q^{\sigma'}_{k',\ell} D_B f = \big[\big(\nabla_b\big(\sigma'\big)^{a_1\cdots a_{k'}}\big)\nabla_{a_1}\cdots\nabla_{a_{k'}} + \big(\sigma'\big)^{a_1\cdots a_{k'}}\nabla_b\nabla_{a_1}\cdots\nabla_{a_{k'}} + lot_{<k'}\big]\Delta^\ell D_B f.$$

We need to discuss here only the first two terms in the square bracket and only $Z^{\bar{b}}_B$-terms according to (8). First, it is easy to see that

$$\big(n+2w'-2\big)Z^{Bb}\big(\nabla_b\big(\sigma'\big)^{a_1\cdots a_{k'}}\big)\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell D_B f = lot_{\leqslant k'+2\ell+1}.$$

(The component $w(n+2w-2)Y_B$ of D_B does not contribute to the right hand side of the previous display at all and the remaining components $(n+2w-2)Z^{\bar{b}}_B\nabla_{\bar{b}}$ and $-X_B\Delta$ contribute by terms of the order $\leqslant k'+2\ell+1$.) Hence it remains to collect $Z^{\bar{b}}_B$-terms of $(\sigma')^{a_1\cdots a_{k'}}\nabla_b\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell D_B f$. Using the form (10) of D_B, we obtain three more summands which are analogues of I.a, I.b and I.c above. A short computation reveals that these are

$$\big(n+2w'-2\big)Z^{Bb}\big(\sigma'\big)^{a_1\cdots a_{k'}}\nabla_b\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell\big[w(n+2w-2)Y_B f\big] = 0, \tag{21}$$

$$\begin{aligned} &\big(n+2w'-2\big)Z^{Bb}\big(\sigma'\big)^{a_1\cdots a_{k'}}\nabla_b\nabla_{a_{1'}}\cdots\nabla_{a_{k'}}\Delta^\ell\big[(n+2w-2)Z^{\bar{b}}_B\nabla_{\bar{b}} f\big] \\ &\quad = \big(n+2w'-2\big)(n+2w-2)Z^{Bb}\big(\sigma'\big)^{a_1\cdots a_{k'}}\nabla_b Z^{\bar{b}}_B\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell\nabla_{\bar{b}} f \\ &\quad = \big(n+2w'-2\big)(n+2w-2)\sigma^{a_1\cdots a_{k'}}\Delta^{\ell+1} f, \end{aligned} \tag{22}$$

$$\begin{aligned}
&(n+2w'-2)Z^{Bb}\sigma^{a_1\cdots a_{k'}}\nabla_b\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell[-X_B\Delta f]\\
&\quad=-(n+2w'-2)Z^{Bb}\sigma^{a_1\cdots a_{k'}}\nabla_b\nabla_{a_1}\cdots\nabla_{a_{k'}}\big[2\ell Z_B^{\bar b}\nabla_{\bar b}\Delta^\ell+X_B\Delta^{\ell+1}\big]f\\
&\quad=-(n+2w'-2)Z^{Bb}(\sigma')^{a_1\cdots a_{k'}}\nabla_b\big[X_B\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}\\
&\qquad+Z_B^{\bar b}\big(2\ell\nabla_{a_1}\cdots\nabla_{a_{k'}}\nabla_{\bar b}\Delta^\ell+k'\boldsymbol{g}_{\bar b a_1}\nabla_{a_2}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}\big)\big]f\\
&\quad=-(n+2w'-2)(2\ell+k'+n)(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}f. \qquad (23)
\end{aligned}$$

Beside the fact that Z^{Bb} contracts nontrivially only with $Z_B^{\bar b}$, we have used (13) to commute Δ^ℓ with $Z_B^{\bar b}$ and X_B, (12) to commute $\nabla_{a_1}\cdots\nabla_{a_{k'}}$ with $Z_B^{\bar b}$ and X_B and (11) to commute ∇_b with $Z_B^{\bar b}$ and X_B.

III. It remains to compute $-X^B\Delta Q^{\sigma'}_{k',\ell}D_Bf$. The computation is analogous to previous cases but getting more tedious. First we observe

$$\begin{aligned}
-X^B\Delta Q^{\sigma'}_{k',\ell}D_Bf=-X^B\big[&\big(\Delta(\sigma')^{a_1\cdots a_{k'}}\big)\nabla_{a_1}\cdots\nabla_{a_{k'}}+2\big(\nabla^p(\sigma')^{a_1\cdots a_{k'}}\big)\nabla_p\nabla_{a_1}\cdots\nabla_{a_{k'}}\\
&+(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta+lot_{\leqslant k'-1}\big]\Delta^\ell D_Bf.
\end{aligned}$$

We shall discuss only the first three terms in the above square bracket. One can compute that

$$-X^B\big[\big(\Delta(\sigma')^{a_1\cdots a_{k'}}\big)\nabla_{a_1}\cdots\nabla_{a_{k'}}+2\big(\nabla^p(\sigma')^{a_1\cdots a_{k'}}\big)\nabla_p\nabla_{a_1}\cdots\nabla_{a_{k'}}\big]\Delta^\ell D_Bf=lot_{\leqslant k'+2\ell+1}$$

so it remains to compute only $-X^B(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}D_Bf$. This yields three summands (analogues of I.a, I.b and I.c above) according to (10). After some computation we obtain

$$\begin{aligned}
&-X^B(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}\big[w(n+2w-2)Y_Bf\big]\\
&\quad=-w(n+2w-2)(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}f, \qquad (24)\\
&-X^B(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}\big[(n+2w-2)Z_B^{\bar b}\nabla_{\bar b}f\big]\\
&\quad=-(n+2w-2)X^B(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\big[-2(\ell+1)Y^B\nabla^{\bar b}\Delta^\ell\nabla_{\bar b}+Z_B^{\bar b}\nabla_{\bar b}\Delta^{\ell+1}\big]f\\
&\quad=-(n+2w-2)\big[-2(\ell+1)-k'\big](\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}f, \qquad (25)\\
&-X^B(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}[-X_B\Delta f]\\
&\quad=X^B\sigma^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\big[-(\ell+1)(n+2\ell)Y_B\Delta^{\ell+1}+2(\ell+1)Z_B^{\bar b}\nabla_{\bar b}\Delta^{\ell+1}+X_B\Delta^{\ell+2}\big]f\\
&\quad=\big[-(\ell+1)(n+2\ell)-2k'(\ell+1)\big](\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}f. \qquad (26)
\end{aligned}$$

The last step of the proof is to sum up the right hand sides of 9 relations (18), (19), (20), (21), (22), (23) and (24), (25), (26) above. That is, we need to compute the scalar

$$\begin{aligned}
&-w'(n+2w'-2)+(n+2w'-2)(n+2w-2)-(n+2w'-2)(2\ell+k'+1),\\
&-w(n+2w-2)+(n+2w-2)(2\ell+k'+2)-(\ell+1)(n+2\ell+2k')
\end{aligned}$$

where $w'=w+\delta'-2\ell-1$. This requires some work, the result is $-(\delta'-\ell)(n+2\delta'+2k'-2\ell-2)$ and the proposition follows. Note the resulting scalar does not depend on w; this is a good verification that computations above are correct. □

Theorem 3.3. *Let $k',\ell\geqslant 0$, $(\sigma')^{a_1\cdots a_{k'}}\in\mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta']$ and $f\in\mathcal{E}[w]$, $\delta',w\in\mathbb{R}$. Then*

$$Q^{\sigma'}_{k',\ell}:=\frac{1}{\beta}D^{B_1}\cdots D^{B_{k'}}Q^{\sigma'}_{k',0}D_{B_{k'}}\cdots D_{B_1}:\mathcal{E}[w]\to\mathcal{E}\big[w+\delta'-2\ell\big]$$

defines, for a nonzero scalar β, *the conformally invariant quantization with the leading term* $(\sigma')^{a_1\cdots a_{k'}} \times \nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell$ *(up to a sign) for every weight* δ' *satisfying*

$$\delta' \notin \Sigma_{k',\ell} := \Sigma_{k',0} \cup \Sigma'_{k',\ell} \cup \Sigma''_{k',\ell} \tag{27}$$

where $\Sigma_{k',0}$ *is given by* (15),

$$\Sigma'_{k',\ell} = \{(j-1) \mid j = 1,\ldots,\ell\}, \qquad \Sigma''_{k',\ell} = \left\{-\frac{1}{2}\big(n+2k'-2j\big) \;\middle|\; j=1,\ldots,\ell\right\} \quad \textit{for } \ell \geqslant 1. \tag{28}$$

We put $\Sigma'_{k',0} = \Sigma''_{k',0} := \emptyset$. *Moreover,* $Q^{\sigma'}_{k',\ell}$ *is strongly invariant in the sense of Theorem* 3.1.

Proof. The set of critical weights $\Sigma_{k',\ell}$ (and the choice of β) easily follows (by induction with respect to ℓ) from Proposition 3.2. Since the tractor D and $Q^{\sigma'}_{k',0}$ are strongly invariant, the last claim is obvious. □

4. Remarks on critical weights

We shall discuss the cases $\delta' \in \Sigma_{k',\ell}$ from (27) in detail. First, a simple calculation shows

Lemma 4.1.

(i) $2\ell \notin \Sigma_{k',\ell}$ *for all* $k', \ell \geqslant 0$.
(ii) *The sets* $\Sigma_{k',0}$ *and* $\Sigma'_{k',\ell} \cup \Sigma''_{k',\ell}$ *are disjoint.* □

The symbols of the quantization $\mathcal{E}[w] \to \mathcal{E}[w]$ (i.e. with zero shift δ) are of a special interest [13]. The flat quantization developed in [13] is never critical for such symbols [13, 3.1], the construction in [11] extends that to the curved setting. The previous lemma (i) recovers this fact for the quantization $Q^{\sigma'}_{k',\ell}$.

The critical weights are closely related to existence of natural linear conformal operators. They are completely classified in the locally flat case [2,3]. Existence of such operators on symbols is related to critical weights δ as follows:

Theorem 4.2. *(See [25].) Consider the quantization* $Q^{\sigma'}_{k',\ell}$ *for symbols*

$$(\sigma')^{a_1\cdots a_k} \in \mathcal{E}^{(a_1\cdots a_{k'})_0}\big[\delta'\big] \subseteq \mathcal{E}^{(a_1\cdots a_{k'}a_{k'+1}\cdots a_{k'+2\ell})_0}\big[\delta'-2\ell\big].$$

If there is a nontrivial linear invariant operator

$$\mathcal{E}^{(a_1\cdots a_{k'})_0}\big[\delta'\big] \to \mathcal{E}^{(a_1\cdots a_p)}\big[\delta'-2\ell\big], \quad 0 \leqslant p \leqslant k'+2\ell-1$$

then the invariant quantization $Q^{\sigma'}_{k',\ell}$ *does not exist.*

Given k' and ℓ, the set of critical weights $\delta' \in \Sigma_{k',\ell}$ corresponds exactly to those δ' for which there is a linear invariant operator as in the theorem. In more detail:

Proposition 4.3. *Assume the manifold* M *is conformally flat. If* $\delta' \in \Sigma_{k',\ell}$ *then there exists a nontrivial natural linear conformal operator on* $\mathcal{E}^{(a_1\cdots a_{k'})_0}[\delta']$ *as follows*

$$\begin{aligned}
&\mathcal{E}^{(a_1\cdots a_{k'})_0}\big[\delta'\big] \to \mathcal{E}^{(a_1\cdots a_{i-1})_0}\big[\delta'\big], \quad \delta' = -\big(n+k'+i-2\big) \in \Sigma_{k',0},\\
&\mathcal{E}^{(a_1\cdots a_{k'})_0}\big[\delta'\big] \to \mathcal{E}^{(a_1\cdots a_{k'+j})_0}\big[\delta'-2j\big], \quad \delta' = j-1 \in \Sigma'_{k',\ell},\\
&\mathcal{E}^{(a_1\cdots a_{k'})_0}\big[\delta'\big] \to \mathcal{E}^{(a_1\cdots a_{k'})_0}\big[\delta'-2j\big], \quad \delta' = -\frac{1}{2}\big(n+2k'-2j\big) \in \Sigma''_{k',\ell}. \qquad □
\end{aligned}$$

Proof. The classification of invariant linear operators between weighted symmetric powers of the tangent bundle (which appear in Theorem 4.2) is easy using the classification [2,3] as presented in [15, Introduction]. From this one easily concludes, that every invariant operator between symmetric weighted tensors is (using the terminology from [15, Introduction]) either the first BGG operator or the last BGG operator or one of singular/nonstandard operators.

The detailed analysis is performed e.g. in [29]. From this one can see that the case $\delta' \in \Sigma_{k',0}$ corresponds to the last BGG operator, the case $\delta' \in \Sigma'_{k',\ell}$ corresponds to the first BGG operator and the case $\delta' \in \Sigma''_{k',\ell}$ to one of singular/nonstandard operators. $\square$

Note the terminology of the proof corresponds to the following: the case $\delta' \in \Sigma_{k',0}$ is a divergence type operator of the order $k' - i + 1$, $\delta' \in \Sigma'_{k',\ell}$ is the generalized conformal Killing operator of the order j and $\delta' \in \Sigma''_{k',\ell}$ yields a power of Laplacian type operator of the order $2j$. Note the operator in the proposition is not unique as generally $\Sigma'_{k',\ell} \cap \Sigma''_{k',\ell} \neq \emptyset$.

We have seen that the set $\Sigma_{k',\ell}$ from Theorem 3.3 agrees exactly with nonexistence results in [25]. That is, Theorem 3.3 together with [25] provides complete classification of critical weights for conformally invariant quantization on densities.

Also note that invariant operators $\mathcal{E}^{(a_1 \cdots a_{k'})_0}[\delta'] \to \mathcal{E}^{(a_1 \cdots a_{k'})_0}[\delta' - 2j]$ from Proposition 4.3 do exist in the conformally flat case but not necessarily in the curved case [19], i.e. this operator might not have a curved analogue. That is, Theorem 4.2 would not provide the full set of critical weights in curved cases.

Now assume $\delta' \in \Sigma_{k',\ell}$ is critical. Then $Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2l]$ cannot exist for all $w \in \mathbb{R}$. Such a quantization can exist, though, for certain w called *resonant weights*. Their classification and relation to linear invariant operators on $\mathcal{E}[w]$ is discussed in [25] in the flat case. It is unclear whether there is a curved analogue of $Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2l]$ for a resonant weight w. Here we show that such curved analogue exists at least for some resonant weights (see the theorem below).

First we recall that, assuming δ' is critical, resonant weights of $Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w + \delta' - 2l]$ are related to existence of following operators on $\mathcal{E}[w]$:

$$
\begin{aligned}
&S_p : \mathcal{E}[p-1] \to \mathcal{E}_{(a_1 \cdots a_p)_0}[p-1], && L_p : \mathcal{E}[-n/2 + p] \to \mathcal{E}[-n/2 - p],\\
&S_p(f) = \nabla_{(a_1} \cdots \nabla_{a_p)_0} f + lot, && L_p(f) = \Delta^p f + lot,
\end{aligned}
$$

for $p \geqslant 1$ (so p is *not* an abstract index here) and *lot* stands for "lower order terms". If n is odd or M is conformally flat, these operators exist for all $p \geqslant 1$. In the curved case for n even, S_p exists for all $p \geqslant 1$ and L_p exists for $1 \leqslant p \leqslant n$, see [9,22,19]. They are strongly invariant (can be given by a strongly invariant formula) in the flat case; in the curved case, S_p is strongly invariant always and L_p only for $p < n$ [18].

Theorem 4.4. *Let $\delta' \in \Sigma_{k',\ell}$ and $f \in \mathcal{E}[w]$ and assume the conformally flat setting. Then there is always a choice of $w \in \mathbb{R}$ for which there is a quantization $Q^{\sigma'}_{k',\ell} : \mathcal{E}[w] \to \mathcal{E}[w + \delta]$ with the leading term $(\sigma')^{a_1 \cdots a_{k'}} \nabla_{a_1} \cdots \nabla_{a_{k'}} \Delta^\ell f$ in the flat case. Explicitly, the quantization is given (up to possibly a nonzero scalar) by formulae*

$$
\begin{aligned}
&Q^{\sigma'}_{k',0} L_\ell : \mathcal{E}[-n/2 + \ell] \to \mathcal{E}\big[\delta' - n/2 - \ell\big], \quad \delta' \in \Sigma'_{k',\ell} \cup \Sigma''_{k',\ell},\\
&D^{B_1} \cdots D^{B_\ell} \iota\big(\sigma'\big) S_{k'} D_{B_1} \cdots D_{B_\ell} : \mathcal{E}\big[k' + \ell - 1\big] \to \mathcal{E}\big[\delta' + k' - \ell - 1\big], \quad \delta' \in \Sigma_{k',0}
\end{aligned}
$$

where $\iota(\sigma')$ is the complete contraction of the image of S_p with σ'.

In the curved case, the statement is true for n odd or $\delta' \in \Sigma_{k',0}$. Assuming further $\ell \leqslant n$ in the curved case, the statement is true also for n even and $\delta' \in \Sigma'_{k',\ell} \cup \Sigma''_{k',\ell}$.

Proof. The conformal invariance is obvious (recall $S_{k'}$ has the source space $\mathcal{E}[k'-1]$ and is strongly invariant). It remains to verify the displayed operators have the required leading term (up to a nonzero multiple). In the case $\delta' \in \Sigma'_{k',\ell} \cup \Sigma''_{k',\ell}$, this follows from the leading term of L_ℓ, properties of $Q^{\sigma'}_{k',0}$ in Theorem 3.3 and Lemma 4.1(ii).

Assume $\delta' \in \Sigma_{k',0}$ and denote by $\bar{Q}^{\sigma'}_{k',\ell}$ the displayed operator for such δ'. We need to compute the leading term of $\bar{Q}^{\sigma'}_{k',\ell}$. Observe the generic quantization $Q^{\sigma'}_{k',\ell}$ is constructed in a similar way as $\bar{Q}^{\sigma'}_{k',\ell}$ – only the subfactor $Q^{\sigma'}_{k',0}$ of $Q^{\sigma'}_{k',\ell}$ (see the display in Theorem 3.3) is replaced by $\iota(\sigma')S_{k'}$ in $\bar{Q}^{\sigma'}_{k',\ell}$. It is mentioned in the proof of Proposition 3.2 that only the term $(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell$ of $Q^{\sigma'}_{k',\ell}$ contributes to the generic leading term $(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^{\ell+1}$ of $\tilde{Q}^{\sigma'}_{k',\ell}$, see Proposition 3.2 for the notation. However $\iota(\sigma')S_{k'}$ has the leading term $(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}$ for $\delta' \in \Sigma_{k',0}$ as well as $Q^{\sigma'}_{k',0}$ for $\delta' \notin \Sigma_{k',0}$. It follows that $\bar{Q}^{\sigma'}_{k',\ell}$ has $(\sigma')^{a_1\cdots a_{k'}}\nabla_{a_1}\cdots\nabla_{a_{k'}}\Delta^\ell$ as the leading term for all $\delta' \in \Sigma'_{k',\ell} \cup \Sigma''_{k',\ell}$. Using Lemma 4.1(ii) the theorem follows. □

Remark 4.5. As one of referees pointed out, nonexistence of operators $L_p : \mathcal{E}[-n/2+p] \to \mathcal{E}[-n/2-p]$ implies nonexistence of certain resonant weights. Consider the $(2p)$th order symbol $\sigma \in \mathcal{E} \subseteq \mathcal{E}^{(a_1\cdots a_{2p})}[-2p]$. The corresponding quantization $Q^\sigma_{0,p} : \mathcal{E}[w] \to \mathcal{E}[w-2p]$ does not exist according to Theorem 3.3 and concerning the possible resonant weights w, we can formulate the following:

(1) If n is even then $\{-n/2+1,\dots,0\}$ is the complete set of resonant weights.
(2) If n is odd or we restrict to conformally flat structures then $\{-n/2+1,\dots,0,\dots\}$ is the complete set of resonant weights.

Indeed, if $Q^\sigma_{0,p} : \mathcal{E}[w] \to \mathcal{E}[w-2p]$ existed, the choice $\sigma = 1$ (the function identically equal to 1) would provide the operator $L_p : \mathcal{E}[w] \to \mathcal{E}[w-2p]$. Thus both (1) and (2) follow from (non)existence of operators L_p discussed before Theorem 4.4.

5. Examples

Explicit formulae (in terms of the Levi-Civita connection and its curvature) for the conformally invariant quantization were computed in [12] for the order $\leqslant 3$, the fourth order case is known explicitly for trace-free symbols $\sigma \in \mathcal{E}^{(abcd)_0}[\delta]$ [28,23]. Here we show tractor formulae for remaining symbols of the order four.

Beside the trace-free part, there are two possible irreducible fourth order symbols: $\sigma' \in \mathcal{E}[\delta']$ where $k' = 0$ and $\ell = 2$ and $(\bar{\sigma}')^{ab} \in \mathcal{E}^{(ab)_0}[\delta']$ where $k' = 2$ and $\ell = 1$. In the first case, Proposition 3.2 means that

$$\tilde{Q}^{\sigma'}_{0,2}(f) = D^A D^B \sigma' D_B D_A f = \delta'(\delta'-1)(n+2\delta'-2)(n+2\delta'-4)\sigma'\Delta^2 f + lot$$

where $f \in \mathcal{E}[w]$. This yields the invariant quantization $Q^{\sigma'}_{0,2}$ for $\delta' \notin \Sigma'_{0,2} = \{0, 1, -\frac{n}{2}+1, -\frac{n}{2}+2\}$. Further, using the explicit formula for the invariant quantization for 2nd order trace-free symbols (see e.g. [12]), we obtain

$$\begin{aligned}\tilde{Q}^{\bar{\sigma}'}_{2,1}(f) = D^A\big[&(n+\delta'+1)(n+\delta'+2)(\bar{\sigma}')^{ab}\nabla_a\nabla_b \\ &- 2(w-2)(n+\delta'+1)(\nabla_a\bar{\sigma}')^{ab}\nabla_b + (w-1)(w-2)(\nabla_a\nabla_b\bar{\sigma}')^{ab}\big]D_A f \\ = -\delta'&(n+2\delta'+2)(n+\delta'+1)(n+\delta'+2)(\bar{\sigma}')^{ab}\nabla_a\nabla_b\Delta f + lot.\end{aligned}$$

That is, the previous display provides the invariant quantization $Q^{\bar{\sigma}'}_{2,1}$ for $\delta' \notin \Sigma'_{2,1} = \{0, -\frac{n}{2}-1, -(n+1), -(n+2)\}$.

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Higher symmetries of the conformal powers of the Laplacian on conformally flat manifolds

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Higher symmetries of the conformal powers of the Laplacian on conformally flat manifolds

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On locally conformally flat manifolds, we describe a construction which maps generalised conformal Killing tensors to differential operators which may act on any conformally weighted tensor bundle; the operators in the range have the property that they are symmetries of any natural conformally invariant differential operator between such bundles. These are used to construct all symmetries of the conformally invariant powers of the Laplacian (often called the GJMS operators) on manifolds of dimension at least 3. In particular, this yields all symmetries of the powers of the Laplacian Δ^k, $k \in \mathbb{Z} > 0$, on Euclidean space $\mathbb{E}^n$. The algebra formed by the symmetry operators is described explicitly. *© 2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3692324]

I. INTRODUCTION

Given a differential operator P, say on functions, it is natural to consider smooth differential operators which locally preserve the solution space of P. A refinement is to seek differential operators S with the property that $P \circ S = S' \circ P$, for some other differential operator S'. In this case, we shall say that S is a symmetry of P. On Euclidean n-space $\mathbb{E}^n$ with $n \geq 3$, the space of first-order symmetries of the Laplacian Δ is finite-dimensional with commutator subalgebra isomorphic to $\mathfrak{so}(n + 1, 1)$, the Lie algebra of conformal motions of $\mathbb{E}^n$. Second-order symmetries have applications in the problem of separation of variables for the Laplacian, see Ref. 41 and references therein; on $\mathbb{E}^3$ the second-order symmetries were classified by Boyer *et al.*[4]

Symmetries are closely related to conformal Killing tensors and their generalisations, see Theorem 2.1 below. Such operators also play a role in physics.[40,44] Partly motivated by these links, Eastwood has recently given a complete algebraic description of the symmetry algebra for the Laplacian on $\mathbb{E}^{n\geq 3}$.[20] His treatment uses conformal geometry and, in particular, a treatment of the conformal Laplacian due to Hughston and Hurd[35] based on the classical model of the conformal n-sphere as the projective image of an indefinite quadratic variety in $\mathbb{R}^{n+2}$. There are close links to the Fefferman-Graham ambient metric,[23,24] which provides a curved version of this model, and the ideas of Maldacena's AdS/CFT correspondence[34,39,45] (as explained in Ref. 20). Eastwood's work was extended in Ref. 21, via similar techniques, where the authors found the symmetry algebra for Δ^2 on $\mathbb{E}^{n\geq 3}$.

Here, the first main result of the article is a simultaneous treatment of all powers of the Laplacian on pseudo-Euclidean space $\mathbb{E}^{s,s'}$ (i.e., $\mathbb{R}^{s+s'}$ equipped with a constant signature (s, s') metric) with $s + s' \geq 3$; we obtain an explicit construction of all symmetries and a description of the algebra these generate. See Theorems 2.1 and 2.5. (In lower dimensions a corresponding result is not to

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be expected as, in that case, the space of conformal Killing vectors is infinite-dimensional). As will shortly be clear, the problem is fundamentally linked to conformal geometry. Thus it is natural to also formulate and treat analogous questions for the conformally invariant generalisations P_k of the powers Δ^k ($k \in \mathbb{Z}_{>0}$) on conformally flat manifolds, and we do this; via Theorem 2.4 and surrounding discussion we see that the algebra is again described by Theorem 2.5. In dimension four, the operators P_k were discussed in Ref. 36. Conformally curved versions in general dimensions ($n \geq 2k$ if even) are due to Paneitz ($k = 2$) (Ref. 42) and Graham-Jenne-Mason-Sparling,[33] and have been the subject of tremendous recent interest in both the mathematics and physics community.[16,17,37] For convenience, we refer to these operators as the GJMS operators.

A main point of the current article is to develop a universal approach to the problem of operator symmetries; the constructions and theory here are designed to be easily adapted to study the symmetries of other classes of differential operators. Indeed with minor adaption our techniques also apply to the entire class of parabolic geometries. First, rather than work on a higher dimensional "ambient" manifold (which is an idea well developed only for conformal geometry and a few other structures), we calculate directly on the n-dimensional space and use tractor calculus, many tools of which apply simultaneously to all parabolic geometries.[1,10,27] Using this machinery, we construct a map which takes solutions of certain overdetermined partial differential equations, PDE's (solutions called generalised conformal Killing tensors) to differential operators which have the universality property that they are symmetries for *any* conformally invariant operator between irreducible bundles. This is Theorem 5.2. These universal symmetry operators form an algebra under formal composition; by construction this is a quotient of the tensor algebra $\bigotimes \mathfrak{so}(s+1, s'+1)$. On the other hand for the case of GJMS operators, Theorem 2.4 states that, conversely, all symmetries arise from the operators in this algebra. Determining the algebra of symmetries of a given order $2k$ GJMS operator P_k then proceeds in two steps. The order $2k$ determines the domain (density) bundle (for P_k and hence) on which the universal symmetry operators should act. From the latter we obtain an ideal of identities satisfied by the universal symmetries; the ideal is specific to the domain. This is the subject of Theorem 7.1. A further ideal is generated by symmetries that are trivial in a sense to be made precise below, see Theorem 7.2. The result is an explicit description in Theorem 2.5 of the ideal, the quotient of $\bigotimes \mathfrak{so}(s+1, s'+1)$ by which yields the symmetry algebra of P_k.

II. THE MAIN THEOREMS

A. Symmetries and triviality

Throughout we shall restrict to conformally flat pseudo-Riemannian manifolds (M, g) of dimension $n \geq 3$ and signature (s, s'), or the conformal structures $(M, [g])$ that these determine. In the spirit of Penrose's *abstract index notation*,[43] we shall denote write $\mathcal{E}^a$ as an alternative notation for TM and $\mathcal{E}_a$ for the dual bundle T^*M. Thus, for example, $\mathcal{E}_{ab} = \otimes^2 T^*M$. According to the context, we may also use *concrete* indices from time to time. That is indices referring to a frame. All manifolds, structures, functions, and tensor fields will be taken to be smooth (i.e., to infinite order) and all differential operators will be linear with smooth coefficients. Since our later treatment generalises easily, we define here the notion of symmetry in greater generality than is strictly needed for our main results. This also serves to indicate the general context for the developments.

Suppose that $P : \mathcal{V} \to \mathcal{W}$ is a smooth differential operator between (section spaces of) irreducible bundles. (In our notation we shall not distinguish bundles from their smooth section spaces.) We shall say that linear differential operators $S : \mathcal{V} \to \mathcal{V}$ and $S' : \mathcal{W} \to \mathcal{W}$ form a (S, S') a *symmetry (pair)* of P if the operator compositions PS and $S'P$ satisfy

$$PS = S'P.$$

An example is the pair (TP, PT), where T is a differential operator $T : \mathcal{W} \to \mathcal{V}$. However for obvious reasons such symmetries shall be termed *trivial*.

Following the treatment of Δ and Δ^2 of Refs. 20 and 21, we note that there is an algebraic structure on the symmetries modulo trivial symmetries as follows. First, the symmetries of P form a vector space under the obvious operations. Then if (S_1, S_1') and (S_2, S_2') are symmetries, then so too is the composition $(S_1 S_2, S_1' S_2')$. So the symmetries of P form an algbera $\tilde{\mathcal{S}}$. Next we say that two symmetries (S_1, S_1') and (S_2, S_2') are equivalent, $(S_1, S_1') \sim (S_2, S_2')$, if and only if $(S_1 - S_2, S_1' - S_2')$ is a trivial symmetry. It is easily verified that trivial symmetries form a two-sided ideal in the algebra $\tilde{\mathcal{S}}$ and the quotient by this yields an algebra $\mathcal{S}$. For the case that P is a GJMS operator it is this algebra that we shall study in detail.

To simplify our discussion, we shall often work with just the first operator $S : \mathcal{V} \to \mathcal{V}$ in a symmetry pair. That is an operator $S : \mathcal{V} \to \mathcal{V}$ shall be called a symmetry if there exists some $S' : \mathcal{W} \to \mathcal{W}$ that makes (S, S') a symmetry as above. (In fact for the main class of operators we treat it is easily verified that S' is uniquely determined by S.) Note that with this language, and in the class of cases satisfying $\mathcal{V} = \mathcal{W}$, the composition PS is a trivial symmetry if and only if S is a symmetry.

B. Symmetries of Δ^k on $\mathbb{E}^{s,s'}$

We shall write $\mathbb{E}^{s,s'}$ to mean $\mathbb{R}^n$, $n = s + s'$, equipped with the standard flat diagonal signature (s, s') metric g; in the $s = n$, $s' = 0$ case this is n-dimensional Euclidean space. Here and throughout we shall make the restriction $n \geq 3$. In this setting the Levi-Civita connection ∇ is flat and, with tensors expressed in terms of the standard $\mathbb{R}^n$ coordinates x^i, the action of ∇_i on these agrees with $\partial/\partial x^i$. We shall use the metric g_{ij} and its inverse g^{ij} to lower and raise indices in the usual way. For example, and capturing also our sign convention for the Laplacian, $\Delta = g^{ij}\nabla_i\nabla_j = \nabla^i\nabla_i$. (We use the summation convention here and below without further mention.)

Recall that a vector field v is a conformal Killing field (or infinitesimal conformal isometry) if $\mathcal{L}_v g = \rho g$ for some function ρ. Otherwise written, this equation is

$$\nabla^i v^j + \nabla^j v^i = \rho g^{ij},$$

and so, for solutions, $\rho = 2\mathrm{div}\, v/n$. Suppose now that φ is a symmetric trace-free covariant tensor satisfying

$$\nabla^{(i} \cdots \nabla^l \varphi^{m \cdots n)} = g^{(ij}\rho^{k \cdots n)}, \qquad \text{with} \qquad |\{i, \cdots, l\}| \text{ an odd integer} \tag{1}$$

for some tensor $\rho^{k\cdots n}$, and where $\phi^{(i\cdots n)}$ indicates the symmetric part of the tensor $\phi^{i\cdots n}$. Then, following Ref. 20, we shall term φ a *generalised conformal Killing tensor*.

In Sec. V below we shall construct a canonical 1-1 map

$$\varphi \mapsto (S_\varphi, S_\varphi'), \tag{2}$$

which takes solutions of (1) to symmetries of Δ^k, see Definition 5.1 and Theorem 5.2 (which, in fact, deal with a far more general setting). Although we defer the construction of (2), let us already term (S_φ, S_φ') the *canonical symmetry* corresponding to φ. Our main classification result is that all symmetries of Δ^k arise this way, and this is established in Theorem 6.4. Putting these results together, on $\mathbb{E}^{s,s'}$ we have the following.

Theorem 2.1: *Let us fix $k \in \mathbb{Z}_+$. For the Laplacian power Δ^k on $\mathbb{E}^{s,s'}$ we have the following. For each φ, a solution of (1), there is canonically associated a symmetry (S_φ, S_φ') for Δ^k with S_φ and S_φ' each having leading term*

$$\varphi^{a_1 \dots a_p}(\nabla_{a_1} \cdots \nabla_{a_p})\Delta^r,$$

$p \in \mathbb{Z}_{\geq 0}$, $r \in \{0, 1, \dots, k - 1\}$.

Modulo trivial symmetries, any symmetry of Δ^k is a linear combination of such pairs (S_φ, S_φ'), with various solutions φ of (1) as above.

C. Conformal geometry and the GJMS operators

Although the question of symmetries of Δ^k is not phrased in terms of conformal geometry, it turns out that there is a deep connection. According to Theorem 2.1 above, all symmetries of Δ^k arise from the solutions of Eq. (1). As we shall explain, these equations are each conformally covariant, and in fact this class of equations can only be fully understood via consideration of their conformal properties. First note that we may alternatively write Eq. (1) as

$$\nabla_{(b_0} \cdots \nabla_{b_{2r}} \varphi_{a_1 \dots a_p)_0} = 0,$$

where we have lowered the indices for convenience, and $(\dots)_0$ indicates the trace-free part over the enclosed indices. For a given (say symmetric) tensor taking the trace-free part is a conformally invariant notion. Then, for example, in the case of $r = 0$ this is the well-known conformal Killing tensor operator. In that case, if (on any pseudo-Riemannian manifold (M, g) of the dimension n) we replace the metric g with the conformally related $\widehat{g} := e^{2\Upsilon} g$, where $\Upsilon \in C^\infty(M)$, and replace φ with $\widehat{\varphi} := e^{2p\Upsilon}\varphi$, then

$$\nabla^{\widehat{g}}_{(b_0} \widehat{\varphi}_{a_1 \dots a_p)_0} = e^{2p\Upsilon} \nabla_{(b_0} \varphi_{a_1 \dots a_p)_0}.$$

One may think of φ here as representing a *density* valued tensor of weight $2p$. Recall that on a smooth manifold, the density bundles $\mathcal{E}[w]$ are the bundles associated with the frame bundle by one-dimensional (real) representations arising as the roots (or powers) of the square of the determinant representation. These representations and the associated bundles are thus naturally parametrised by *weights* w from $\mathbb{R}$. These weights are normalised so that $\mathcal{E}[-2n] \cong (\Lambda^n T^* M)^2$, and with this normalisation the weights are often called *conformal weights*. Note that $(\Lambda^n T^* M)^2$ is trivialised by a choice of metric and hence so are all the line bundles $\mathcal{E}[w]$. There is a section $\tilde{\varphi}$ of $\mathcal{E}_{(a_1 \cdots a_p)_0}[2p] = \mathcal{E}_{(a_1 \cdots a_p)_0} \otimes \mathcal{E}[2p]$ which, in the trivialisation of $\mathcal{E}[2p]$ afforded by g, has the component φ, while $\tilde{\varphi}$ has the component $\widehat{\varphi} = e^{2p}\varphi$ with respect to the trivialisation from $\widehat{g}$. Since the Levi-Civita connection (for any metric g) may be viewed as a connection on the principal frame bundle it follows immediately that it yields a connection on density weighted tensor bundles. Thus dropping the tilde, for $\varphi \in \mathcal{E}_{(a_1 \cdots a_p)_0}[2p]$ we have $\nabla^{\widehat{g}}_{(b_0} \varphi_{a_1 \dots a_p)_0} = \nabla_{(b_0} \varphi_{a_1 \dots a_p)_0}$. This means that the operator descends to a well-defined differential operator on a conformal manifold (M, c). Here (M, c) means a manifold equipped with just an equivalence class of conformally related metrics: if $g, \widehat{g} \in c$, then $\widehat{g} = e^{2\Upsilon} g$ for some $\Upsilon \in C^\infty(M)$.

Henceforth, it will be convenient to use the notation and language of conformal densities, for further details and conventions see, e.g., Refs. 9 or 30. In particular, below we shall use the *conformal metric* $\boldsymbol{g}_{ab}$ to raise and lower indices. On a conformal manifold, this is a tautological section of $\mathcal{E}_{(ab)}[2] = \mathcal{E}_{(ab)} \otimes [2]$ which gives an isomorphism $\mathcal{E}^a = \mathcal{E}^a[0] \cong \mathcal{E}_b[2]$. In particular, via the conformal metric, we shall identify $\mathcal{E}_{(a_1 \dots a_p)_0}[2p + 2r]$ and $\mathcal{E}^{(a_1 \dots a_p)_0}[2r]$. Note also that with these conventions the Laplacian Δ is given by $\Delta = \mathbf{g}^{ab}\nabla_a \nabla_b = \nabla^b \nabla_b$ and so this carries a conformal weight of -2. (That is, the conformal Laplacian lowers the conformal weight by 2.)

From the partial classification of conformally invariant operators given in Ref. 22 (which uses heavily the algebraic results of Ref. 3), one easily extracts the following result.

Proposition 2.2: For each pair (p, r), of non-negative integers, there is a conformally invariant operator

$$\begin{aligned} &\mathcal{E}_{(a_1 \dots a_p)_0}[2p + 2r] \to \mathcal{E}_{(b_0 \dots b_{2r} a_1 \dots a_p)_0}[2p + 2r] \\ &\varphi_{a_1 \dots a_p} \mapsto \nabla_{(b_0} \cdots \nabla_{b_{2r}} \varphi_{a_1 \dots a_p)_0} + lot, \end{aligned} \tag{3}$$

where "lot" denotes lower order terms.

In fact, there is a larger class of similar operators, but we shall not need the even order analogues of the operators above for our current discussion. An algorithm for generating explicit formulas for these operators is given in Ref. 25 (in dimension four but same formulas hold in all dimensions,[26] see also Refs. 13 and 7). The lower order terms are given by Ricci curvature and its derivatives; in particular on $\mathbb{E}^{s,s'}$ we recover the operator of (1). On any manifold, we shall term φ in the kernel of (3) a *(generalised) conformal (Killing) tensor*. (The terminology generalised conformal Killing tensor was introduced in Ref. 21 for solutions of (3) in the case $p = 3$. We use the same terminology for solutions of (3) in the general case.)

By construction the GJMS operator, P_k is conformally invariant.[33] This means that it is a natural operator on pseudo-Riemannian manifolds M that descends to a well-defined differential operator on conformal structures

$$P_k : \mathcal{E}[k - \frac{n}{2}] \to \mathcal{E}[-k - \frac{n}{2}].$$

Recall that we say (M, g) is *locally conformally flat*, if locally there is a metric $\widehat{g}$, conformally related to g, so that on this neighbourhood $(M, \widehat{g})$ is isometric to $\mathbb{E}^{s,s'}$. If (M, g) is locally conformally flat, then in all dimensions $n \geq 3$ the operators P_k exist for every $k \geq 1$.

Definition 2.3: Let us fix a conformal manifold (M, c). Suppose that (S, S') is a pair of differential operators

$$S : \mathcal{E}[k - \frac{n}{2}] \to \mathcal{E}[k - \frac{n}{2}] \quad \text{and} \quad S' : \mathcal{E}[-k - \frac{n}{2}] \to \mathcal{E}[-k - \frac{n}{2}]$$

on the given conformal manifold (M, c). If locally (i.e., in contractable neighbourhoods) on (M, c), we have agreement of the compositions as follows:

$$P_k S = S' P_k,$$

as operators on $\mathcal{E}[k - \frac{n}{2}]$, then we shall say that (S, S') is a conformal symmetry (pair) of P_k on (M, c).

Note that the definition does not require/impose naturality properties of the pair (S, S'). They are simply required to be well-defined differential operators on the given (M, c).

For a given conformal manifold, and suitable natural number k, we may ask for some description of all conformal symmetries of P_k. From Theorem 2.1, we have the following theorem. Here and below, we use $\mathcal{E}_r^{(p)_0}$ as shorthand for the bundle $\mathcal{E}^{(a_1 \ldots a_p)_0}[2r]$ (and its section space). We will often write φ_r^p to denote some section of this bundle.

Theorem 2.4: *Let (M, c) be a (locally) conformally flat manifold of signature (s, s'). For each nonzero $\varphi \in \mathcal{E}_r^{(p)_0}$, $p \in \mathbb{Z}_{\geq 0}$, $r \in \{0, 1, \ldots, k-1\}$, a solution of (3), there is canonically associated a nontrivial conformal symmetry (S_φ, S'_φ) for P_k, with S_φ and S'_φ each having leading term*

$$\varphi_r^{a_1 \ldots a_p}(\nabla_{a_1} \cdots \nabla_{a_p})\Delta^r.$$

Modulo trivial symmetries, locally any conformal symmetry of P_k is a linear combination of such pairs (S_φ, S'_φ), for various solutions φ of (3), with p and r in the range assumed here.

The question of conformal symmetries is not *a priori* the same question as that addressed in Theorem 2.1. However, using that S, S' and P_k are well defined on (M, c), we may use any metric $g \in c$ to calculate. This is a choice similar to choosing coordinates in order to calculate; indeed g gives a trivialisation of the density bundles. Now, by working locally and choosing a flat metric, the result here follows immediately from Theorem 2.1, since by the definition of the canonical symmetries in Definition 5.1 and Theorem 5.2, they are well defined on locally conformally flat conformal manifolds.

D. Algebraic structure

Let us denote by $\mathcal{A}_k$ the algebra of symmetries of Δ^k on $\mathbb{E}^{s,s'}$ modulo trivial symmetries. As usual we write $n = s + s'$. It follows from Theorem 7.1, we have the vector space isomorphism

$$\mathcal{A}_k \cong \bigoplus_{j=0}^{\infty} \bigoplus_{i=0}^{k-1} \mathcal{K}_i^j, \tag{4}$$

where $\mathcal{K}_i^j \subseteq \mathcal{E}_i^{(j)_0}$ is the space of solutions of (3) with $r = j$ and $p = i$.

Now we turn to the algebra structure of $\mathcal{A}_k$. It is well known,[38,14] and given explicitly by (23) below, that the (finite-dimensional) spaces $\mathcal{K}_i^j$ are isomorphic to irreducible $\mathfrak{g} := \mathfrak{so}_{s+1,s'+1}$–modules

$$\mathcal{K}_i^j \cong \overbrace{\ydiagram{2,2}\cdots\ydiagram{1,1}}^{j}\overbrace{\ydiagram{2}\cdots\ydiagram{1}}^{2i}{}_0 \tag{5}$$

in the notation of Young diagrams. (Using the highest weights, expressed as a vector of coefficients over the Dynkin diagram as in Ref. 2, $\mathcal{K}_i^j$ corresponds to the coefficient $2i$ over the first node, the coefficient j over the second one and with remaining coefficients equal to zero. At least this applies in dimensions at least five, but there is an obvious adjustment in lower dimensions.)

We follow Ref. 21 in the discussion of the algebraic structure of $\mathcal{A}_k$. Decomposing the tensor product of two copies of $g = \ydiagram{1,1}$, we obtain

$$\mathfrak{g} \otimes \mathfrak{g} = \underbrace{\ydiagram{2,2}_0 \oplus \ydiagram{2}_0 \oplus \mathbb{R} \oplus \ydiagram{1,1,1,1}}_{\mathfrak{g} \odot \mathfrak{g}} \oplus \underbrace{\ydiagram{2,1,1}_0 \oplus \ydiagram{1,1}}_{\mathfrak{g} \wedge \mathfrak{g}} \tag{6}$$

where $\odot$ is the symmetric tensor product. All these components occur with multiplicity one. We shall need notation for the projections of $V_1 \otimes V_2 \in \mathfrak{g} \otimes \mathfrak{g}$ to some of the irreducible components on the right-hand side of the previous display. In particular, we put

$$V_1 \boxtimes V_2 \in \ydiagram{2,2}_0, \quad V_1 \bullet V_2 \in \ydiagram{2}_0, \quad \langle V_1, V_2\rangle \in \mathbb{R} \quad \text{and} \quad [V_1, V_2] \in \ydiagram{1,1}, \tag{7}$$

and we write the same notation for the projections. Here, the $\boxtimes$ denotes the Cartan product, $\langle,\rangle$ the Killing form on $\mathfrak{g}$ (normalised as in Ref. 21), and [,] is the Lie bracket. These projections are described explicitly in (41) below. There is also the inclusion

$$\boxtimes^{2k}\square = \underbrace{\ydiagram{2}\cdots\ydiagram{1}}_{2k}{}_0 \hookrightarrow \underbrace{\ydiagram{1,1} \odot \ydiagram{1,1} \odot \cdots \odot \ydiagram{1,1}}_{2k} \subset \underbrace{\ydiagram{1,1} \otimes \ydiagram{1,1} \otimes \cdots \otimes \ydiagram{1,1}}_{2k},$$

see (44) for the explicit form. That is, there is an (obviously unique) irreducible component in $\bigodot^{2k}\mathfrak{g}$ of the type specified on the left-hand side.

With this notation, we obtain the following generalisation of Theorem 3 Ref. 21:

Theorem 2.5: *The algebra $\mathcal{A}_k$ is isomorphic to the tensor algebra $\bigotimes \mathfrak{g}$ modulo the two-sided ideal generated by*

$$V_1 \otimes V_2 - V_1 \boxtimes V_2 - V_1 \bullet V_2 - \frac{1}{2}[V_1, V_2] + \frac{(n-2k)(n+2k)}{4n(n+1)(n+2)}\langle V_1, V_2\rangle, \quad V_1, V_2 \in \mathfrak{g} \tag{8}$$

and the image of $\boxtimes^{2k}\square$ in $\otimes^{2k}\mathfrak{g}$.

Note that, from Theorem 2.4, $\mathcal{A}_k$ is also the algebra of local symmetries of P_k on any conformally flat conformal manifold of dimension n.

III. CONFORMAL TRACTOR CALCULUS

We first recall the basic elements of tractor calculus following Refs. 9 and 30.

A. Tractor bundles

Let M be a smooth manifold of dimension $n \geq 3$ equipped with a conformal structure (M, c) of signature (s, s'). Since the Levi-Civita connection is torsion-free, the (Riemannian) curvature $R_{ab}{}^c{}_d$ is given by $[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d$, where $[\ \cdot\ ,\ \cdot\ ]$ indicates the commutator bracket. The Riemannian curvature can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to $R_{abcd} = C_{abcd} + 2\mathbf{g}_{c[a}P_{b]d} + 2\mathbf{g}_{d[b}P_{a]c}$, where $[\dots]$ indicates antisymmetrisation over the enclosed indices. We shall write $J := P^a{}_a$. The *Cotton tensor* is defined by

$$A_{abc} := 2\nabla_{[b}P_{c]a}.$$

The standard tractor bundle over $(M, [g])$ is a vector bundle of rank $n + 2$ defined, for each $g \in c$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\widehat{g} = e^{2\Upsilon}g$ $(\Upsilon \in C^\infty(M))$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\widehat{\alpha}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{E}^A]_{\widehat{g}}$ by the transformation

$$\begin{pmatrix} \widehat{\alpha} \\ \widehat{\mu}_a \\ \widehat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_c\Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix}, \tag{9}$$

where $\Upsilon_a := \nabla_a \Upsilon$. These identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{T}$, or $\mathcal{E}^A$ in an abstract index notation, over the conformal manifold. The bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(s + 1, s' + 1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. In a conformal scale g, these are given by

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \nabla_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a\alpha - \mu_a \\ \nabla_a\mu_b + \mathbf{g}_{ab}\tau + P_{ab}\alpha \\ \nabla_a\tau - P_{ab}\mu^b \end{pmatrix}. \tag{10}$$

It is readily verified that both of these are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence, we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A$, $\mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the "projectors" from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$, and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are, respectively, denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$, and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\mathbf{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we immediately see that

$$Y_A X^A = 1, \quad Z_{Ab}Z^A{}_c = \mathbf{g}_{bc}, \tag{11}$$

and that all other quadratic combinations that contract the tractor index vanish. In (9) note that $\widehat{\alpha} = \alpha$ and hence X^A is conformally invariant. Using this notation the tractor V^A given by

$$[V^A]_g = \begin{pmatrix} \alpha \\ \mu_a \\ \tau \end{pmatrix}$$

may be written

$$V^A = \alpha Y^A + \mu^a Z^A{}_a + \tau X^A. \tag{12}$$

The curvature Ω of the tractor connection is defined by

$$[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E$$

for $V^C \in \mathcal{E}^C$. Using (10) and the formulas for the Riemannian curvature yield

$$\Omega_{abCE} = Z_C{}^c Z_E{}^e C_{abce} - 2X_{[C}Z_{E]}{}^e A_{eab}. \tag{13}$$

In the following, we shall also need two-form tractors, that is $\Lambda^2\mathcal{T}$, or in abstract indices $\mathcal{E}_{[AB]}$. To simplify notation, we shall set the rule that indices labelled sequentially by a superscript are implicitly skewed over and then denote skew pairs with a bold multi-index. Here, we shall need this only for valence two forms. This convention does not apply to subscripts. That is, A^0A^1 means $[A^0A^1] = \mathbf{A}$ but, e.g., the notation $A_1A_2A_3$ does not assume any implicit projection to a tensor part. The same convention will be used for tensor indices, i.e., $[a^0a^1]$ means $a^0a^1 = \mathbf{a}$.

With $\mathcal{E}^k[w]$ denoting the space of k-forms of weight w, the structure of $\mathcal{E}_\mathbf{A} = \mathcal{E}_{A^0A^1}$ is Refs. 6 and 31,

$$\mathcal{E}_\mathbf{A} = \mathcal{E}^1[2] \oplus\!\!\!\!+ \left(\mathcal{E}^2[2] \oplus \mathcal{E}[0]\right) \oplus\!\!\!\!+ \ \mathcal{E}^1[0]; \tag{14}$$

this means that in a choice of scale the *semidirect sums* $\oplus\!\!\!\!+$ may be replaced by direct sums and otherwise they indicate the composition series structure arising from the tensor powers of (9).

In a choice of metric g from the conformal class, the projectors (or splitting operators) X, Y, Z for $\mathcal{E}_A$ determine corresponding projectors $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}$ for $\mathcal{E}_\mathbf{A}$, These execute the splitting of this space into four components and are given as follows:

$$\begin{aligned}
\mathbb{Y} &= \mathbb{Y}_{A^0A^1}{}^{a^1} = Y_{A^0}Z_{A^1}^{a^1} \in \mathcal{E}_\mathbf{A}^{a^1}[-2],\\
\mathbb{Z} &= \mathbb{Z}_{A^1A^2}^{a^1a^2} = Z_{A^1}^{a^1}Z_{A^2}^{a^2} \in \mathcal{E}_\mathbf{A}^\mathbf{a}[-2],\\
\mathbb{W} &= \mathbb{W}_{A^0A^1} = X_{A^0}Y_{A^1} \in \mathcal{E}_\mathbf{A}[0],\\
\mathbb{X} &= \mathbb{X}_{A^0A^1}{}^{a^1} = X_{A^0}Z_{A^1}^{a^1} \in \mathcal{E}_\mathbf{A}^{a^1}[0].
\end{aligned}$$

Further, they satisfy $\mathbb{X}^\mathbf{A}_a\mathbb{Y}^c_\mathbf{A} = \frac{1}{2}\delta^c_a$, $\mathbb{Z}^\mathbf{A}_\mathbf{a}\mathbb{Z}^\mathbf{c}_\mathbf{A} = \delta^{c^1}_{a^1}\delta^{c^2}_{a^2}$, and $\mathbb{W}^\mathbf{A}\mathbb{W}_\mathbf{A} = -\frac{1}{2}$ id, the remaining contractions are zero. The explicit formula for the tractor connection is then determined by how it acts on these (cf. Refs. 31 and 6),

$$\begin{aligned}
\nabla_p\mathbb{Y}_{A^0A^1}{}^{a^1} &= P_{pa_0}\mathbb{Z}_{A^0A^1}^{a^0a^1} + P_p{}^{a^1}\mathbb{W}_{A^0A^1},\\
\nabla_p\mathbb{Z}_{A^0A^1}^{a^0a^1} &= -2\delta_p^{a^0}\mathbb{Y}_{A^0A^1}{}^{a^1} - 2P_p{}^{a^0}\mathbb{X}_{A^0A^1}{}^{a^1},\\
\nabla_p\mathbb{W}_{A^0A^1} &= -\mathbf{g}_{pa^1}\mathbb{Y}_{A^0A^1}{}^{a^1} + P_{pa^1}\mathbb{X}_{A^0A^1}{}^{a^1},\\
\nabla_p\mathbb{X}_{A^0A^1} &= \mathbf{g}_{pa^0}\mathbb{Z}_{A^0A^1}^{a^0a^1} - \delta_p^{a^1}\mathbb{W}_{A^0A^1}.
\end{aligned} \tag{15}$$

B. Key differential operators

Given a choice of conformal scale, Thomas' *tractor-D operator*[1] $D_A : \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w - 1]$ is defined by

$$D_AV := (n + 2w - 2)wY_AV + (n + 2w - 2)Z_{Aa}\nabla^aV - X_A(\Delta V + w\mathrm{J})V. \tag{16}$$

This is conformally invariant, as can be checked directly using the formulas above (or alternatively there are conformally invariant constructions of D, see, e.g., Ref. 27). Acting on sections of weight $w \neq 1 - n/2$ (16) is a *differential splitting operator* since there is a bundle homomorphism which inverts D. In this case, it is a multiple of $X^A : \mathcal{E}_{AB\cdots E}[w - 1] \to \mathcal{E}_{B\cdots E}[w]$; X^AD_A is a multiple of the identity on the domain space. This splitting operator is particularly important on $\mathcal{E}[1]$, the densities of weight 1: for non-vanishing $\sigma \in \mathcal{E}[1]$, $g := \sigma^{-2}\mathbf{g}$ is Einstein if and only if $D_A\sigma$ is parallel for the tractor connection. The point is that the tractor connection (10) gives a prolonged system essentially equivalent to the equation $\nabla_{(a}\nabla_{b)_0}\sigma + P_{(ab)_0}\sigma = 0$ which controls whether the metric $g \in c$ is Einstein.[1]

The GJMS operators on conformally flat manifolds can easily be constructed using the tractor D-operator. It turns out

$$(-1)^kX_{A_1}\ldots X_{A_k}P_k = D_{A_1}\ldots D_{A_k} \quad \text{on} \quad \mathcal{E}_\bullet[-n/2 + k],$$

see Ref. 27 for details. Here $\bullet$, in $\mathcal{E}_\bullet$, denotes any system of tractor indices (or $\mathfrak{so}(h)$ tensor part thereof).

In addition to the tractor-*D* operator D_A, one has also the conformally invariant *double-D* operator $\mathbb{D}_\mathbf{A}$ and its "square" $\mathbb{D}^2_{AB} = -\mathbb{D}_{(A}{}^P\mathbb{D}_{|P|B)}$ defined as

$$\begin{aligned} \mathbb{D}_\mathbf{A} &= 2(w\mathbb{W}_\mathbf{A} + \mathbb{X}_\mathbf{A}^a\nabla_a) : \mathcal{E}_\bullet[w] \longrightarrow \mathcal{E}_\mathbf{A}\otimes\mathcal{E}_\bullet[w], \quad w\in\mathbb{R}, \\ \mathbb{D}^2_{AB} &= -(wh_{AB} + X_{(A}D_{B)}) : \mathcal{E}_\bullet[w] \longrightarrow \mathcal{E}_{(AB)}\otimes\mathcal{E}_\bullet[w], \quad w\in\mathbb{R}. \end{aligned} \tag{17}$$

The operator $\mathbb{D}_\mathbf{A}$ (but with the opposite sign) was originally defined in Ref. 28. Note that, $2X_{[A^0}D_{A^1]} = (n+2w-2)\mathbb{D}_\mathbf{A}$ on $\mathcal{E}_\bullet[w]$. We shall also need the commutation relation on $\mathcal{E}_\bullet[w]$,

$$[D_A, X_B] = -2\mathbb{D}_{AB} + (n+2w)h_{AB} \tag{18}$$

from Ref. 27; alternatively this may be viewed as defining $\mathbb{D}$ as (one half of) the skew part of the left-hand side.

Finally, some points of notation: In the following, we shall sometimes write ∇^q to denote the composition of q applications of ∇. By context it will be clear that q is not to be interpreted as an abstract index. Next if $\mathcal{V}$ is a tensor bundle, or a tensor product of the standard tractor bundle, then for $F\in\mathcal{V}$ we shall write $F|_\boxtimes$ to denote the projection of the section F to the Cartan component (with respect to the $\mathfrak{co}(g)$ structure, or $\mathfrak{so}(h)$ tensor structure, respectively) of the bundle $\mathcal{V}$. For example, on $\mathbb{E}^{s,s'}$ equipped with the standard flat diagonal signature (s, s') metric Eq. (3) may be expressed as $[\nabla^{2r+1}\varphi]|_\boxtimes = 0$.

IV. THE DOUBLE-D AND CONFORMALLY INVARIANT OPERATORS

We work on $(M, [g])$, assumed to be locally conformally flat. We outline a rather general picture here. The theorem below provides a general technique for the construction of symmetries of *any* conformally invariant operator that acts between irreducible natural bundles. Moreover, since the tools used are general in nature, this result indicates how to deal with symmetries of invariant operators on a bigger class of structures, the so-called parabolic geometries.[12] This will be taken up elsewhere.

Consider a conformally invariant differential operator $P:\mathcal{V}\to\mathcal{W}$ between irreducible (or completely reducible will suffice) natural bundles $\mathcal{V}$ and $\mathcal{W}$. More specifically, we restrict only to subbundles of $(\bigotimes\mathcal{E}_a)\otimes(\bigotimes\mathcal{E}^b)\otimes\mathcal{E}[w]$ which we shall term tensor bundles. The case of spinor bundles is, however, completely analogous.

Assume for a moment the general (i.e., possibly curved) conformal setting. Following Ref. 10, the double-D operator $\mathbb{D}_\mathbf{A}$ can be extended to all irreducible bundles (see the discussion on the fundamental derivative below for details). This extension obeys the Leibniz rule, and since (17) describes $\mathbb{D}_\mathbf{A}$ on $\mathcal{E}_\bullet[w]$, it remains to understand the action of $\mathbb{D}_\mathbf{A}$ on $\mathcal{E}_a\cong\mathcal{E}^b[-2]$. In this case, we obtain

$$\mathbb{D}_\mathbf{B} f_a = -2\mathbb{W}_\mathbf{B} f_a + 2\mathbb{Z}_\mathbf{B}^\mathbf{b}\mathbf{g}_{b^0a}f_{b^1} + 2\mathbb{X}_\mathbf{B}^b\nabla_b f_a \quad \text{for} \quad f_a\in\mathcal{E}_a, \tag{19}$$

where $\mathbf{B}$ is a multi-index, following the convention introduced in Sec. III.

Our use of $\mathbb{D}$ is linked to the following proposition. For a tangent vector $\varphi^a\in\mathcal{E}^a$, we denote by L_φ the Lie derivative on sections of natural bundles. Recall $\mathcal{E}[w]$ is such a natural bundle, cf. the definition of $\mathcal{E}[w]$ in Sec. II, as well as $\mathcal{E}_a$ and $\mathcal{E}^b$.

Proposition 4.1: Let M be any conformally flat manifold and assume $\varphi^a\in\mathcal{E}^a$ is a conformal Killing vector (i.e., a solution of (3)). Then there is a unique parallel tractor $I_\varphi^\mathbf{A}\in\mathcal{E}^\mathbf{A}$ such that $\varphi^a = 2\mathbb{X}_\mathbf{A}^a I_\varphi^\mathbf{A}$,[31] cf. Ref. (43). Then

$$I_\varphi^\mathbf{A}\mathbb{D}_\mathbf{A} = L_\varphi \quad \text{on} \quad (\bigotimes\mathcal{E}_b)\otimes(\bigotimes\mathcal{E}^c)\otimes\mathcal{E}[w].$$

Proof: It is sufficient to verify the theorem on $\mathcal{E}[w]$ and $\mathcal{E}^a$ since both operators L_φ and $I_\varphi^\mathbf{A}\mathbb{D}_\mathbf{A}$ obey the Leibniz rule and $\mathcal{E}_b\cong\mathcal{E}^a[-2]$. Using (17) and (43), we have $I_\varphi^\mathbf{A}\mathbb{D}_\mathbf{A} = \varphi^a\nabla_a - \frac{2}{n}(\nabla_a\varphi^a)$ on

$\mathcal{E}[2]$. Thus using (19) (and (43) below), we obtain

$$\begin{aligned} I^{\mathbf{B}}_{\varphi}\mathbb{D}_{\mathbf{B}} f_a &= \varphi^b \nabla_b f_a + (\nabla_{[a}\varphi_{b]}) f^b - \frac{1}{n}(\nabla_b \varphi^b) f_a \\ &= \varphi^b \nabla_b f_a - f^b \nabla_b \varphi_a + f^b \Big[\frac{1}{2}(\nabla_b \varphi_a + \nabla_a \varphi_b) - \frac{1}{n}\mathbf{g}_{ab}\nabla^c \varphi_c\Big] \end{aligned}$$

on $f_a \in \mathcal{E}_a[2] \cong \mathcal{E}^b$. The square bracket in the display is the conformal Killing operator, and thus vanishes. The equality of L_φ and $I^{\mathbf{A}}_\varphi \mathbb{D}_{\mathbf{A}}$ on $\mathcal{E}[w]$ is even simpler, and hence the general case follows. □

Note it obvious from the proof that the proposition does not hold without the assumption that $\varphi^a \in \mathcal{E}^a$ is a conformal Killing vector.

The conformal invariance of the operator $P : \mathcal{V} \to \mathcal{W}$ (between completely reducible, bundles $\mathcal{V}$ and $\mathcal{W}$) is given by the property $L_\varphi P = PL_\varphi$ for every conformal Killing field $\varphi^a \in \mathcal{E}^a$. That is, every conformal Killing vector φ^a provides a symmetry of the operator P.

As is well known, conformal invariance can equivalently be verified from a formula for the operator P. In particular for each conformally invariant operator, and a choice of metric from the conformal class, there is a formula in terms of the Levi-Civita connection ∇, its curvature, and various algebraic projections which express the operator as a natural (pseudo-)Riemmanian differential operator. The hallmark of conformal invariance is then that this operator is unchanged if we use the same formula when starting with a different metric form the conformal class. Now, given such a formula for $P : \mathcal{V} \to \mathcal{W}$, we have also the (tractor coupled) operator $P^\nabla : \mathcal{V} \otimes \mathcal{E}_\bullet \to \mathcal{W} \otimes \mathcal{E}_\bullet$ given by the same formula where ∇ is now assumed to be coupled Levi-Civita-tractor connection. Then P^∇ is also conformally invariant. We shall often write P instead of P^∇ to simplify the notation.

Theorem 4.2: *On a conformally flat manifold, let $P : \mathcal{V} \to \mathcal{W}$ be a conformally invariant operator between completely reducible tensor natural bundles $\mathcal{V}$ and $\mathcal{W}$. Then*

$$P^\nabla \mathbb{D}_{\mathbf{A}_1} \cdots \mathbb{D}_{\mathbf{A}_p} = \mathbb{D}_{\mathbf{A}_1} \cdots \mathbb{D}_{\mathbf{A}_p} P : \mathcal{V} \to \mathcal{E}_{\mathbf{A}_1 \ldots \mathbf{A}_p} \otimes \mathcal{W}.$$

Proof: It is sufficient to prove the theorem in the (globally) flat case. First assume $p = 1$ and consider a conformal Killing field $\varphi^a \in \mathcal{E}^a$. Then I_φ is parallel (see, e.g., Ref. 29 but this follows here easily from the fact the standard tractor connection is flat). Then $[P^\nabla, I^{\mathbf{A}}_\varphi] = 0$ and using Proposition 4.1 plus the fact that $L_\varphi P = PL_\varphi$, from conformal the invariance of P, means that $I^{\mathbf{A}}_\varphi [\mathbb{D}_{\mathbf{A}}, P^\nabla] = 0$ for every conformal Killing vector φ^a. The space of conformal Killing fields on the conformally flat manifolds has the maximal dimension, i.e., the dimension of the bundle $\mathcal{E}_{\mathbf{A}}$. Therefore, $[\mathbb{D}_{\mathbf{A}}, P^\nabla] = \mathbb{D}_{\mathbf{A}} P - P^\nabla \mathbb{D}_{\mathbf{A}} = 0$ on $\mathcal{V}$. Now it follows from the definition of $\mathbb{D}$ that the formulas for $[\mathbb{D}_{\mathbf{A}}, P^\nabla]$ on $\mathcal{V}$ and $\mathcal{E}_\bullet \otimes \mathcal{V}$ formally coincide. Since $[\mathbb{D}_{\mathbf{A}}, P^\nabla] = 0$ on $\mathcal{V}$, and the tractor connection is flat, this formula yields a zero operator on every bundle $\mathcal{E}_\bullet \otimes \mathcal{V}$. Using an obvious induction, the theorem follows. □

Below we shall identify two-form tractor fields $F_{\mathbf{A}} = F_{A^1 A^2}$ with endomorphism fields of the standard tractor bundle according to the rule $(F \sharp f)_B := F_B{}^P f_P$ for $f_B \in \mathcal{E}_B$. This also defines the notation $\sharp$. Moreover, we shall define $\sharp$ to be trivial on the bundles $\mathcal{E}_a$ and $\mathcal{E}[w]$, and then extend this action to tensor products of $\mathcal{E}_A$, $\mathcal{E}_a$, and $\mathcal{E}[w]$ by the Leibniz rule. Note that since F is skew it yields an (pseudo-)orthogonal action pointwise and hence preserves the $SO(p+1, q+1)$ decompositions of tractor bundles.

Theorem 4.2: above is one of the primarily tools for our subsequent construction of symmetries. However, there are some conceptual gains in linking this to some related results and so we complete this section with these observations.

The double-D operator discussed above reflects a more general operator called *fundamental derivative* from Ref. 10 (where it is called the fundamental-D operator). The specialisation of this to conformal geometry provides, for any natural bundle $\mathcal{V}$, a conformally invariant differential operator $\mathcal{D} : \mathcal{V} \to \mathcal{A} \otimes \mathcal{V}$, where $\mathcal{A} = \Lambda^2 \mathcal{T}$ is often called the *adjoint tractor bundle* (because it is modelled on $\mathfrak{g} = \mathfrak{so}_{s+1,s'+1}$). Since there is a natural inclusion $\mathcal{A} \hookrightarrow \operatorname{End}\mathcal{E}_\bullet$ via $\sharp$, we may form $((-1)$–times) the symmetrisation of the contracted composition, to be denoted by

$$\mathcal{D}^2 : \mathcal{V} \to (\operatorname{End}\mathcal{V}) \otimes \mathcal{V}.$$

In the abstract index notation, we write $\mathcal{D}_A{}^B$ (or $\mathcal{D}_{\mathbf{A}}$, using the identification $\mathcal{A} \cong \mathcal{E}_{A^1A^2}$) for the fundamental derivative and so $\mathcal{D}^2_{AB} = -\mathcal{D}^C{}_{(A}\mathcal{D}_{B)C}$.

We shall use $\mathcal{D}$ only on weighted tensor bundles $\mathcal{V} \subseteq (\bigotimes \mathcal{E}_a) \otimes (\bigotimes \mathcal{E}^b) \otimes \mathcal{E}_\bullet[w]$. Recall the fundamental derivative obeys the Leibniz rule and actually $\mathcal{D}_{\mathbf{A}} = \mathbb{D}_{\mathbf{A}}$ on irreducible bundles. (In fact, the double-D was defined in such way in Ref. 10.) To show the difference between $\mathbb{D}$ and $\mathcal{D}$ and, more generally, the analogue of (17) we shall need certain special tractor sections and their corresponding algebraic actions on tractor bundles as follows:

$$\begin{aligned} &\mathbb{H}_{\mathbf{AB}} = h_{A^0B^0}h_{A^1B^1}, && \mathbb{H}_{\mathbf{A}}\sharp = h_{A^0B^0}h_{A^1B^1}\sharp_{\mathbf{B}}, \\ &\widetilde{\mathbb{H}}_{AD\mathbf{BC}} = h_{(A|B^0|}h_{D)C^0}h_{B^1C^1}, && \widetilde{\mathbb{H}}_{AD}\sharp\sharp = h_{(A|B^0|}h_{D)C^0}h_{B^1C^1}\sharp_{\mathbf{B}}\sharp_{\mathbf{C}} \end{aligned} \tag{20}$$

where, as usual, we skew over the index pairs A^0A^1, B^0B^1, and C^0C^1. Here, the subscript of $\sharp$ indicates which skew symmetric component is considered as an endomorphism. That is, for example, $(\mathbb{H}_{\mathbf{A}}\sharp f)_C = h_{A^0C}f_{A^1}$ for $f_C \in \mathcal{E}_C$, and this extends to tensor powers of the tractor bundle by the Leibniz rule. It also indicates the order of applications of these endomorphisms in the case of $\widetilde{\mathbb{H}}$.

We need $\mathcal{D}$ only up to a (nonzero) scalar multiple and our choice will differ from Ref. 8 by -1. Explicit formulas of $\mathcal{D}$ and $\mathcal{D}^2$ on weighted tractor bundles $\mathcal{E}_\bullet[w]$ are given by

$$\begin{aligned} \mathcal{D}_{\mathbf{A}} &= 2(w\,\mathbb{W}_{\mathbf{A}} + \mathbb{X}_{\mathbf{A}}^a\nabla_a + \mathbb{H}_{\mathbf{A}}\sharp), \\ \mathcal{D}^2_{AD} &= -(wh_{AD} + X_{(A}D_{D)} + 4h_{(A|B^0|}\mathbb{D}_{D)B^1}\sharp_{\mathbf{B}} - 4\widetilde{\mathbb{H}}_{AD}\sharp\sharp), \end{aligned} \tag{21}$$

where we skew over $[B^0B^1]$ and $\sharp_{\mathbf{B}}$ indicates the skewed symmetric component which is considered as an endomorphism. That is, $\mathcal{D}_{\mathbf{A}} = \mathbb{D}_{\mathbf{A}} + 2\mathbb{H}_{\mathbf{A}}\sharp$.

Corollary 4.3: Assume the locally conformally flat setting. Let $P : \mathcal{V} \to \mathcal{W}$ be a conformally invariant operator between irreducible weighted tensor bundles $\mathcal{V}$ and $\mathcal{W}$. Then

$$P^\nabla \mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p} = \mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p}P : \mathcal{V} \to \mathcal{E}_{\mathbf{A}_1\ldots\mathbf{A}_p} \otimes \mathcal{W}.$$

Proof: We shall use an induction. The case $p = 1$ is obvious as $\mathcal{D}_{\mathbf{A}} = \mathbb{D}_{\mathbf{A}}$ on $\mathcal{V}$ and $\mathcal{W}$. Assume the corollary holds for a fixed integer p. Then

$$\mathcal{D}_{\mathbf{A}_0}\mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p} = \mathbb{D}_{\mathbf{A}_0}\mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p} + 2\mathbb{H}_{\mathbf{A}_0}\sharp\,\mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p}.$$

The operator P commutes with the first term on the right-hand side using $[P, \mathbb{D}_{\mathbf{A}_0}] = 0$ and the inductive assumption. Since the second term involves only $\mathcal{D}_{\mathbf{A}_1}\cdots\mathcal{D}_{\mathbf{A}_p}$ with some additional trace factors, P commutes with the second term (using the induction) as well. □

Lemma 4.4: Assume the locally conformally flat setting. Then $[\mathcal{D}_{\mathbf{A}}, \mathbb{D}_{\mathbf{B}}] = 0$ on $\mathcal{V} \otimes \mathcal{E}_\bullet$ for $\mathcal{V}$ irreducible.

Proof: From (17) and (21), we obtain

$$[\mathcal{D}_{\mathbf{A}}, \mathbb{D}_{\mathbf{B}}] = [\mathcal{D}_{\mathbf{A}}, \mathcal{D}_{\mathbf{B}}] - 2\mathcal{D}_{\mathbf{A}}\mathbb{H}_{\mathbf{B}}\sharp + 2\mathbb{H}_{\mathbf{B}}\sharp\mathcal{D}_{\mathbf{A}} = [\mathcal{D}_{\mathbf{A}}, \mathcal{D}_{\mathbf{B}}] + 4h_{B^0A^0}\mathcal{D}_{B^1A^1}.$$

Thus contracting arbitrary sections $I^{\mathbf{A}} \in \mathcal{E}^{\mathbf{A}}$, $\bar{I}^{\mathbf{B}} \in \mathcal{E}^{\mathbf{B}}$ into the previous display we get

$$I^{\mathbf{A}}\bar{I}^{\mathbf{B}}[\mathcal{D}_{\mathbf{A}}, \mathbb{D}_{\mathbf{B}}] = I^{\mathbf{A}}\bar{I}^{\mathbf{B}}[\mathcal{D}_{\mathbf{A}}, \mathcal{D}_{\mathbf{B}}] + 4I^{A^1P}\bar{I}_P{}^{B^1}\mathcal{D}_{A^1B^1}.$$

We put $[I, \bar{I}]^{\mathbf{C}} := 4I^{C^0P}\bar{I}_P{}^{C^1}$. On the one hand, $I^{\mathbf{A}}\bar{I}^{\mathbf{B}}[\mathcal{D}_{\mathbf{A}}, \mathcal{D}_{\mathbf{B}}]$ is given in, Proposition, p. 21 of Ref. 10. On the other hand, a direct computation verifies the statement on $\mathcal{E}_\bullet[w]$, cf. (40) below. Therefore by restricting to this case (of $\mathcal{E}_\bullet[w]$), it follows that our notation $[I, \bar{I}]$ coincides precisely with $\{I, \bar{I}\}$ used in Ref. 10. Thus using, Proposition, p. 21 of Ref. 10 on $\mathcal{V} \otimes \mathcal{E}_\bullet[w]$, the lemma follows.

□

Remark 4.5: There is also a more conceptual proof of the previous corollary (thus also of Theorem 4.2). Motivated by, Theorem 3.3 of Ref. 10, we note that, at each point $x \in M$, the section

$$\overline{\mathbb{D}}^{(k)}\sigma := (\sigma, \mathbb{D}\sigma, \mathbb{D}^{(2)}\sigma = \mathbb{D}\mathbb{D}\sigma, \dots, \mathbb{D}^{(k)}\sigma) \in \overline{\mathcal{A}}^{(k)}(\mathcal{V}) \subseteq \mathcal{V} \oplus \mathcal{E}_{\mathbf{A}} \otimes \mathcal{V} \oplus \dots \oplus \bigotimes^{k} \mathcal{E}_{\mathbf{A}} \otimes \mathcal{V}$$

contains the data of the entire k-jet of $\sigma \in \mathcal{V}$. Note although here we assume $\mathcal{V}$ is irreducible, the operator $\overline{\mathbb{D}}^{(k)}$ is defined also on bundles of the form $\mathcal{V} \otimes \mathcal{E}_{\bullet}$. From the general theory, the subbundle $\overline{\mathcal{A}}^{(k)}(\mathcal{V})$ (defined in the obvious way by the display) is an induced bundle of a principle H–bundle where $H \subseteq SO(s + 1, s' + 1)$ is a parabolic subgroup. It is straightforward to argue that any conformally invariant k-order operator on $\mathcal{V}$ is given by $\overline{\mathbb{D}}^{(k)}$ followed by a suitable H-homomorphism Φ on this subbundle. We denote this homomorphism by Φ_P in the case of the operator P.

Our aim is to commute $P = \Phi_P \circ \overline{\mathbb{D}}^{(k)}$ and $\mathbb{D}_{\mathbf{B}}$. More precisely, we put

$$P^{\nabla} := (\mathrm{id}\,|_{\mathcal{E}_{\mathbf{B}}} \otimes \Phi_P) \circ \overline{\mathbb{D}}^{(k)} : \mathcal{E}_{\mathbf{B}} \otimes \mathcal{V} \to \mathcal{E}_{\mathbf{B}} \otimes \mathcal{W}.$$

Observe the formulas for $\overline{\mathbb{D}}^{(k)} : \mathcal{V} \to \overline{\mathcal{A}}^{(k)}(\mathcal{V})$ and $\overline{\mathbb{D}}^{(k)} : \mathcal{E}_{\mathbf{B}} \otimes \mathcal{V} \to \mathcal{E}_{\mathbf{B}} \otimes \overline{\mathcal{A}}^{(k)}(\mathcal{V})$ are formally the same. (Note the implicit ∇ is interpreted as the coupled Levi-Civita-tractor connection in the latter case). That means also the formulas for $P : \mathcal{V} \to \mathcal{W}$ and $P^{\nabla} : \mathcal{E}_{\mathbf{B}} \otimes \mathcal{V} \to \mathcal{E}_{\mathbf{B}} \otimes \mathcal{W}$ are given by the same formal expression. Hence our definition of P^{∇} coincides with that given before Theorem 4.2.

Now we are ready to show that $\mathcal{D}_{\mathbf{B}} P = P^{\nabla} \mathcal{D}_{\mathbf{B}}$ on $\mathcal{V}$, i.e.,

$$(\Phi_p \otimes \mathrm{id}\,|_{\mathcal{E}_{\mathbf{B}}}) \circ \overline{\mathbb{D}}^{(k)} \mathcal{D}_{\mathbf{B}} = \mathcal{D}_{\mathbf{B}}\big(\Phi_p \circ \overline{\mathbb{D}}^{(k)}\big) : \mathcal{V} \to \mathcal{E}_{\mathbf{B}} \otimes \mathcal{W}.$$

Clearly $\mathcal{D}_{\mathbf{B}} \Phi_P = (\Phi_P \otimes \mathrm{id}\,|_{\mathcal{E}_{\mathbf{B}}}) \mathcal{D}_{\mathbf{B}}$. Since $[\mathcal{D}_{\mathbf{B}}, \mathbb{D}_{\mathbf{A}}] = 0$ from Lemma 4.4 and $\mathcal{D}_{\mathbf{B}}$ preserves subbundles (of the space $\mathcal{D}_{\mathbf{B}}$ acts on), $(\Phi_P \otimes \mathrm{id}\,|_{\mathcal{E}_{\mathbf{B}}}) \mathbb{D}_{\mathbf{A}_1} \dots \mathcal{D}_{\mathbf{B}} \dots \mathbb{D}_{\mathbf{A}_i}$ is conformally invariant and the previous display follows.

Henceforth we shall write P instead of P^{∇} for simplicity. Finally note although we have shown $[\mathcal{D}_{\mathbf{B}}, P] = 0$ only on an irreducible $\mathcal{V}$, the same reasoning shows $[\mathcal{D}_{\mathbf{B}}, P] = 0$ also on bundles $\mathcal{V} \otimes \mathcal{E}_{\bullet}$. Therefore, this remark offers an alternative proof of the previous corollary (thus also of Theorem 4.2).

The previous results provide an obvious way to construct symmetries of conformally invariant operators. Assume the section

$$I^{\mathbf{A}_1 \dots \mathbf{A}_p B_1 B_1' \dots B_r B_r'} \in \mathcal{E}^{\mathbf{A}_1 \dots \mathbf{A}_p B_1 B_1' \dots B_r B_r'}$$

is parallel. Then from Theorem 4.2 and Corollary 4.3 the differential operators

$$\begin{aligned} S &= I^{\mathbf{A}_1 \dots \mathbf{A}_p B_1 B_1' \dots B_r B_r'} \mathcal{D}_{\mathbf{A}_1} \dots \mathcal{D}_{\mathbf{A}_p} \mathcal{D}^2_{B_1 B_1'} \dots \mathcal{D}^2_{B_r B_r'} \quad \text{and} \\ \mathbb{S} &= I^{\mathbf{A}_1 \dots \mathbf{A}_p B_1 B_1' \dots B_r B_r'} \mathbb{D}_{\mathbf{A}_1} \dots \mathbb{D}_{\mathbf{A}_p} \mathbb{D}^2_{B_1 B_1'} \dots \mathbb{D}^2_{B_r B_r'} \end{aligned} \tag{22}$$

commute with P. That is S and $\mathbb{S}$ are symmetries of the operator P.

Proposition 4.6: Assume the tractor $I^{\mathbf{A}_1 \dots \mathbf{A}_p B_1 B_1' \dots B_r B_r'}$ is parallel and irreducible, $I = I|_{\boxtimes}$. Then $S = \mathbb{S}$ on $\mathcal{E}[w]$.

Proof: Consider the parallel and irreducible tractor $I^{\mathbf{A}_1 \dots \mathbf{A}_p B_1 B_1' \dots B_r B_r'}$ and the symmetry S from (22). Since $\mathcal{D}_{\mathbf{A}} = \mathbb{D}_{\mathbf{A}} + 2\mathbb{H}_{\mathbf{A}}\sharp$, the difference

$$\mathcal{D}_{\mathbf{A}_1} \mathcal{D}_{\mathbf{A}_2} \dots \mathcal{D}_{\mathbf{A}_p} \mathcal{D}^2_{B_1 B_1'} \dots \mathcal{D}^2_{B_r B_r'} - \mathbb{D}_{\mathbf{A}_1} \mathcal{D}_{\mathbf{A}_2} \dots \mathcal{D}_{\mathbf{A}_p} \mathcal{D}^2_{B_1 B_1'} \dots \mathcal{D}^2_{B_r B_r'}$$

lives in the trace part of $\mathcal{E}_{\mathbf{A}_1 \dots \mathbf{A}_p B_1 B_1' \dots B_r B_r'}[w]$, cf. (20). Therefore, this difference is killed after contraction with $I_{\varphi}^{\mathbf{A}_1 \dots \mathbf{A}_p B_1 B_1' \dots B_r B_r'}$. Repeating this argument for $\mathcal{D}_{\mathbf{A}_2}, \dots, \mathcal{D}_{\mathbf{A}_p}$, we obtain

$$S = I^{\mathbf{A}_1 \dots \mathbf{A}_p B_1 B_1' \dots B_r B_r'} \mathbb{D}_{\mathbf{A}_1} \dots \mathbb{D}_{\mathbf{A}_p} \mathcal{D}^2_{B_1 B_1'} \dots \mathcal{D}^2_{B_r B_r'} : \mathcal{E}[w] \to \mathcal{E}[w].$$

Now we replace $\mathcal{D}^2_{B_1B'_1}$ in the previous display by $\mathbb{D}^2_{B_1B'_1}$. Note $I^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B'_1\ldots B_rB'_r}$ commutes with $\mathbb{D}_{\mathbf{A}_i}$ and consider $I^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B'_1\ldots B_rB'_r}$ contracted with

$$\begin{aligned}\mathcal{D}^2_{B_1B'_1}\mathcal{D}^2_{B_2B'_2}\cdots\mathcal{D}^2_{B_rB'_r} - \mathbb{D}^2_{B_1B'_1}\mathcal{D}^2_{B_2B'_2}\cdots\mathcal{D}^2_{B_rB'_r}\\ = -\big(4h_{(B_1|C^0|}\mathbb{D}_{B'_1)C^1}\sharp_{\mathbf{C}} - 4\widetilde{\mathbb{H}}_{B_1B'_1}\sharp\sharp\big)\mathcal{D}^2_{B_2B'_2}\cdots\mathcal{D}^2_{B_rB'_r},\end{aligned}$$

where we have used (21) and (17). The second term in the round brackets on the right-hand side vanishes after the contraction (using trace-freeness of I again) so it remains to contract $I^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B'_1\ldots B_rB'_r}$ with

$$\begin{aligned}4h_{C^0(B_1}\mathbb{D}_{B'_1)C^1}\sharp_{\mathbf{C}}\mathcal{D}^2_{(B_2B'_2}\cdots\mathcal{D}^2_{B_rB'_r)} = 4(r-1)h_{(B_2B_1}\mathbb{D}_{B'_1}{}^{P}\mathcal{D}^2_{|P|B'_2}\cdots\mathcal{D}^2_{B_rB'_r)}\\ -4(r+1)\mathbb{D}_{(B'_1B_2}\mathcal{D}^2_{B_1B'_2}\cdots\mathcal{D}^2_{B_rB'_r)}.\end{aligned}$$

Here, we have used the fact that the indices $B_1B'_1\ldots B_rB'_r$ of I are symmetric (because I is irreducible). Now the second term on the right-hand side is zero due to skew symmetry of indices of $\mathbb{D}_{B'_1B_2}$ and the first term vanishes after contraction with I which is trace-free. Repeating the same argument for $\mathcal{D}^2_{B_1B'_2}, \ldots, \mathcal{D}^2_{B_rB'_r}$, the proposition follows. □

Note an analogous statement to the proposition above holds, where $\mathcal{E}[w]$ is replaced by any irreducible bundle $\mathcal{V}$. This may be proved along the same lines as in the treatment above. However, since the details are technical and not required here, this proof is omitted.

Finally note the operators given by (22) are also well defined on bundles $\mathcal{E}_\bullet[w]$. In this setting, however, they yield generally different operators $\mathcal{E}_\bullet[w] \to \mathcal{E}_\bullet[w]$.

V. A CONSTRUCTION OF SYMMETRIES

We are now ready to construct canonical symmetries. For a section $\varphi_r^{a_1\ldots a_p} \in \mathcal{E}^{(a_1\ldots a_p)_0}[2r]$, we shall define the operators (S_φ, S'_φ) where S_φ and S'_φ have leading term $\varphi_r^{a_1\ldots a_p}\nabla_{a_1}\cdots\nabla_{a_p}\Delta^r$. To do this, we use the bijective correspondence between the linear space of solutions of (3) and certain finite-dimensional $\mathfrak{g}$–modules, cf. the discussion around (4). Explicitly, this is given by differential prolongation in the form of a differential splitting operator $\mathcal{E}^{(a_1\ldots a_p)_0}[2r] \to \mathcal{E}^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B'_1\ldots B_rB'_r}|_{\boxtimes}$. There are many ways of constructing this, but for our current purposes the splitting operator can be conveniently expressed using the fundamental derivative. There is a certain operator $\mathcal{C}$ known as the *curved Casimir*[15] which is given by $h^{AB}\mathcal{D}^2_{AB}$. (Properties of the splitting operators coming from $\mathcal{C}$ will be used in Proposition 6.1.) This acts on any natural bundle and, in particular, on weighted tractor bundles. It can thus be iterated and we shall use operators polynomial in $\mathcal{C}$. In particular, one gets the splitting operator as

$$\varphi_r^{a_1\ldots a_p} \mapsto \mathbb{Y}^{\mathbf{A}_1}{}_{a_1}\cdots\mathbb{Y}^{\mathbf{A}_p}{}_{a_p}Y^{B_1}Y^{B'_1}\cdots Y^{B_r}Y^{B'_r}\varphi_r^{a_1\ldots a_p} \xrightarrow{Q} \mathcal{E}^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B'_1\ldots B_rB'_r}, \tag{23}$$

where Q is an operator polynomial in $\mathcal{C}$, and hence is polynomial in $\mathcal{D}$, see Refs. 15 and 32. We shall denote the image by $I_\varphi^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B'_1\ldots B_rB'_r} \in \mathcal{E}^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B'_1\ldots B_rB'_r}|_{\boxtimes}$. The main point we need is that the tractor I_φ is parallel if and only if φ is a solution of the operator (3).

Definition 5.1: Given $\varphi = \varphi_r^{(a_1\ldots a_p)_0} \in \mathcal{E}^{(a_1\ldots a_p)_0}[2r]$, $r, p \geq 0$, we shall associate a differential operator S_φ as follows. Let I_φ denote the tractor corresponding to φ, in the sense of the discussion surrounding (23) above. Then via (22),

$$S_\varphi := I_\varphi^{\mathbf{A}_1\ldots\mathbf{A}_pB_1B'_1\ldots B_rB'_r}\mathbb{D}_{\mathbf{A}_1}\ldots\mathbb{D}_{\mathbf{A}_p}\mathbb{D}^2_{B_1B'_1}\ldots\mathbb{D}^2_{B_rB'_r}, \tag{24}$$

is a well-defined differential operator $S_\varphi : \mathcal{V} \to \mathcal{V}$, for any weighted tensor-tractor bundle $\mathcal{V}$.

Assume φ is a solution of (3), and so the tractor I_φ is parallel. It follows immediately from Theorem 4.2, and the fact that I_φ is parallel, that S_φ is a universal symmetry operator. That is, using also that $\varphi \mapsto I_\varphi$ is a splitting operator, we have the following.

Theorem 5.2: *On a conformally flat manifold, let $P : \mathcal{V} \to \mathcal{W}$ be a conformally invariant operator between irreducible tensor bundles $\mathcal{V}$ and $\mathcal{W}$, and suppose that $\varphi = \varphi_r^{(a_1 \ldots a_p)_0} \in \mathcal{E}^{(a_1 \ldots a_p)_0}[2r]$, $r, p \geq 0$ is a solution of (3). Then with $S_\varphi : \mathcal{V} \to \mathcal{V}$ and $S'_\varphi : \mathcal{W} \to \mathcal{W}$ given by (24), the pair (S_φ, S'_φ) is symmetry of P. Assuming P is the GJMS operator P_k then for $\varphi \neq 0$ and $r < k$, this is a nontrivial symmetry.*

Proof: It remains to prove the last claim. Note that acting on any density bundle, φ is the leading symbol of the operator (24). This follows from the construction of S_φ and is also shown by Proposition 6.1 (which we will come to later). Thus the leading term does not have Δ^k as the right factor for $r < k$. □

Note that S_φ and S'_φ are not the same differential operators. The point is that (24) really defines a family of differential operators parametrised by the space of domain bundles.

We shall henceforth only pursue the case that P is a GJMS operator. As mentioned in the proof of the theorem, φ is then the leading symbol of the operator (24). Also note that in this case the use of $\mathcal{D}$ and $\mathcal{D}^2$ rather than $\mathbb{D}$ and $\mathbb{D}^2$ (respectively) in (24) yields the same symmetry, as follows from Proposition 4.6.

Remark: Consider an operator $F : \mathcal{E}[w] \to \mathcal{E}[w]$, of order $\tilde{p} \geq 0$, on a smooth conformal manifold manifold $(M, [g])$ and its symbol $\tilde{\varphi}^{(a_1 \ldots a_{\tilde{p}})} \in \mathcal{E}^{(a_1 \ldots a_{\tilde{p}})}$. Then, via the conformal structure $[g]$ we may decompose $\tilde{\varphi}$ into irreducibles. Each irreducible component φ of $\tilde{\varphi}$ can be realised as $\varphi^{(a_1 \ldots a_p)_0} \in \mathcal{E}^{(a_1 \ldots a_p)_0}[2r]$ where $p = \tilde{p} - 2r$. Thus we have also the operator S_φ, constructed as above except that we here do not require φ to solve (3). We may then take the difference $F - S_\varphi : \mathcal{E}[w] \to \mathcal{E}[w]$. Now the whole procedure can be repeated for the operator $F - S_\varphi$. It is clear that after a finite number of steps we obtain the form $F = \sum_{\varphi \in U} S_\varphi$ for a (finite) index set $U \subseteq \mathbb{N}$. That is, given an operator $F : \mathcal{E}[w] \to \mathcal{E}[w]$ on a smooth manifold M, any conformal structure on M yields a decomposition of F as a sum of canonical operators S_φ.

In the other direction, the operators S_φ provide the conformally invariant quantisation introduced in Ref. 18, in particular the special case.[18], 3.1] Also note Sec. IV shows how to rewrite the general construction[11] using an affine connection.

VI. CLASSIFICATION OF LEADING TERMS OF SYMMETRIES

According to the discussion following Theorem 2.4, the problem of conformal symmetries for the GJMS operators (on locally conformally flat manifolds) is reduced to the setting of Theorem 2.1. So throughout this section, we work on $\mathbb{E}^{s,s'}$ equipped with the standard flat diagonal signature (s, s') metric g with $s + s' =: n \geq 3$.

All linear differential operators $L : \mathcal{E}[w] \to \mathcal{E}[w]$ may be expressed as sums of the form

$$L = \sum_{p,r \geq 0} \varphi_r^{a_1 \ldots a_p} (\nabla_{a_1} \cdots \nabla_{a_p}) \Delta^r, \quad \varphi_r^{a_1 \ldots a_p} \in \mathcal{E}^{(a_1 \ldots a_p)_0}[2r] = \mathcal{E}_r^{(p)_0}. \tag{25}$$

We shall describe the right-hand side here as a *standard expression* for L. Moreover, we shall typically use the notation $\varphi_r^p (\odot^p \nabla) \Delta^r$ as a shorthand for the operator $\varphi_r^{a_1 \ldots a_p} (\nabla_{a_1} \cdots \nabla_{a_p}) \Delta^r$ in the displayed sum (as the details of the internal index contractions are not important for our arguments).

We use the standard expressions as above to analyse the structure of potential symmetries and their compositions with Δ^k. In particular, we shall use the following properties/descriptions of a given coefficient φ_r^p. We shall write $o(\varphi_r^p) = p + 2r$ and term this the *formal order* of φ_r^p and $\ell(\varphi_r^p) = p + r$ which will be termed *level* of φ_r^p. (These reflect properties of terms $\varphi_r^{a_1 \ldots a_p} (\nabla_{a_1} \cdots \nabla_{a_p}) \Delta^r$ and how they appear naturally in appropriate tractor formulas. However, these quantities are fully determined by the coefficients φ_r^p, so it is sufficient to consider formal order and level of coefficients.) We also say $\left[\begin{smallmatrix} p \\ r \end{smallmatrix} \right]$ is the *type* of φ_r^p. We shall write $o(R) = a$ and $\ell(R) = b$ if all terms of a differential operator $R : \mathcal{E}[w] \to \mathcal{E}[w]$ are of the formal order at most a, respectively, level at most b. Finally, if L is a symmetry of Δ^k, then we shall say L is a *normal* symmetry (of Δ^k) if $r < k$ for all terms in the standard expression (25). Modulo trivial symmetries, any symmetry of Δ^k may be represented by a normal symmetry. (More generally, this holds for all operators on functions, cf. the remark following Proposition 6.1.)

Further, we shall need a suitable ordering of the terms in a standard expression. This will be defined via the coefficients as follows:

$$\varphi^p_r \lhd \psi^{p'}_{r'} \quad \text{iff} \quad \ell(\varphi^p_r) < \ell(\psi^{p'}_{r'}) \text{ or } \big(\ell(\varphi^p_r) = \ell(\psi^{p'}_{r'})\big) \wedge \big(o(\varphi^p_r) < o(\psi^{p'}_{r'})\big). \tag{26}$$

Since the coefficient φ^p_r determines a corresponding term in the standard expression completely, we shall use the ordering $\lhd$ for both coefficients and terms of an operator (25).

In the following, we shall use the terminology the *greatest term* (or coefficient) with respect to the ordering $\lhd$, the *leading term* (i.e., the term of the highest formal order o) and the *term of highest level*, which refers to the quantity ℓ defined above. We would like to emphasise that all these characteristics of terms are generally different.

First, we shall study the canonical symmetries. Since these are constructed using tractor operators we need a further weight type measure as follows. In the tractor formulas, we use strings of the symbols X, Y, Z and $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ and $\mathbb{W}$ from Sec. III A. We define the *homogeneity* $\mathrm{h}(\omega)$ of a string $\omega \in \{X, Y, Z, \mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}\}$ by

$$\mathrm{h}(Y) = 1, \mathrm{h}(Z) = 0, \mathrm{h}(X) = -1, \mathrm{h}(\mathbb{Y}) = 1, \mathrm{h}(\mathbb{Z}) = h(\mathbb{W}) = 0, \mathrm{h}(\mathbb{X}) = -1, \tag{27}$$
$$\text{and} \quad \mathrm{h}(\omega_1\omega_2) := \mathrm{h}(\omega_1) + \mathrm{h}(\omega_2),$$

where $\omega_1\omega_2$ means a concatenation of the strings ω_1 and ω_2.

Now we are set to describe properties of the canonical symmetries (and more generally operators of the form (24)), as follows.

Proposition 4.1: Consider $\varphi = \varphi^p_r \in (\odot^p TM) \otimes \mathcal{E}[2r]$ and the corresponding operators $S_\varphi : \mathcal{E}[w] \to \mathcal{E}[w]$ and $S'_\varphi : \mathcal{E}[w'] \to \mathcal{E}[w']$, $w, w' \in \mathbb{R}$ given by (24). Then, in the standard expressions for S_φ and S'_φ, the following properties hold:

(i) *S_φ and S'_φ have the same leading term φ.*

(ii) *$\ell(S'_\varphi) = \ell(S_\varphi) = r + p = \ell(\varphi^p_r)$, that is every term ψ of S_φ or S'_φ satisfies $\ell(\psi) \leq p + r$. Moreover, the greatest terms of S_φ and S'_φ have the coefficient φ.*

(iii) *$o(S'_\varphi) = o(S_\varphi) = p + 2r = o(\varphi^p_r)$, that is every term ψ of S_φ or S'_φ satisfies $o(\psi) \leq p + 2r$. Moreover, the equality happens only for $\psi = \varphi$.*

(iv) *Every term ψ of type $\left[\begin{smallmatrix}\bar{p}\\ \bar{r}\end{smallmatrix}\right]$ of S_φ or S'_φ satisfies $r \geq \bar{r}$.*

Remark: We shall actually use the proposition only in the case $[\nabla^{2r+1}\varphi]|_{\boxtimes} = 0$, i.e., when (S_φ, S'_φ) is the symmetry pair. But note that part (iv) means, in particular, that any operator L on functions satisfies, modulo trivial symmetries of Δ^k that $r < k$ for all terms in the standard expression (25) of L.

Proof: First note that because S_φ and S'_φ are given by the same operator (24) acting on different density bundles, it turns out to be sufficient to establish facts only for S_φ. From (24) S_φ is defined as the contraction of the parallel tractor $I_\varphi^{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r}$, corresponding to φ, with the operator

$$\widetilde{\mathbb{D}}_{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r} := \mathbb{D}_{\mathbf{A}_1} \ldots \mathbb{D}_{\mathbf{A}_p} \mathbb{D}^2_{B_1 B'_1} \ldots \mathbb{D}^2_{B_r B'_r} : \mathcal{E}_\bullet[w] \longrightarrow \mathcal{E}_{\bullet\,\mathbf{A}_1\ldots\mathbf{A}_p B_1 B'_1 \ldots B_r B'_r}[w].$$

We need some broad facts about the structure of the tractor formulas for I_φ and $\widetilde{\mathbb{D}}$. When working in a metric scale and using (12), (10), (15), and (17) it follows that terms of these are built, respectively, from tensor fields and tensor valued differential operators contracted into "projectors"

$$\omega \in \mathcal{B}.$$

Here, $\mathcal{B}$ is a set of fields taking values in the appropriate tractor bundle tensor product with an irreducible weighted trace-free tensor bundle. Each element $\omega \in \mathcal{B}$ is an appropriate projection (onto the irreducible part with respect to the tensor indices) of a p-fold tensor product of elements from $\{\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \mathbb{W}\}$ with a $2r$-fold tensor product of elements from $\{X, Y, Z\}$, and we may take $\mathcal{B}$ to be all such. Similarly, the elements of $\mathcal{B}$ can be considered as "injectors," i.e., a mapping going in the opposite direction. For example, since I_φ is obtained from φ by a splitting operator, it has the form

$$I_\varphi^{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B_1'\ldots B_r B_r'} = \sum_{\omega\in\mathcal{B}} \omega^{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B_1'\ldots B_r B_r'} \cdot F_\omega(\varphi) \tag{28}$$

where, for each $\omega \in \mathcal{B}$, $F_\omega(\varphi)$ is the result of a (weighted tensor valued) differential operator F_ω acting on φ (a section of $(\odot^p TM)\otimes\mathcal{E}[2r]$) and "$\cdot$" indicates a contraction of tensor indices (which are suppressed); cf. (43) below which shows I_φ for $\varphi^a \in \mathcal{E}^a$ explicitly. Note also that we sum over all strings in $\mathcal{B}$ in the previous display, so many of the F_ω will be zero. Similarly, it follows from the definition of $\widetilde{\mathbb{D}}$ that

$$\widetilde{\mathbb{D}}_{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B_1'\ldots B_r B_r'} = \sum_{\omega\in\mathcal{B}} \omega_{\mathbf{A}_1\ldots\mathbf{A}_p B_1 B_1'\ldots B_r B_r'} \cdot G_\omega, \tag{29}$$

where G_ω is a (weighted tensor valued) differential operator acting on densities and, again, "$\cdot$" denotes contraction of (suppressed) tensor indices. See (17) and (39) for explicit examples. Contracting the last two displays, we obtain the canonical symmetry S_φ as in (24). Thus, using (11) and the surrounding observations, we have

$$S_\varphi = \sum_{\substack{\omega,\,\omega'\,\in\,\mathcal{B},\\ \mathrm{h}(\omega)+h(\omega')=0}} (F_\omega(\varphi))\cdot G_{\omega'},$$

where "$\cdot$" indicates the contraction of suppressed tensor indices. Note pairs (ω, ω') not satisfying $\mathrm{h}(\omega) + \mathrm{h}(\omega') = 0$ have dropped out of the sum by properties of the tractor metric. (Also note that the same property implies that if the tensor indices of F_ω and $G_{\omega'}$ are not compatible for complete contraction, then the term $(F_\omega(\varphi))\cdot G_{\omega'}$ is necessarily zero.)

The differential order of F_ω (and similarly $G_{\omega'}$) is exactly the maximal number of ∇'s in the corresponding expression in the splitting operator. (We consider formulas for splitting operators obtained using the curved Casimir $\mathcal{C} = h^{AB}\mathcal{D}^2_{AB}$ here.) Denoting the differential order of F_ω and $G_{\omega'}$ (in (28) and (29)) by, respectively, $o(F_\omega)$ and $o(G_{\omega'})$, we have

$$\mathrm{h}(\omega) + o(F_\omega) = p + 2r \quad \text{and} \quad \mathrm{h}(\omega') + o(G_{\omega'}) = 0, \quad \omega, \omega' \in \mathcal{B}.$$

Here, the first equality follows from (23) and the properties of splitting operators. The second follows from the definition of $\widetilde{\mathbb{D}}$ (in particular, from the tractor expressions for $\mathbb{D}$ and $\mathbb{D}^2$ in (17), (10), and (15). Summing up the equalities in the previous display we see that

$$S_\varphi = \sum_{\substack{\omega,\,\omega'\,\in\,\mathcal{B},\\ o(F_\omega)+o(G_{\omega'})=p+2r}} (F_\omega(\varphi))\cdot G_{\omega'}. \tag{30}$$

Note that all tractor indices have been eliminated, the formula (30) for S_φ is expressed using tensor operators and contractions only. Now consider a summand $(F_\omega(\varphi))\cdot G_{\omega'}$ of S_φ as in (30). First, $o(F_\omega) + o(G_{\omega'}) = p + 2r$ implies $o(G_{\omega'}) \le p + 2r$; moreover, the equality can happen only if $F_\omega = \mathrm{id}$ (up to a nonzero scalar multiple), since (23) is a differential splitting operator. For the same reason, this term does occur. In the previous display the term with $F_\omega = \mathrm{id}$ clearly recovers the highest order term, i.e., the leading term. Therefore (i) follows.

Now by assumption $F_\omega(\varphi)$ is irreducible. Since $S_\varphi : \mathcal{E}[w] \to \mathcal{E}[w]$, it follows from (25) that in the standard expression $(F_\omega(\varphi))\cdot G_{\omega'} = \gamma^{a_1\ldots a_{\bar p}}\nabla_{a_1}\ldots\nabla_{a_{\bar p}}\Delta^{\bar r}$, $\bar p, \bar r \ge 0$, where γ is symmetric and trace-free. In fact, it follows from the form of I_φ and $\widetilde{\mathbb{D}}$ that $F_\omega(\varphi) = \gamma^{a_1\ldots a_{\bar p}}$ and $G_{\omega'} = \nabla_{a_1}\ldots\nabla_{a_{\bar p}}\Delta^{\bar r}$. We denote the type of $F_\omega(\varphi)$ by $\left[\begin{smallmatrix}\bar p\\ \bar r\end{smallmatrix}\right]$. From this we get $o(G_{\omega'}) = \bar p + 2\bar r$ and, since F_ω takes φ of the type $\left[\begin{smallmatrix}p\\ r\end{smallmatrix}\right]$ to a section of the type $\left[\begin{smallmatrix}\bar p\\ \bar r\end{smallmatrix}\right]$, we get $o(F_\omega) \ge |p - \bar p|$. (The point is that each application of the Levi-Civita connection may increase or decrease the rank by 1, and this is the only way the rank may change.) These properties hold for every (irreducible) term $(F_\omega(\varphi)) \cdot G_{\omega'}$ in (30). Therefore,

$$p + 2r = o(F_\omega) + o(G_{\omega'}) \geq |p - \bar{p}| + \bar{p} + 2\bar{r}$$

using (30). We prove (ii), (iii), and (iv) separately in cases $p \geq \bar{p}$ and $p \leq \bar{p}$. If $p \geq \bar{p}$, then the previous display says $p + 2r \geq p + 2\bar{r}$ hence $r \geq \bar{r}$. This implies $p + r \geq \bar{p} + \bar{r}$ and $p + 2r \geq \bar{p} + 2\bar{r}$. If $p \leq \bar{p}$, then the previous display means $2p + 2r \geq 2\bar{p} + 2\bar{r}$ hence $p + r \geq \bar{p} + \bar{r}$. The latter inequality with $p \leq \bar{p}$ yields $r \geq \bar{r}$ and so $p + 2r \geq \bar{p} + 2\bar{r}$. This show (iv) and the inequalities in (ii) and (iii). Now when equality holds in (iii) then $p + 2r = \bar{p} + 2\bar{r}$. But then $p = \bar{p}$ from the previous display thus also $r = \bar{r}$. This means $o(G_{\omega'}) = p + 2r$ and $o(F_\omega) = 0$. Hence $F_\omega = id$, up to a multiple, and so if the term is nontrivial we recover the leading term. It remains to discuss the greatest term of S_φ. But since we have already proved the inequality in (ii), according to the ordering of (26) we need to consider the order of terms of level $p + r$. The maximal order is then characterised by (iii). □

Note the part (iii) of the previous proposition means that the canonical symmetry (S_φ, S'_φ), $\varphi_r^p \in (\otimes^p TM) \otimes \mathcal{E}[2r]$ is nontrivial for P_k, $k > r$. (The statement (iii) is actually stronger: no term in S_φ has Δ^k, $k > r$ as the right factor.)

Our strategy for classifying the leading terms of symmetries uses the ordering (26). We shall start with the greatest term and study what the symmetry condition imposes on its coefficient. We obtain the following.

Claim: Let $\varphi_i^j \in \mathcal{E}_i^{(j)_0}$ is the greatest coefficient of a symmetry T. Then $[\nabla^{2i+1}\varphi_i^j]_\boxtimes = 0$.

The claim forms the basis for an inductive procedure, as if $[\nabla^{2i+1}\varphi_i^j]_\boxtimes = 0$, then the greatest term of $T - S_{\varphi_i^j}$ is strictly smaller (with respect to $\rhd$) than φ_i^j, and using Proposition 6.1, we can replace T by $T - S_{\varphi_0^p}$ and apply the previous claim again.

The claim is proved as Proposition 6.3, and then the detailed inductive procedure is in the proof of Theorem 6.4. The proof of Proposition 6.3 requires a detailed analysis of certain terms. To demonstrate the technique, let us discuss an example first. Assume that (T, T') is a symmetry of $P_4 = \Delta^4$ of order p, i.e.,

$$\Delta^4 T = T'\Delta^4, \quad T = \sum_{2i+j \leq p, i<4} \varphi_i^j (\odot^j \nabla)\Delta^i,$$

where we have displayed the standard expression of T. Note we have not included terms with $i \geq 4$ as they may be eliminated by the addition of trivial symmetries of Δ^4. It is useful to write the terms of T in a table as follows:

$$\begin{array}{lllll}
\text{order } p: & \varphi_0^p(\odot^p\nabla) + & \varphi_1^{p-2}(\odot^{p-2}\nabla)\Delta^1 + & \varphi_2^{p-4}(\odot^{p-4}\nabla)\Delta^2 + & \varphi_3^{p-6}(\odot^{p-6}\nabla)\Delta^3 + \\
\text{order } p-1: & \varphi_0^{p-1}(\odot^{p-1}\nabla) + & \varphi_1^{p-3}(\odot^{p-3}\nabla)\Delta^1 + & \varphi_2^{p-5}(\odot^{p-5}\nabla)\Delta^2 + & \varphi_3^{p-7}(\odot^{p-7}\nabla)\Delta^3 + \\
\text{order } p-2: & \varphi_0^{p-2}(\odot^{p-2}\nabla) + & \varphi_1^{p-4}(\odot^{p-4}\nabla)\Delta^1 + & \varphi_2^{p-6}(\odot^{p-6}\nabla)\Delta^2 + & \varphi_3^{p-8}(\odot^{p-8}\nabla)\Delta^3 + \\
\text{order } p-3: & \varphi_0^{p-3}(\odot^{p-3}\nabla) + & \varphi_1^{p-5}(\odot^{p-5}\nabla)\Delta^1 + & \varphi_2^{p-7}(\odot^{p-7}\nabla)\Delta^2 + & \varphi_3^{p-9}(\odot^{p-9}\nabla)\Delta^3 + \\
\text{order } p-4: & \varphi_0^{p-4}(\odot^{p-4}\nabla) + & \varphi_1^{p-6}(\odot^{p-6}\nabla)\Delta^1 + & \varphi_2^{p-8}(\odot^{p-8}\nabla)\Delta^2 + & \varphi_3^{p-10}(\odot^{p-10}\nabla)\Delta^3 + \\
\vdots & \vdots \quad + & \vdots \quad + & \vdots \quad + & \vdots \quad +
\end{array}$$

Every line shows terms of the same formal order and moreover every antidiagonal shows terms of the same level. So the ordering (26) in this case means

$$\varphi_0^p \rhd \varphi_1^{p-2} \rhd \varphi_0^{p-1} \rhd \varphi_2^{p-4} \rhd \varphi_1^{p-3} \rhd \varphi_0^{p-2} \rhd \varphi_3^{p-6} \rhd \cdots .$$

Observe the level $\ell(R)$ of an operator R is increased by k under composition with Δ^k,

$$\ell(\Delta^k R) = \ell(R) + k.$$

Moreover, only terms of the highest level in R can contribute to terms of the highest level in $\Delta^k R$.

The greatest coefficient (with respect to $\rhd$) is φ_0^p. Recall $o(T) = p$ so we can assume $\ell(T) = p$ which means $\ell(\Delta^4 T) = p + 4$. Now we consider terms of the level $p + 4$ of $\Delta^4 T$. First, we commute all covariant derivatives ∇ to the right. In fact, it is sufficient for our purpose to consider only certain terms. First, we restrict to terms of the level $p + 4$ without a right factor Δ^4 and then take the candidate for the greatest among these. This is $(\nabla^1\varphi_0^p)(\odot^{p+1}\nabla)\Delta^3$. Since this does not have a right factor Δ^4, it has to vanish since T is a symmetry. Hence $(\nabla^1\varphi_0^p)_{\boxtimes} = 0$, which means that φ_0^p is a conformal Killing tensor. Now we replace the symmetry T by $T - S_{\varphi_0^p}$; this is also a symmetry. The greatest coefficient of $T - S_{\varphi_0^p}$ is now strictly smaller (with respect to $\rhd$) than the greatest coefficient of T. (Here we have adjusted $S_{\varphi_0^p}$ so the leading term is precisely $\varphi_0^p(\odot^p\nabla)$ rather than some nonzero multiple. We will not comment further when this sort of maneuver is used below.) It is φ_1^{p-2} according to (26). So now we may rename $T - S_{\varphi_0^p}$ as T and continue with the argument.

The next step is to assume $\varphi_0^p = 0$ and study differential conditions imposed on φ_1^{p-2}. Here, we skip this and several other steps and we assume the greatest coefficient of T is φ_3^{p-6}. So suppose that $\varphi_i^j = 0$ for $\ell(\varphi_i^j) > p - 3 = \ell(\varphi_3^{p-6})$. Then $\ell(T) = p - 3$ and so $\ell(\Delta^4 T) = p + 1$. We shall examine those terms of the operator $\Delta^4 T$ of the (highest) level $p + 1$ and such that they are without a right factor Δ^4. To find these it is sufficient to consider

$$\Delta^4\left[\varphi_3^{p-6}(\odot^{p-6}\nabla)\Delta^3 + \varphi_2^{p-5}(\odot^{p-5}\nabla)\Delta^2 + \varphi_1^{p-4}(\odot^{p-4}\nabla)\Delta^1 + \varphi_0^{p-3}(\odot^{p-3}\nabla)\right].$$

We use the Leibniz rule to move Δ^4 to the right in the previous display. We need to know the form of (level $p + 1$) terms of types $\left[{p-2 \atop 3}\right]$, $\left[{p-1 \atop 2}\right]$, $\left[{p \atop 1}\right]$, and $\left[{p+1 \atop 0}\right]$. The simplest case is the type $\left[{p+1 \atop 0}\right]$, we obtain only the term $2^4(\nabla^4\varphi_0^{p-3}) \odot^{p+1} \nabla$. The operator $\odot^{p+1}\nabla$ does not arise in any other way, so the given term must vanish through φ_0^{p-3} satisfying the obvious equation. In the case of the type $\left[{p \atop 1}\right]$, we similarly get the equation

$$2^4(\nabla^4\varphi_1^{p-4})(\odot^p\nabla)\Delta + 2^3\cdot 4(\nabla^3\varphi_0^{p-3})(\odot^p\nabla)\Delta = 0.$$

Here, $2^3\cdot 4 = 2^3\binom{4}{1} = 2^3\binom{4}{3}$; generally we put $C^s(4) = 2^s\binom{4}{s}$. The types $\left[{p-2 \atop 3}\right]$ and $\left[{p-1 \atop 2}\right]$ yield two more equations which give conditions for the coefficients φ_0^{p-3}, φ_1^{p-4}, φ_2^{p-5}, and φ_3^{p-6}. Together these four equations yield the following differential equations for the coefficients φ_i^j:

$$\begin{array}{llllll}
\text{type } \left[{p-2 \atop 3}\right]: & C^4(4)\nabla^4\varphi_3^{p-6} & + C^3(4)\nabla^3\varphi_2^{p-5} & + C^2(4)\nabla^2\varphi_1^{p-4} & + C^1(4)\nabla^1\varphi_0^{p-3} & = 0 \\
\text{type } \left[{p-1 \atop 2}\right]: & 0 & + C^4(4)\nabla^4\varphi_2^{p-5} & + C^3(4)\nabla^3\varphi_1^{p-4} & + C^2(4)\nabla^2\varphi_0^{p-3} & = 0 \\
\text{type } \left[{p \atop 1}\right]: & 0 & + \quad 0 & + C^4(4)\nabla^4\varphi_1^{p-4} & + C^3(4)\nabla^3\varphi_0^{p-3} & = 0 \\
\text{type } \left[{p+1 \atop 0}\right]: & 0 & + \quad 0 & + \quad 0 & + C^4(4)\nabla^4\varphi_0^{p-3} & = 0
\end{array}$$

Here, we implicitly consider the symmetric trace-free parts in every equation. Now applying ∇^3 to the first equation, ∇^2 to the second, and ∇ to the third, and then taking the trace-free symmetric part in all cases, we obtain a linear system in variables $[\nabla^7\varphi_3^{p-6}]_{\boxtimes}$, $[\nabla^6\varphi_2^{p-5}]_{\boxtimes}$, $[\nabla^5\varphi_1^{p-4}]_{\boxtimes}$, and $[\nabla^4\varphi_0^{p-3}]_{\boxtimes}$. The matrix of (integer) coefficient is

$$\begin{pmatrix} C^4(4) & C^3(4) & C^2(4) & C^1(4) \\ 0 & C^4(4) & C^3(4) & C^2(4) \\ 0 & 0 & C^4(4) & C^3(4) \\ 0 & 0 & 0 & C^4(4) \end{pmatrix}.$$

This is non-singular. So all the variables must vanish, and in particular $[\nabla^7\varphi_3^{p-6}]_\boxtimes = 0$, which is what we wanted to prove.

This was the case with greatest coefficient φ_3^{p-6}. It suggests a route to solving the remaining cases, as they yield linear systems in the same way. Actually, it turns out that in each of the cases with the greatest terms between φ_0^p and φ_3^{p-6} (which were skipped above), the matrix of coefficients includes a square "upper right" submatrix of the matrix above, i.e., a matrix obtained by removing the first q columns and the last q rows for some choice of q, that is sufficient if non-degenerate. That is it suffices to prove that determinants of these matrices are nonzero. This necessitates analysing the combinatorial coefficients $C^s(4)$ in more detail.

The general case is analogous; in the case of Δ^k, $k \in \mathbb{N}$ we shall need the scalars

$$C^s(k) := 2^s\binom{k}{s}, \quad C^s(k) := 0 \text{ for } s > k$$

and matrices

$$\begin{aligned} &\boldsymbol{C}(k;d) \in \mathrm{Mat}_{k-d},\ 0 \le d \le k-1, \quad \text{where} \\ &\boldsymbol{C}(k;d)_{s,t} = C^{k-d+s-t}(k),\ 1 \le s,t \le k-d. \end{aligned} \tag{31}$$

The matrices $\boldsymbol{C}(k,0)$ are upper diagonal with $C^k(k)$ on the diagonal; the matrix $\boldsymbol{C}(4,0)$ appeared in the previous example. In fact, $\boldsymbol{C}(k,d)$ is obtained from $\boldsymbol{C}(k,0)$ by removing d first columns and d last rows. Note also that considering (any) diagonal of $\boldsymbol{C}(k,d)$, all the coefficients are the same.

Clearly, the $\boldsymbol{C}(k,0)$ are regular.

Theorem 6.2: *The matrices $\boldsymbol{C}(k,d)$, $k \in \mathbb{N}$, $0 \le d \le k - 1$ are regular.*

The following proof of this theorem is due to J. Kadourek, of Masaryk University.

Proof: First observe that for $d = 0$, the matrix $\boldsymbol{C}$ is upper triangular with nonzero entries on the diagonal. Thus, it is regular so it is sufficient to assume $1 \le d \le k - 1$. Also to simplify the notation we put $k_d := k - d$. Clearly $1 \le k_d \le k - 1$.

It turns out to be useful to consider also the closely related matrix

$$\begin{aligned} &\widetilde{\boldsymbol{C}}(k;d) \in \mathrm{Mat}_{k_d},\ 0 \le d \le k-1, \quad \text{where} \\ &\widetilde{\boldsymbol{C}}(k;d)_{s,t} = \binom{k}{k_d+s-t},\ 1 \le s,t \le k_d, \end{aligned} \tag{32}$$

where the latter is taken to be 0 if $s - t > d$. That is, the entries of $\boldsymbol{C}$ and $\widetilde{\boldsymbol{C}}$ differ by a power of 2. Now writing the determinant as a sum (over permutations of $\{1,\ldots,k_d\}$)) of products of entries of a matrix, one easily shows that determinants of $\boldsymbol{C}$ and $\widetilde{\boldsymbol{C}}$ differ by a power of 2. That is, the matrix $\boldsymbol{C}$ is regular if and only if $\widetilde{\boldsymbol{C}}$ is regular. We shall prove regularity for the latter.

First recall the well-known relation

$$\binom{q}{m} + \binom{q}{m+1} = \binom{q+1}{m+1}, \quad q,m \ge 0. \tag{33}$$

Henceforth we fix the values k, d from the allowed range. The proof now consists of several series of row or column elementary operations which change the determinant by a nonzero multiple. During certain stages of this process we shall obtain matrices D_1, D_2, D_3, $D_4 \in \mathrm{Mat}_{k_d}$ whose determinants differ from each other only by nonzero multiples. The last of these, D_4 is upper triangular with nonzero entries on the diagonal, and so this concludes the proof.

The construction of D_1 from $\widetilde{\boldsymbol{C}}$ consists of $k_d - 1$ steps; in each of these we undertake a series of elementary column operations, as follows. In the first step, we add the second column to the first one, then the third column to the second and so on; finally, we add the last column to the last but one. In the second step, we add the second column to the first one, then the third column to the second and so on but finish by adding the $(k_d - 1)$th column to the $(k_d - 2)$th column. Continuing in this way, in the last step (i.e., the step number $k_d - 1$) we add only the second column to the first one. Note the determinants of D_1 and $\widetilde{C}$ differ by a nonzero multiple.

Overall we obtain the matrix

$$D_1(s,t) = \binom{k+k_d-t}{k_d+s-t} = \frac{(k+k_d-t)!}{(k_d+s-t)!(k-s)!}; \tag{34}$$

note $1 \leq k_d + s - t \leq k + k_d - t$. The reasoning uses (33) in every addition of two binomial numbers and goes as follows. Consider how the (s, t)-entry changes during the procedure described in the previous paragraph. First observe that after the ith step of elementary column operations, this entry has the form $\binom{a_i}{k_d+s-t}$. That is, the "denominator" of the binomial number on the position (s, t) does not change during this procedure. This follows from (33). Second, the "numerator" of the binomial number on the (s, t)-position increases by 1 if we add the $(s, t+1)$–entry, see (33). Thus the "numerator" depends on the number of additions of the $(t+1)$st column, as stated in (34).

Now we modify the matrix D_1 as follows. First, we multiple the tth column by $\frac{1}{(k+k_d-t)!}$, where we note that $k + k_d - t \geq k \geq 1$. Then, we multiply the sth row by $(k - s)!$ where $k - s \geq 1$ because $s \leq k_d \leq k - 1$. We obtain the matrix D_2, the determinants of D_1 and D_2 differ by a nonzero multiple. It follows from the fractional form of entries of D_1 in (35) that

$$D_2(s,t) = \frac{1}{(k_d+s-t)!}. \tag{35}$$

We continue with the following modification of D_2. First, we multiply the sth row by $(k_d + s - 1)! \geq 1$. Then, we multiply the tth column by $\frac{1}{(t-1)!}$, $t - 1 \geq 0$ (thus $(t - 1)! \geq 1$). The result is a matrix D_3, the determinants of D_3 and D_2 differ by nonzero multiple. It follows from (35) that

$$D_3(s,t) = \frac{(k_d+s-1)!}{(k_d+s-t)!(k-1)!} = \binom{k_d+s-1}{k_d+s-t}. \tag{36}$$

In the last stage we apply the following $k_d - 1$ steps of elementary row transformations to the matrix D_3. Observe that the first column of D_3 has all its entries equal to 1. In the first step, we subtract the $(k_d - 1)$-st row from the k_d-th row, then we subtract $(k_d - 2)$-nd row from the $(k_d - 1)$-st row and so on; finally, we subtract the first row from the second one. Thus, the first column has now 1 as its top entry and 0's below this. In the second step, we subtract the $(k_d - 1)$-st row from the k_d-th row, then we subtract $(k_d - 2)$-nd row from the $(k_d - 1)$-st row and so on, as before except in this step we finish at the point of subtracting the 2nd row from the 3rd row. Continuing in this way, in the last step we subtract only $(k_d - 1)$-st row from the k_d-th row. We shall denote the resulting matrix by D_4.

It turns out D_4 is upper triangular with all entries on the diagonal equal to 1. To show this note we use (33) at every step of the above procedure. In fact, the final form of D_4 can be foreseen already from the first step, after which we obtain a matrix that we shall denote $O \in \mathrm{Mat}_{k_d}$. We already know the first column of O is $(1, 0, \ldots, 0)^T$. From this it follows that in the second step we effectively work only with submatrix of O with entries (s, t), $2 \leq s, t \leq k_d$. Since

$$O(s,t) = \binom{k_d+s-2}{k_d+s-t} = D_3(s-1, t-1), \quad 2 \leq s, t \leq k_d$$

using (33), we see this submatrix of O is exactly the submatrix of D_3 without the last row and the last column. Applying the second step to the displayed submatrix corresponds to applying the first step to the corresponding submatrix of D_3 (the last row and column clearly have no influence on the previous ones). These observations yield an inductive procedure which demonstrates the claimed form of D_4. □

Proposition 6.3: Let (T, T') be a normal symmetry of Δ^k and suppose that, in a standard expression for T, $\varphi_r^p(\odot^p\nabla)\Delta^r$ is the greatest nonzero term of T with respect to $\rhd$. Then $[\nabla^{2r+1}\varphi_r^p]|_{\boxtimes} = 0$.

Proof: The ordering $\lhd$ can be equivalently described as $\varphi_i^j \lhd \varphi_{i'}^{j'}$ if and only if either $i + j < i' + j'$ or $i + j = i' + j'$ and $i < i'$. Thus

$$\Delta^k T = T'\Delta^k, \quad T = \varphi_r^p(\odot^p\nabla)\Delta^r + \sum_{\substack{i<k \\ i+j<r+p \text{ or} \\ (i+j=r+p)\wedge(i<r)}} \varphi_i^j(\odot^j\nabla)\Delta^i.$$

Note φ_r^p might not be a leading term of T.

Note, $\ell(T) = p + r$ and $\ell(\Delta^k T) = p + r + k$. We shall discuss the terms of the highest level in $\Delta^k T$. For this it is sufficient to apply Δ^k only to level $p + r$ terms of T. That is, we need to understand the right-hand side of

$$\begin{aligned}\Delta^k &\left[\varphi_r^p(\odot^p\nabla)\Delta^r + \varphi_{r-1}^{p+1}(\odot^{p+1}\nabla)\Delta^{r-1} + \ldots + \varphi_0^{p+r}(\odot^{p+r}\nabla)\right] - F\Delta^k \\ &= \psi_{k-1}^{p+r+1}(\odot^{p+r+1}\nabla)\Delta^{k-1} + \psi_{k-2}^{p+r+2}(\odot^{p+r+2}\nabla)\Delta^{k-2} + \ldots + \psi_0^{p+r+k}(\odot^{p+r+k}\nabla) + \text{llt},\end{aligned}$$

where F is a differential operator. Here "llt" denotes terms of the level at most $p + r + k - 1$ (with powers of Δ strictly less than k) and ψ_i^j is of type $\left[\begin{smallmatrix} j \\ i \end{smallmatrix}\right]$. Since $i < k$ for every ψ_i^j on the right-hand side, imposing the symmetry condition, each of these terms has to vanish. This yields k differential conditions

$$\psi_{k-1}^{p+r+1}(\odot^{p+r+1}\nabla)\Delta^{k-1} = 0, \psi_{k-2}^{p+r+2}(\odot^{p+r+2}\nabla)\Delta^{k-2} = 0, \ldots, \psi_0^{p+r+k}(\odot^{p+r+k}\nabla) = 0.$$

Thus $\psi_{k-q-1}^{p+r+q+1} = 0$ for $q \in \{0, \ldots, k - 1\}$. For our purposes it turns out to be sufficient to take q in the (in general smaller) range $\{0, \ldots, r\}$. So we have $r + 1$ differential conditions. Now fix such a q; we have more explicitly

$$\psi_{k-q-1}^{p+r+q+1} = \left[a_{q,0}\nabla^{r+q+1}\varphi_r^p + a_{q,1}\nabla^{r+q}\varphi_{r-1}^{p+1} + \ldots + a_{q,r}\nabla^{q+1}\varphi_0^{p+r}\right]|_{\boxtimes}$$

for some integer coefficients $a_{q,q'}$, $q' \in \{0, \ldots, r\}$. Via the Leibniz rule and a counting argument, it is straightforward to verify that $a_{q,q'} = C^{r+q-q'+1}(k)$. Recall $\psi_{k-q-1}^{p+r+q+1} = 0$ hence the right-hand side of the previous display vanishes. Finally, let us apply ∇^{r-q} to both sides of the previous display. Projecting to the Cartan component, we obtain

$$\left[C^{r+q+1}(k)(\nabla^{2r+1}\varphi_r^p) + C^{r+q}(k)(\nabla^{2r}\varphi_{r-1}^{p+1}) + \ldots + C^{q+1}(k)(\nabla^{r+1}\varphi_0^{p+r})\right]|_{\boxtimes} = 0.$$

This is a linear equation in the $r + 1$ variables $(\nabla^{2r+1}\varphi_r^p)|_{\boxtimes}$, $(\nabla^{2r}\varphi_{r-1}^{p+1})|_{\boxtimes}$, $\ldots$, $(\nabla^{r+1}\varphi_0^{p+r})|_{\boxtimes}$. These variables obviously do not depend on q. That is for every $q \in \{0, \ldots, r\}$ we obtain one equation in these variables. Overall we have a system of $r + 1$ linear equations in $r + 1$ variables $(\nabla^{2r+1}\varphi_r^p)|_{\boxtimes}$, $(\nabla^{2r}\varphi_{r-1}^{p+1})|_{\boxtimes}$, $\ldots$, $(\nabla^{r+1}\varphi_0^{p+r})|_{\boxtimes}$. The integer coefficients are $a_{q,q'} = C^{r+q-q'+1}(k) = C^{(r+1)+(q+1)-(q'+1)}(k)$, $q, q' \in \{0, \ldots, r\}$ thus the $(r + 1) \times (r + 1)$ matrix of integer coefficients is exactly $\boldsymbol{C}(k, d)$ for $d = k - r - 1$ from (31). (Note $r < k$ hence $d \in \{0, \ldots, k - 1\}$.) But matrices $\boldsymbol{C}(k, d)$ are regular according to Theorem 6.2. Therefore, this linear system has only the zero solution, i.e.,

$$(\nabla^{2r+1}\varphi_r^p = 0)|_{\boxtimes} = 0, \nabla^{2r}(\varphi_{r-1}^{p+1})|_{\boxtimes} = 0, \ldots, (\nabla^{r+1}\varphi_0^{p+r})|_{\boxtimes} = 0.$$

In particular $(\nabla^{2r+1}\varphi_r^p)|_{\boxtimes} = 0$, which is what we wanted to prove. □

Finally, we have the key theorem of this section. By an obvious induction this establishes the second part of Theorem 2.1.

Theorem 6.4: *Let (S, S') be a normal symmetry of Δ^k and suppose that, in a standard expression for S, $\varphi_r^p(\odot^p\nabla)\Delta^r$, $r < k$ is a leading term. Then $[\nabla^{2r+1}\varphi_r^p]|_{\boxtimes} = 0$. This establishes the second part of Theorem 2.1. Note that using the conformal metric, we can view all $p + 2r + 1$ abstract indices* *of* $\nabla^{2r+1}\varphi_r^p$ *as contravariant. Then the projection to the Cartan component in* $[\nabla^{2r+1}\varphi_r^p]|_\boxtimes = 0$ *simply means taking the symmetric trace-free part.*

Proof: Consider the coefficients of the maximal level $\ell(S)$ of S; among them, denote by ψ_i^j the term of the highest order. In the other words, ψ_i^j is the greatest coefficient in S with respect to $\lhd$. Now $[\nabla^{2i+1}\psi_i^j]|_\boxtimes = 0$ according to Proposition 6.3 hence ψ_i^j yields the corresponding canonical symmetry (S_ψ, S'_ψ) of Δ^k. Therefore, $(S - S_\psi, S' - S'_\psi)$ is also a symmetry of Δ^k.

First observe using Proposition 6.1 (iii) that the leading terms of S and $S - S_\psi$ can differ only if $\psi_i^j(\odot^j\nabla)\Delta^i$ is a leading term of S. But in that case we have proved the theorem for $\psi_i^j(\odot^j\nabla)\Delta^i$. Therefore, it is sufficient to prove the theorem for $S - S_\psi$. So we can take $S := S - S_\psi$ and continue inductively.

Proposition 6.1: (ii) guarantees that the greatest term of $S := S - S_\psi$ is smaller than the greatest term of S. Hence this induction with respect to $\lhd$ is finite. □

VII. ALGEBRA OF SYMMETRIES

Here, we shall prove Theorem 2.5. Recall that the finite-dimensional space of solutions of (3) may be realised as a standard linear "matrix" representation of $\mathfrak{g} = \mathfrak{so}_{s+1,s'+1}$ via the map from solutions to parallel tractors $\varphi \mapsto I_\varphi$. In the case of conformal Killing vectors (i.e., (3) with $p = 1$, $r = 0$) the range space is $\mathfrak{g}$, on which $\mathfrak{g}$ acts by the adjoint representation. Then the identification of $\mathfrak{g}$ with differential symmetries is given by the mapping $\mathfrak{g} \ni I_\varphi \mapsto S_\varphi = I_\varphi^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}$, as a special case of (24). The mapping $S_\varphi = I_\varphi^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}$ extends to

$$\mathfrak{g}\otimes\mathfrak{g}\otimes\cdots\mathfrak{g} \ni I_{\varphi_1}\otimes\cdots\otimes I_{\varphi_m} \mapsto S_{\varphi_1}\cdots S_{\varphi_m}, \quad m \ge 1, \tag{37}$$

and hence to the full tensor algebra $\bigotimes\mathfrak{g}$ by linearity.

The first step in the proof of Theorem 2.5 is to express the composition $S_\varphi S_{\bar\varphi}$ for $I_\varphi, I_{\bar\varphi} \in \mathfrak{g}$ in terms of canonical symmetries. This is done , Theorem 5.1 of Ref. 21] and necessarily our results must agree with those from their construction (as uniqueness of the low order symmetries involved is easily verified). We present the details here to keep this text self-contained and also because we derive the formulas for all conformally flat manifolds.

Putting $I := I_\varphi$, $\bar I := I_{\bar\varphi}$ to simplify the notation, one has

$$S_\varphi S_{\bar\varphi} = I^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}\bar I^{\mathbf{B}}\mathbb{D}_{\mathbf{B}} = I^{\mathbf{A}}\bar I^{\mathbf{B}}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}, \tag{38}$$

on $\mathcal{E}[w]$, since I is parallel. This gives an explicit and key link between the algebraic structure of symmetries $\mathcal{A}_k$ and operations on the tensor algebra $\bigotimes\mathfrak{g}$. We shall consider the displayed operator acting on $\mathcal{E}[w]$ for all $w \in \mathbb{R}$ at this stage.

We need to decompose $\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}$ into irreducible components. Using the definition of $\mathbb{D}_{\mathbf{A}}$, a direct computation shows that

$$\begin{aligned}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}} f &= 4w^2\mathbb{W}_{\mathbf{A}}\mathbb{W}_{\mathbf{B}} f - 4w\mathbb{X}_{\mathbf{A}}^a\mathbb{Y}_{\mathbf{B}}^b\mathbf{g}_{ab} f\\ &\quad + 4(w-1)\mathbb{X}_{\mathbf{A}}^a\mathbb{W}_{\mathbf{B}}\nabla_a f + 4w\mathbb{W}_{\mathbf{A}}\mathbb{X}_{\mathbf{B}}^b\nabla_b f + 4\mathbb{X}_{\mathbf{A}}^a\mathbb{Z}_{\mathbf{B}}^{\mathbf{b}}\mathbf{g}_{ab^0}\nabla_{b^1} f\\ &\quad + 4\mathbb{X}_{\mathbf{A}}^a\mathbb{X}_{\mathbf{B}}^b(\nabla_a\nabla_b + wP_{ab})f.\end{aligned} \tag{39}$$

From this, one easily verifies that

$$\begin{aligned}\frac{1}{2}(\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}} + \mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{A}}) &= \frac{1}{2}(\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}} + \mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{A}})|_\boxtimes + \frac{4}{n}h_{A^0B^0}\mathbb{D}^2_{(A^1B^1)_0}\\ &\quad + \frac{2}{(n+1)(n+2)}h_{A^0B^0}h_{A^1B^1}\mathbb{D}^{\mathbf{A}}\mathbb{D}_{\mathbf{A}},\\ \frac{1}{2}(\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}} - \mathbb{D}_{\mathbf{B}}\mathbb{D}_{\mathbf{A}}) &= 3h_{A^0[A^1}\mathbb{D}_{\mathbf{B}]} = -2h_{A^0B^0}\mathbb{D}_{A^1B^1}.\end{aligned} \tag{40}$$

Hence, we need the irreducible components $\yng(2,2)_0$, $\yng(2)_0$, $\yng(1,1)$ and $\mathbb{R}$ of $I^{\mathbf{A}}\bar{I}^{\mathbf{B}}$, cf. (6). Explicitly, we put

$$\begin{aligned}
\langle I, \bar{I}\rangle &:= -4nI^{\mathbf{A}}\bar{I}_{\mathbf{A}} \in \mathbb{R},\\
[I, \bar{I}]^{\mathbf{A}} &:= 4I^{A^0P}\bar{I}_p{}^{A^1} \in \yng(1,1),\\
(I \bullet \bar{I})^{BB'} &:= \frac{4}{n}I^{P(B}\bar{I}_P{}^{B')_0} \in \yng(2)_0
\end{aligned} \tag{41}$$

and we denote by $(I \boxtimes \bar{I})^{\mathbf{AB}}$ the trace-free part of the Young projection $\yng(2,2)$ applied to $I^{\mathbf{A}}\bar{I}^{\mathbf{B}}$. Using this notation, the projection and decomposition of $I^{\mathbf{A}} \otimes \bar{I}^{\mathbf{B}}$ into its irreducible components in $\yng(2,2)$, $\mathbb{R}$, $\yng(1,1)$ and $\yng(2)_0$ is given by

$$\begin{aligned}
\yng(1,1)\otimes\yng(1,1) \ni I^{\mathbf{A}} \otimes \bar{I}^{\mathbf{B}} \mapsto &(I \boxtimes \bar{I})^{\mathbf{AB}} - \frac{1}{2n(n+1)(n+2)}h^{A^0B^0}h^{A^1B^1}\langle I, \bar{I}\rangle\\
&+ \frac{1}{n}h^{A^0B^0}[I, \bar{I}]^{A^1B^1} + h^{A^0B^0}(I \bullet \bar{I})^{A^1B^1}.
\end{aligned} \tag{42}$$

Using the computation above, we easily recover, Theorem 5.1 of Ref. 21].

Theorem 7.1: *Let $\varphi^a, \bar{\varphi}^a \in \mathcal{E}^a$ be conformal Killing fields corresponding to $I^{\mathbf{A}} := I^{\mathbf{A}}_{\varphi}$ and $\bar{I}^{\mathbf{A}} := I^{\mathbf{A}}_{\bar{\varphi}}$ in $\mathfrak{g} = \mathfrak{so}_{s+1,s'+1}$. Then*

$$S_{\varphi}S_{\bar{\varphi}}f = (I \boxtimes \bar{I})^{\mathbf{AB}}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}f + (I \bullet \bar{I})^{BB'}\mathbb{D}^2_{BB'}f + \frac{1}{2}[I, \bar{I}]^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}f + \frac{w(n+w)}{n(n+1)(n+2)}\langle I, \bar{I}\rangle f$$

for $f \in \mathcal{E}[w]$, cf. (7). The four summands on the right-hand side are canonical symmetries, explicitly

- $(I \boxtimes \bar{I})^{\mathbf{AB}}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}} = S_{\Phi}$ for $\mathcal{E}^{(ab)_0} \ni \Phi^{ab} = \varphi^{(a}\bar{\varphi}^{b)_0}$,
- $(I \bullet \bar{I})^{BB'}\mathbb{D}^2_{BB'} = S_{\Phi}$ for $\mathcal{E}[2] \ni \Phi = \frac{1}{n}\varphi^a\bar{\varphi}_a$,
- $[I, \bar{I}]^{\mathbf{A}}\mathbb{D}_{\mathbf{A}} = S_{\Phi}$ for $\mathcal{E}^a \ni \Phi^a = \varphi^b\nabla_b\bar{\varphi}^a - \bar{\varphi}^b\nabla_b\varphi^a$ *(the Lie bracket of vector fields)*,
- $\mathbb{R} \ni \langle I, \bar{I}\rangle = -4nI^{\mathbf{A}}\bar{I}_{\mathbf{A}} = -2[\varphi^a\nabla_a\nabla_b\bar{\varphi}^b + \bar{\varphi}^a\nabla_a\nabla_b\varphi^b] + n(\nabla_a\varphi^b)(\nabla_b\bar{\varphi}^a) - \frac{n-2}{n}(\nabla_a\varphi^a)(\nabla_b\bar{\varphi}^b) - 4nP_{ab}\varphi^a\bar{\varphi}^b$.

In all these cases, the section Φ is a solution of the corresponding equation (3).

Proof: The statement puts together the previous computations. Following (38), we need to decompose $I^{\mathbf{A}}\bar{I}^{\mathbf{B}}\mathbb{D}_{\mathbf{A}}\mathbb{D}_{\mathbf{B}}$ into canonical symmetries. This is provided by contracting right-hand sides of (42) and (40). Using in addition $\mathbb{D}^{\mathbf{A}}\mathbb{D}_{\mathbf{A}}f = -2w(n+w)f$ for $f \in \mathcal{E}[w]$ (which easily follows from (39)), the right-hand side of $S_{\varphi}S_{\bar{\varphi}}$ in the display above follows.

The components $I \boxtimes \bar{I}$, $I \bullet \bar{I}$, $[I, \bar{I}]$, and $\langle I, \bar{I}\rangle$ are parallel (and irreducible) thus their projecting parts Φ are solutions of the corresponding equation from the family (3). To prove the theorem, it remains to identify howthese solutions are built from $\varphi^a, \bar{\varphi}^a \in \mathcal{E}^a$. Note

$$I^{\mathbf{A}} = \mathbb{Y}^{\mathbf{A}}_a\varphi^a + \frac{1}{2}\mathbb{Z}^{\mathbf{A}}_{\mathbf{a}}\nabla^{a^0}\varphi^{a^1} + \frac{1}{n}\mathbb{W}^{\mathbf{A}}\nabla_a\varphi^a + \mathbb{X}^{\mathbf{A}}_a[\frac{1}{n}\nabla_a\nabla_b\varphi^b + P_{ab}\varphi^b] \tag{43}$$

and similarly for $\bar{I}^{\mathbf{A}}$.[31] Now the explicit form of such Φ for irreducible components of $I^{\mathbf{A}} \otimes \bar{I}^{\mathbf{B}}$ is easily obtained from (41) for $I \bullet \bar{I}$, $[I, \bar{I}]$, and $\langle I, \bar{I}\rangle$. Since $\frac{1}{2}(I^{\mathbf{A}}\bar{I}^{\mathbf{B}} + I^{\mathbf{B}}\bar{I}^{\mathbf{A}})$ has the projecting part $\varphi^{(a}\bar{\varphi}^{b)}$, the case $I \boxtimes \bar{I}$ follows by irreducibility. □

To finish the proof of Theorem 2.5, observe the following. First, we have an associative algebra morphism

$$\bigotimes \mathfrak{g} \to \mathcal{A}_k$$

determined by (37). That this is surjective and an easy consequence of Theorem 2.4 since the canonical symmetries S_ϕ of (24) clearly arise in the range of (37). We want to find all corresponding relations, that is identify the two-sided ideal annihilated by this map. The ideal certainly contains (8), as follows from Theorem 7.1 with $w = -\frac{n}{2} + k$. That it also contains $\boxtimes^{2k}\Box$ is due to the following result.

Lemma 7.2: Assume $I \in \boxtimes^{2k} square$ *is parallel. Then* $I = I_\varphi$ *for* $\varphi \in \mathcal{E}[2k]$ *and* $S_\varphi = \varphi P_k : \mathcal{E}[-\frac{n}{2}+k] \to \mathcal{E}[-\frac{n}{2}+k]$.

Proof: $I \in \boxtimes^{2k} square$ means $I^{A_1A_1'\cdots A_kA_k'} \in \mathcal{E}^{(A_1A_1'\cdots A_kA_k')_0}$ and $I = I_\varphi$ for $\varphi \in \mathcal{E}[2k]$ is due to the irreducibility of I and the fact that is parallel. Then

$$S_\varphi = I_\varphi^{A_1A_1'\cdots A_kA_k'} \mathcal{D}^2_{A_1A_1'} \cdots \mathcal{D}^2_{A_kA_k'}.$$

Now observe $\mathcal{D}^2_{(CD)_0} = -X_{(C}D_{D)_0}$ and $X_{(C}D_{D)_0} = D_{(C}X_{D)_0}$, cf. (18). On the other hand $D_{(A_1}\cdots D_{A_k)_0} = (-1)^k X_{(A_1}\cdots X_{A_k)_0}P_k$ on $\mathcal{E}[-\frac{n}{2}+k]$.[27,30] Thus $\mathcal{D}^2_{A_1A_1'}\cdots\mathcal{D}^2_{A_kA_k'} = X_{A_1}X_{A_1'}\cdots X_{A_k}X_{A_k'}P_k$ on $\mathcal{E}[-\frac{n}{2}+k]$. The rest follows from the relation between φ and I_φ in (23). □

We have found the generators of the ideal in $\bigotimes \mathfrak{g}$ described in Theorem 2.5; it remains to show that this ideal large enough to have $\mathcal{A}_k$ as the resulting quotient. Essentially, we follow Refs. 20 and 21, where cases $k = 1$ and $k = 2$ are studied. We assume $k \geq 1$ here. Since we know $\mathcal{A}_k$, as a vector space, from (4), it is sufficient to consider the corresponding graded algebra (i.e., the symbol algebra of $\mathcal{A}_k$.) The corresponding graded ideal contains $I_1 \otimes I_2 - I_1 \boxtimes I_2 - I_1 \bullet I_2$ for $I_1, I_2 \in \mathfrak{g}$, cf. (8), hence it contains $\mathfrak{g} \wedge \mathfrak{g}$. Therefore, we can pass to $\bigodot \mathfrak{g}$ and we write $\mathcal{I}$ for the ideal in $\bigodot \mathfrak{g}$ which is the image of the ideal of Theorem 2.5. We claim that as a graded structure $\mathcal{A}_k = \bigoplus \mathcal{A}_{k,t}$, where the $\mathcal{A}_{k,t}$ are defined as the submodules satisfying

$$\mathcal{A}_{k,t} = \Big\{ X \in \overbrace{\begin{array}{|c|c|c|c|}\hline & & \cdots & \\ \hline & & \cdots & \\ \hline \end{array}}^{t} \quad \text{s.t.} \ \underbrace{\mathrm{trace}(\ldots(\mathrm{trace}}_{k}(X)..)) = 0 \Big\} \subseteq \bigodot^t \mathfrak{g}.$$

The traces are taken via the tractor metric and note that the trace condition arises from Lemma 7.2 above. As a vector space this is the right answer as, by standard representation theory, $\mathcal{A}_{k,t} = \bigoplus_{j+2i=t} \mathcal{K}_i^j$, $t \geq 1$. To finish the proof, we need to show $\bigodot^t \mathfrak{g} = \mathcal{A}_{k,t} \oplus \mathcal{I}_t$ (as vector spaces) where $\mathcal{I}_t = \mathcal{I} \cap \bigodot^t \mathfrak{g}$, $t \geq 1$. This is based on the following.

Lemma 7.3: Assume $t \geq 3$, $k \geq 1$. Then

$$\left(\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \otimes \mathcal{A}_{k,t-1} \right) \cap \left(\mathcal{A}_{k,t-1} \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \right) = \begin{cases} \mathcal{A}_{k,t} & t \neq 2k \\ \mathcal{A}_{k,t} \oplus \boxtimes^{2k}\Box & t = 2k. \end{cases}$$

Proof: The case $t < 2k$ follows from, Theorem 2 of Ref. 19 or can be easily checked directly. Assume $t > 2k$. The inclusion "$\supseteq$" is obvious. To show "$\subseteq$" consider the tensor $F^{\mathbf{A}_1\ldots\mathbf{A}_t}$ in the left-hand side of the display. Then

$$F^{\mathbf{A}_1\ldots\mathbf{A}_i\ldots\mathbf{A}_j\ldots\mathbf{A}_t} = F^{\mathbf{A}_1\ldots\mathbf{A}_j\ldots\mathbf{A}_i\ldots\mathbf{A}_t}$$

for any $1 \leq i < j \leq t$. From this it easily follows that the skew symmetrisation over any three indices of F is zero. (This and the last display also follow from, Theorem 2 of Ref. 19.) Now any composition of k traces applied to F affects $2k$ indices among $2t$ indices $A_1^0, A_1^1, \ldots, A_t^0, A_t^1$, i.e., at most $2k$ form indices among $\mathbf{A}_1, \ldots, \mathbf{A}_t$. Thus, there is a free form index $\mathbf{A}_i$ (as $t > 2k$) and the inclusion "$\subseteq$" follows from the symmetry given by the previous display.

Assume $t = 2k$. Following the previous case "$\subseteq$," the difference appears only if a composition of k traces affects all $2k$ form indices of F. After taking of such composition of traces we obtain a tensor in $\bigodot^t \square$ and one easily sees this tensor is trace-free. On the other hand, for any symmetric trace-free tensor $G^{A_1^0 \cdots A_{2k}^0} \in \boxtimes^{2k}\square$, one has

$$G^{A_1^0 \cdots A_{2k}^0} h^{A_1^1 \cdots A_{2k}^1} \in \left(\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \otimes \mathcal{A}_{k,t-1} \right) \cap \left(\mathcal{A}_{k,t-1} \otimes \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \right) \tag{44}$$

which can be easily verified by direct computation. Here, $h^{A_1^1 \cdots A_{2k}^1} = h^{(A_1^1 A_2^1} \cdots h^{A_{2k-1}^1 A_{2k}^1)}$ and recall we implicitly skew over the couples $A_i^0 A_i^1$ for $1 \le i \le 2k$. $\square$

The final step is to use that for each s, there is (by standard theory) a projection $\odot^s \mathfrak{g} \to \mathcal{A}_{k,s}$ and that the induced projections $P_t : \bigodot^t \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{A}_{k,t-1}$ and $Q_t : \bigodot^t \mathfrak{g} \to \mathcal{A}_{k,t-1} \otimes \mathfrak{g}$ have kernel in, respectively, $\mathfrak{g} \otimes \mathcal{I}_{t-1}$ and $\mathcal{I}_{t-1} \otimes \mathfrak{g}$ (and hence in both cases in $\mathcal{I}_t$) where for each non-negative integer s, $\mathcal{I}_s = \mathcal{I} \cap \bigodot^s \mathfrak{g}$. Therefore, by obvious dimensional considerations,

$$\bigodot^t \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} = (\operatorname{im} P_t \cap \operatorname{im} Q_t) \oplus (\ker P_t + \ker Q_t), \quad t \ge 3 \tag{45}$$

and the claim above and then Theorem 2.5 follow by induction.

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Second Order Symmetries of the Conformal Laplacian

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Abstract. Let (M, g) be an arbitrary pseudo-Riemannian manifold of dimension at least 3. We determine the form of all the conformal symmetries of the conformal (or Yamabe) Laplacian on (M, g), which are given by differential operators of second order. They are constructed from conformal Killing 2-tensors satisfying a natural and conformally invariant condition. As a consequence, we get also the classification of the second order symmetries of the conformal Laplacian. Our results generalize the ones of Eastwood and Carter, which hold on conformally flat and Einstein manifolds respectively. We illustrate our results on two families of examples in dimension three.

Key words: Laplacian; quantization; conformal geometry; separation of variables

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1 Introduction

We work over a pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$, with Levi-Civita connection ∇ and scalar curvature Sc. Our main result is the classification of all the differential operators D_1 of second order such that the relation

$$\Delta_Y D_1 = D_2 \Delta_Y \tag{1.1}$$

holds for some differential operator D_2, where $\Delta_Y := \nabla_a \mathrm{g}^{ab} \nabla_b - \frac{n-2}{4(n-1)}\mathrm{Sc}$ is the Yamabe Laplacian. Such operators D_1 are called conformal symmetries of order 2 of Δ_Y. They preserve the kernel of Δ_Y, i.e. the solution space of the equation $\Delta_Y \psi = 0$, $\psi \in \mathcal{C}^\infty(M)$. Under a conformal change of metric, $\hat{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$, $\Upsilon \in \mathcal{C}^\infty(M)$, the Yamabe Laplacian transforms as

$$\widehat{\Delta_Y} = e^{-\frac{n+2}{2}\Upsilon} \circ \Delta_Y \circ e^{\frac{n-2}{2}\Upsilon},$$

so that each conformal symmetry D_1 of Δ_Y gives rise to one of $\widehat{\Delta_Y}$ given by

$$\widehat{D_1} = e^{-\frac{n-2}{2}\Upsilon} \circ D_1 \circ e^{\frac{n-2}{2}\Upsilon}.$$

This emphasizes the conformal nature of the problem and justify our choice of the Yamabe Laplacian, rather than the more usual Laplace–Beltrami one, $\Delta := \nabla_a \mathrm{g}^{ab} \nabla_b$. Over flat pseudo-Euclidean space, the classification of conformal symmetries up to second order is due to Boyer, Kalnins and Miller [7], who use it to study the R-separation of variables of the Laplace equation $\Delta\Psi = 0$. More generally, Kalnins and Miller provide an intrinsic characterization for R-separation of this equation on (M, g) in terms of second order conformal symmetries [19]. Thus, classifying those symmetries happens to be a basic problem in the theory of separation of variables. A new input into the quest of conformal symmetries has been given by the work of Eastwood [15]. He classified indeed the conformal symmetries of any order over the conformally flat space and exhibited their interesting algebraic structure. This leads to a number of subsequent works, dealing with other invariant operators [16, 18, 31].

Using principal symbol maps, one can extract two informations from the equation (1.1): the operators D_1 and D_2 have the same principal symbol and the latter is a conformal Killing 2-tensor, i.e. a constant of motion of the geodesic flow, restricted to the null cone. One looks then for a right inverse to the principal symbol maps, called a quantization map, which associates with each conformal Killing tensor a conformal symmetry of Δ_Y. For Killing vector fields this is trivial. If K is a 2-tensor, Carter proves that if the minimal prescription

$$K \mapsto \nabla_a K^{ab} \nabla_b$$

satisfies $[\Delta_Y, \nabla_a K^{ab}\nabla_b] = 0$, then K is Killing. Moreover, he shows that if (M, g) is Einstein, i.e. if $\mathrm{Ric} = \frac{1}{n}\mathrm{Sc}\mathrm{g}$ with Ric the Ricci tensor, the fact that K is Killing is sufficient to ensure that the minimal prescription above is a symmetry of Δ_Y (for application to the separation of variables, see [3]). Besides, in [15], Eastwood defines conformally invariant operators on an arbitrary pseudo-Riemannian manifold, which coincide with the conformal symmetries of Δ_Y on the flat space. These operators are given by means of the natural and conformally invariant quantization $\mathcal{Q}_{\lambda_0,\lambda_0}$ (where $\lambda_0 = \frac{n-2}{2n}$), developed in [9, 24, 29, 30]. Explicitly, if X is a vector field and K a symmetric trace-less 2-tensor, $\mathcal{Q}_{\lambda_0,\lambda_0}(X)$ and $\mathcal{Q}_{\lambda_0,\lambda_0}(K)$ are differential operators acting between λ_0-densities defined in the following way:

$$\mathcal{Q}_{\lambda_0,\lambda_0}(X) = X^a\nabla_a + \frac{n-2}{2n}(\nabla_a X^a),$$
$$\mathcal{Q}_{\lambda_0,\lambda_0}(K) = K^{ab}\nabla_a\nabla_b + \frac{n}{n+2}\left((\nabla_a K^{ab})\nabla_b + \frac{n-2}{4(n+1)}(\nabla_a\nabla_b K^{ab})\right) - \frac{n+2}{4(n+1)}\mathrm{Ric}_{ab}K^{ab}.$$

In the conformally flat case, all the conformal symmetries of second order are of the type $\mathcal{Q}_{\lambda_0,\lambda_0}(K + X + c)$, where $c \in \mathbb{R}$, X is a conformal Killing vector field and K is a conformal Killing 2-tensor. Thanks to the conformal covariance of Δ_Y one can show that $\mathcal{Q}_{\lambda_0,\lambda_0}(X)$ is still a conformal symmetry of Δ_Y on an arbitrary pseudo-Riemannian manifold, if X is a conformal Killing vector field. However, as pointed out by Eastwood in [15], it is unclear whether $\mathcal{Q}_{\lambda_0,\lambda_0}(K)$ is a conformal symmetry when K is a conformal Killing 2-tensor.

Our strategy relies on the properties of the quantization map $\mathcal{Q}_{\lambda_0,\lambda_0}$ and on the classification of natural and conformally invariant operators acting on prescribed subspaces of symbols. This method has been developed first on conformally flat manifolds, in [26]. In that case, the map $\mathcal{Q}_{\lambda_0,\lambda_0}$ is a conformally equivariant quantization [12], and the author proved that it is precisely the bijective map between conformal Killing tensors and conformal symmetries of Δ_Y, discovered by Eastwood. The description of conformal symmetries on arbitrary pseudo-Riemannian manifolds is more involved, even at order 2. Namely, there exists a conformal symmetry with principal symbol K if and only if K is a conformal Killing tensor and $\mathbf{Obs}(K)^\flat$ is an exact one-form. Here, **Obs** is a natural and conformally invariant operator which reads, in abstract index notation, as

$$\mathbf{Obs}(K)^a = \frac{2(n-2)}{3(n+1)}\left(\mathrm{C}^r{}_{st}{}^a\nabla_r - 3\mathrm{A}_{st}{}^a\right)K^{st},$$

where C denotes the Weyl tensor and A the Cotton–York tensor. If $\mathbf{Obs}(K)^\flat$ is equal to the exact one-form $-2df$, with $f \in \mathcal{C}^\infty(M)$, then the operators

$$\mathcal{Q}_{\lambda_0,\lambda_0}(K + X + c) + f$$

are conformal symmetries of Δ_Y for all conformal Killing vector field X and constant $c \in \mathbb{R}$. As a consequence, $\mathcal{Q}_{\lambda_0,\lambda_0}(K)$ is a conformal symmetry of Δ_Y if and only if $\mathbf{Obs}(K) = 0$.

We illustrate our results on two examples in dimension three. In the first one, the space $\mathbb{R}^3$ is endowed with the most general Riemannian metric admitting a Killing 2-tensor K, which is diagonal in orthogonal coordinates [28]. Then, $\mathbf{Obs}(K)^\flat$ is a non-trivial exact 1-form and, up to our knowledge, the symmetry of Δ_Y that we obtain is new. In the second one, we consider a conformal Stäckel metric g on $\mathbb{R}^3$ with one ignorable coordinate. Such a metric admits an irreducible conformal Killing tensor K. Using the generic form of g and K given in the reference [8], we obtain that $\mathbf{Obs}(K)^\flat$ is a non-exact 1-form in general. This means there are no conformal symmetries of Δ_Y with principal symbol K in general.

We detail now the content of the paper.

In Section 2, we introduce the basic spaces: the one of tensor densities $\mathcal{F}_\lambda(M)$ of weight $\lambda \in \mathbb{R}$, the one of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ acting between λ- and μ-densities, the one of symbols $\mathcal{S}_\delta(M)$ with $\delta = \mu - \lambda$. Then, we define the Yamabe Laplacian Δ_Y as an element of $\mathcal{D}_{\lambda_0,\mu_0}(M)$, with $\lambda_0 = \frac{n-2}{2n}$ and $\mu_0 = \frac{n+2}{2n}$, so that it becomes a conformally invariant operator. Finally we introduce our main tool, namely the natural and conformally invariant quantization

$$\mathcal{Q}_{\lambda,\mu} : \ \mathcal{S}_{\mu-\lambda}(M) \ \to \ \mathcal{D}_{\lambda,\mu}(M),$$

and we provide explicit formulas for it.

In Section 3, we classify the natural and conformally invariant operators between some subspaces of symbols. Among the operators we obtain (and which are crucial for understanding of 2nd order symmetries), one of them, $\mathbf{G}$, is classical, whereas another one, $\mathbf{Obs}$, acting on symbols of second degree, is new and admits no counterpart on flat space. We obtain also an analogous classification for higher order trace-free symbols where the situation is much more complicated. Note that the discovered operators act between source and target spaces of well-known conformally invariant operators, which appear in the generalized BGG sequence [10]. It would be interesting to understand better the relations between all these conformal operators.

In Section 4 lies our main result. After defining the spaces of conformal symmetries and of conformal Killing tensors, we prove that, on symbols K of degree 2, we have

$$(\mathcal{Q}_{\lambda_0,\mu_0})^{-1} \left(\Delta_Y \mathcal{Q}_{\lambda_0,\lambda_0}(K) - \mathcal{Q}_{\mu_0,\mu_0}(K)\Delta_Y\right) = 2\mathbf{G}(K) + \mathbf{Obs}(K).$$

The kernel of $\mathbf{G}$ is precisely the space of conformal Killing tensors, whereas $\mathbf{Obs}(K)$ is the obstruction for a conformal Killing 2-tensor to provide a conformal symmetry of the form $\mathcal{Q}_{\lambda_0,\lambda_0}(K)$. The full description of conformal symmetries of 2nd order of Δ_Y easily follows. Using that $\mathcal{Q}_{\lambda_0,\lambda_0}(K) = \mathcal{Q}_{\mu_0,\mu_0}(K)$ for Killing 2-tensors, we deduce also the classification of second order symmetries of Δ_Y, which satisfy by definition $[\Delta_Y, D_1] = 0$.

In Section 5, we provide two examples illustrating our main result. In the first one, the Killing tensor K is such that $\mathbf{Obs}(K)^\flat$ is a non-vanishing but exact one-form. In the second example, we provide several conformal Killing tensors K such that $\mathbf{Obs}(K)^\flat$ is a non-exact one-form. Hence, there is no conformal symmetry with such K as principal symbols.

2 Conformal geometry, differential operators, and their symbols

Throughout this paper, we employ the abstract index notation from [27]. That is, on a smooth manifold M, v^a denotes a section of the tangent bundle TM, v_a a section of the cotangent bundle

T^*M and e.g. $v^{ab}{}_c$ a section of $TM \otimes TM \otimes T^*M$. The letters a, b, c, d and r, s, t are reserved for abstract indices. Repetition of an abstract index in the covariant and contravariant position means contraction, e.g. v_b^{ab} is a section of TM. In few places we use concrete indices attached to a coordinate system. This is always explicitly stated and we denote such indices by letters i, j, k, l to avoid confusion with abstract indices. We always use the Einstein's summation convention for indices, except if stated otherwise.

2.1 Basic objects

Let M be a n-dimensional smooth manifold. If $\lambda \in \mathbb{R}$, the vector bundle of λ-*densities*, $F_\lambda(M) \to M$, is a line bundle associated with P^1M, the linear frame bundle over M:

$$F_\lambda(M) = P^1M \times_\rho \mathbb{R},$$

where the representation ρ of the group $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}$ is given by

$$\rho(A)e = |\det A|^{-\lambda} e, \qquad \forall\, A \in \mathrm{GL}(n, \mathbb{R}), \quad \forall\, e \in \mathbb{R}.$$

We denote by $\mathcal{F}_\lambda(M)$ the space of smooth sections of this bundle. Since $F_\lambda(M)$ is associated with P^1M, the space $\mathcal{F}_\lambda(M)$ is endowed with canonical actions of $\mathrm{Diff}(M)$ and $\mathrm{Vect}(M)$. If $(x^1, \dots, x^n)$ is a coordinate system on M, we denote by $|Dx|^\lambda$ the local λ-density equal to $[(\mathrm{Id}, 1)]$, where Id is the identity frame in the coordinates system $(x^1, \dots, x^n)$.

Actually, a λ-density φ at a point $x \in M$ can be viewed as a map on $\wedge^n T_xM$ with values in $\mathbb{R}$ such that

$$\varphi(cX_1 \wedge \dots \wedge X_n) = |c|^\lambda \varphi(X_1 \wedge \dots \wedge X_n)$$

for all $X_1, \dots, X_n \in T_xM$ and $c \in \mathbb{R}$. The λ-density $|Dx|^\lambda$ is then the λ-density equal to one on $\partial_1 \wedge \dots \wedge \partial_n$, where $\partial_1, \dots, \partial_n$ denotes the canonical basis of T_xM corresponding to the coordinate system $(x^1, \dots, x^n)$.

If a λ-density φ reads locally $f|Dx|^\lambda$, where f is a local function, then the Lie derivative of φ in the direction of a vector field X reads locally

$$L_X^\lambda \varphi = \big(X.f + \lambda\big(\partial_i X^i\big) f\big)|Dx|^\lambda. \tag{2.1}$$

It is possible to define the multiplication of two densities. If φ_1 reads locally $f|Dx|^\lambda$ and if φ_2 reads locally $g|Dx|^\delta$, then $\varphi_1\varphi_2$ reads locally $fg|Dx|^{\lambda+\delta}$.

On a pseudo-Riemannian manifold (M, g), it is possible to define in a natural way a λ-density. In a coordinate system, this λ-density reads

$$|\mathrm{Vol_g}|^\lambda = |\det \mathrm{g}|^{\frac{\lambda}{2}} |Dx|^\lambda,$$

where $|\det \mathrm{g}|$ denotes the absolute value of the determinant of the matrix representation of g in the coordinate system.

We shall denote by $\mathcal{D}_{\lambda,\mu}(M)$ the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. It is the space of linear maps between $\mathcal{F}_\lambda(M)$ and $\mathcal{F}_\mu(M)$ that read in trivialization charts as differential operators. The actions of $\mathrm{Vect}(M)$ and $\mathrm{Diff}(M)$ on $\mathcal{D}_{\lambda,\mu}(M)$ are induced by the actions on tensor densities: if $\mathcal{L}_X D$ denotes the Lie derivative of the differential operator D in the direction of the vector field X, we have

$$\mathcal{L}_X D = L_X^\mu \circ D - D \circ L_X^\lambda, \qquad \forall\, D \in \mathcal{D}_{\lambda,\mu}(M) \qquad \text{and} \qquad \forall\, X \in \mathrm{Vect}(M).$$
$$\phi \cdot D = \phi \circ D \circ \phi^{-1}, \qquad \forall\, D \in \mathcal{D}_{\lambda,\mu}(M) \qquad \text{and} \qquad \forall\, \phi \in \mathrm{Diff}(M).$$

The space $\mathcal{D}_{\lambda,\mu}(M)$ is filtered by the order of differential operators. We denote by $\mathcal{D}^k_{\lambda,\mu}(M)$ the space of differential operators of order k. It is well-known that this filtration is preserved by the action of local diffeomorphisms.

On a pseudo-Riemannian manifold (M, g), it is easy to build an isomorphism between $\mathcal{D}_{\lambda,\mu}(M)$ and $\mathcal{D}(M)$, the space of differential operators acting between functions. Indeed, thanks to the canonical densities built from $|\mathrm{Vol_g}|$, all operators $D \in \mathcal{D}_{\lambda,\mu}(M)$ can be pulled-back on functions as follows

$$\begin{tikzcd} \mathcal{F}_\lambda(M) \arrow[r, "D"] & \mathcal{F}_\mu(M) \\ \mathcal{C}^\infty(M) \arrow[u, "|\mathrm{Vol_g}|^\lambda"] \arrow[r, "|\mathrm{Vol_g}|^{-\mu}\circ D\circ|\mathrm{Vol_g}|^\lambda"'] & \mathcal{C}^\infty(M) \arrow[u, "|\mathrm{Vol_g}|^\mu"'] \end{tikzcd} \tag{2.2}$$

The space of *symbols* is the graded space associated with $\mathcal{D}_{\lambda,\mu}(M)$: it is then equal to

$$\mathrm{gr}\mathcal{D}_{\lambda,\mu}(M) := \bigoplus_{k=0}^{\infty} \mathcal{D}^k_{\lambda,\mu}(M)/\mathcal{D}^{k-1}_{\lambda,\mu}(M).$$

The canonical projection $\sigma_k : \mathcal{D}^k_{\lambda,\mu}(M) \to \mathcal{D}^k_{\lambda,\mu}(M)/\mathcal{D}^{k-1}_{\lambda,\mu}(M)$ is called the principal symbol map. As the actions of $\mathrm{Diff}(M)$ and $\mathrm{Vect}(M)$ preserve the filtration of $\mathcal{D}_{\lambda,\mu}(M)$, they induce actions of $\mathrm{Diff}(M)$ and $\mathrm{Vect}(M)$ on the space of symbols.

Let $\delta = \mu - \lambda$ be the shift of weights. If the sum of the k-order terms of $D \in \mathcal{D}^k_{\lambda,\mu}$ in a coordinate system $(x^1, \ldots, x^n)$ reads

$$D^{i_1\ldots i_k}\partial_{i_1}\cdots\partial_{i_k}$$

and if (x^i, p_i) is the coordinate system on T^*M canonically associated with $(x^1, \ldots, x^n)$, then we get the following identification:

$$\sigma_k(D) \qquad \longleftrightarrow \qquad D^{i_1\ldots i_k}p_{i_1}\cdots p_{i_k}.$$

Thus, the space of symbols of degree k can be viewed as the space $\mathcal{S}^k_\delta(M) := \mathrm{Pol}^k(T^*M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{F}_\delta(M)$, where $\mathrm{Pol}^k(T^*M)$ denotes the space of real functions on T^*M which are polynomial functions of degree k in the fibered coordinates of T^*M. The algebra $\mathcal{S}(M) := \mathrm{Pol}(T^*M)$ is clearly isomorphic to the algebra $\Gamma(STM)$ of symmetric tensors and depending on the context we will refer to its elements as symbols, functions on T^*M or symmetric tensors on M.

Let us recall that, if $S_1, S_2 \in \mathcal{S}(M)$, then the Poisson bracket of S_1 and S_2, denoted by $\{S_1, S_2\}$, is defined in a canonical coordinate system (x^i, p_i) of T^*M in the following way:

$$\{S_1, S_2\} = (\partial_{p_i}S_1)\big(\partial_{x^i}S_2\big) - (\partial_{p_i}S_2)(\partial_{x^i}S_1). \tag{2.3}$$

We conclude this subsection by two properties of the principal symbol map linked to the composition and to the commutator of differential operators. For all $k, l \in \mathbb{N}$, we have:

$$\sigma_{k+l}(A \circ B) = \sigma_k(A)\sigma_l(B), \tag{2.4}$$
$$\sigma_{k+l-1}([A, B]) = \{\sigma_k(A), \sigma_l(B)\}, \tag{2.5}$$

where A and B are elements of $\mathcal{D}(M)$ of order k and l respectively.

2.2 Pseudo-Riemannian and conformal geometry

Let (M, g) be a pseudo-Riemannian manifold. The isometries Φ of (M, g) are the diffeomorphisms of M that preserve the metric g, i.e. $\Phi^*\mathrm{g} = \mathrm{g}$. Their infinitesimal counterparts $X \in \mathrm{Vect}(M)$ are called Killing vector fields, they satisfy $L_X\mathrm{g} = 0$, with $L_X\mathrm{g}$ the Lie derivative of g along X.

Given the Levi-Civita connection ∇ corresponding to the metric g, the Riemannian curvature tensor, which reads as $\mathrm{R}_{ab}{}^c{}_d$ in abstract index notation, is given by $[\nabla_a, \nabla_b]v^c = \mathrm{R}_{ab}{}^c{}_d v^d$ for a tangent vector field v^c. Then, one gets the Ricci tensor by taking a trace of the Riemann tensor, which is indicated by repeated indices: $\mathrm{Ric}_{bd} = \mathrm{R}_{ab}{}^a{}_d$. By contraction with the metric, the Ricci tensor leads to the scalar curvature $\mathrm{Sc} = \mathrm{g}^{ab}\,\mathrm{Ric}_{ab}$.

A conformal structure on a smooth manifold M is given by the conformal class [g] of a pseudo-Riemannian metric g, where two metrics g and $\hat{\mathrm{g}}$ are conformally related if $\hat{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$, for some function $\Upsilon \in \mathcal{C}^\infty(M)$. The conformal diffeomorphisms Φ of $(M, [\mathrm{g}])$ are those which preserve the conformal structure [g], i.e. there exists $\Upsilon \in \mathcal{C}^\infty(M)$ such that $\Phi^*\mathrm{g} = e^{2\Upsilon}\mathrm{g}$. Their infinitesimal counterparts $X \in \mathrm{Vect}(M)$ are called conformal Killing vector fields, they satisfy $L_X\mathrm{g} = f_X\mathrm{g}$, for some function $f_X \in \mathcal{C}^\infty(M)$.

Let (x^i, p_i) be a canonical coordinate system on T^*M. If M is endowed with a metric g, we define the metric symbol and the trace operator by, respectively,

$$H = \mathrm{g}^{ij}p_ip_j \qquad \text{and} \qquad \mathrm{Tr} = \mathrm{g}_{ij}\partial_{p_i}\partial_{p_j}.$$

Note that the symbol $|\mathrm{Vol}_\mathrm{g}|^{2/n}H \in \mathcal{S}_{2/n}$ and the operator $|\mathrm{Vol}_\mathrm{g}|^{-2/n}\,\mathrm{Tr} : \mathcal{S}_\delta \to \mathcal{S}_{\delta-2/n}$ are conformally invariant. In consequence, we get a conformally invariant decomposition

$$S^kTM = \bigoplus_{0\leq 2s\leq k} S^{k,s}TM, \tag{2.6}$$

where $S \in \mathcal{S}^{k,s}(M) := \Gamma(S^{k,s}TM)$ is of the form $S = H^sS_0$ with $\mathrm{Tr}\,S_0 = 0$.

2.3 The conformal Laplacian

Starting from a pseudo-Riemannian manifold (M, g) of dimension n, one can define the Yamabe Laplacian, acting on functions, in the following way:

$$\Delta_Y := \nabla_a\mathrm{g}^{ab}\nabla_b - \frac{n-2}{4(n-1)}\,\mathrm{Sc},$$

where ∇ denotes the Levi-Civita connection of g and Sc the scalar curvature. For the conformally related metric $\hat{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$, the associated Yamabe Laplacian is given by

$$\widehat{\Delta_Y} = e^{-\frac{n+2}{2}\Upsilon} \circ \Delta_Y \circ e^{\frac{n-2}{2}\Upsilon}.$$

According to the transformation law $|\mathrm{Vol}_{\hat{\mathrm{g}}}| = e^{n\Upsilon}|\mathrm{Vol}_\mathrm{g}|$ and to the diagram (2.2), this translates into the conformal invariance of Δ_Y viewed as an element of $\mathcal{D}_{\lambda_0,\mu_0}(M)$, for the specific weights

$$\lambda_0 = \frac{n-2}{2n}, \qquad \mu_0 = \frac{n+2}{2n} \qquad \text{and} \qquad \delta_0 = \mu_0 - \lambda_0 = \frac{2}{n}. \tag{2.7}$$

Thus, the data of a conformal manifold $(M, [\mathrm{g}])$ is enough to define $\Delta_Y \in \mathcal{D}_{\lambda_0,\mu_0}(M)$. We write it below as $\Delta_Y^M(\mathrm{g})$ and we refer to it as the Yamabe or conformal Laplacian. One easily gets

Proposition 2.1. *The conformal Laplacian is a natural conformally invariant operator, i.e.*

- *it satisfies the naturality condition:*

$$\Delta_Y^N(\Phi^*\mathrm{g}) = \Phi^*\left(\Delta_Y^M(\mathrm{g})\right), \tag{2.8}$$

for all diffeomorphisms $\Phi : N \to M$ and for all pseudo-Riemannian metric g on M,

- *it is conformally invariant, $\Delta_Y^M(e^{2\Upsilon}\mathrm{g}) = \Delta_Y^M(\mathrm{g})$ for all $\Upsilon \in \mathcal{C}^\infty(M)$.*

More generally, a natural operator over pseudo-Riemannian manifolds is an operator that acts between natural bundles, is defined over any pseudo-Riemannian manifold (M, g) and satisfies an analogue of the naturality condition (2.8). It is said to be conformally invariant if it depends only on the conformal class of g. For a general study of natural operators in the pseudo-Riemannian setting, see the book [20].

From Proposition 2.1, we deduce that the conformal Laplacian Δ_Y is invariant under the action of conformal diffeomorphisms, which reads infinitesimally as

$$L_X^{\mu_0} \circ \Delta_Y = \Delta_Y \circ L_X^{\lambda_0}, \tag{2.9}$$

for all conformal Killing vector fields X. Here, as introduced in (2.1), L^{λ_0} and L^{μ_0} denote the Lie derivatives of λ_0- and μ_0-densities. If the manifold $(M, [\mathrm{g}])$ is locally conformally flat, then, up to multiplication by a scalar, Δ_Y is the unique second order operator acting on densities which is invariant under the action (2.9) of conformal Killing vector fields.

2.4 Natural and conformally invariant quantization

Recall first the definition of a quantization on a smooth manifold M.

Definition 2.2. Let $\lambda, \mu \in \mathbb{R}$ and $\delta = \mu - \lambda$. A quantization on M is a linear bijection $\mathcal{Q}^M_{\lambda,\mu}$ from the space of symbols $\mathcal{S}_\delta(M)$ to the space of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ such that

$$\sigma_k\big(\mathcal{Q}^M_{\lambda,\mu}(S)\big) = S, \qquad \forall\, S \in \mathcal{S}^k_\delta(M), \quad \forall\, k \in \mathbb{N}.$$

On locally conformally flat manifolds $(M, [\mathrm{g}])$, for generic weights λ, μ, there exists a unique conformally equivariant quantization [12], i.e. a unique quantization which intertwines the actions of the conformal Killing vector fields on $\mathcal{S}_\delta(M)$ and on $\mathcal{D}_{\lambda,\mu}(M)$. In the following, we need an extension of the conformally equivariant quantization to arbitrary conformal manifolds. This is provided by the notion of *natural and conformally invariant quantization.* The definition and the conjecture of the existence of such a quantization were given for the first time in [22].

Definition 2.3. A natural and conformally invariant quantization is the data for every pseudo-Riemannian manifold (M, g) of a quantization $\mathcal{Q}^M_{\lambda,\mu}(\mathrm{g})$, which satisfies

- the naturality condition:

$$\mathcal{Q}^N_{\lambda,\mu}(\Phi^*\mathrm{g})(\Phi^* S) = \Phi^*(\mathcal{Q}^M_{\lambda,\mu}(\mathrm{g})(S)), \qquad \forall\, S \in \mathcal{S}_\delta(M), \tag{2.10}$$

for all diffeomorphisms $\Phi : N \to M$ and for all pseudo-Riemannian metric g on M.

- the conformal invariance: $\mathcal{Q}^M_{\lambda,\mu}(e^{2\Upsilon}\mathrm{g}) = \mathcal{Q}^M_{\lambda,\mu}(\mathrm{g})$ for all $\Upsilon \in \mathcal{C}^\infty(M)$.

In the following we refer to a quantization map by $\mathcal{Q}_{\lambda,\mu}$, the dependence in the chosen pseudo-Riemannian manifold (M, g) being understood. Accordingly, we drop the reference to M in the spaces of densities $\mathcal{F}_\lambda$, symbols $\mathcal{S}_\delta$ and differential operators $\mathcal{D}_{\lambda,\mu}$.

The concept of natural and conformally invariant quantization is an extension to quantizations of the more usual one of natural conformally invariant operator, introduced in the previous section. Restricting to conformally flat manifolds $(M,[\mathrm{g}])$ and to $\Phi\in\mathrm{Diff}(M)$ preserving $[\mathrm{g}]$, the naturality condition (2.10) reads as conformal equivariance of the quantization map $\mathcal{Q}_{\lambda,\mu}$. Thus, the problem of the natural and conformally invariant quantization on an arbitrary manifold generalizes the problem of the conformally equivariant quantization on conformally flat manifolds.

Remark that the bundles S^kTM are natural bundles over $(M,[\mathrm{g}])$. Hence, one can consider natural and conformally invariant quantization restricted to the subspaces of symbols $\mathcal{S}^k_\delta$ or $\mathcal{S}^{\leq k}_\delta=\bigoplus_{j\leq k}\mathcal{S}^j_\delta$. In a first step, the proofs of the existence of a natural and conformally invariant quantization at the second and the third orders were given respectively in [13] and [23], together with explicit formulas. We provide the one at order 2, which we will need later on.

Theorem 2.4 ([13])**.** *Let $\delta\notin\{\frac{2}{n},\frac{n+2}{2n},1,\frac{n+1}{n},\frac{n+2}{n}\}$. A natural and conformally invariant quantization $\mathcal{Q}_{\lambda,\mu}:\mathcal{S}^{\leq 2}_\delta\to\mathcal{D}^2_{\lambda,\mu}$ is provided, on a pseudo-Riemannian manifold (M,g) of dimension n, by the formulas*

$$\begin{aligned}
&\mathcal{Q}_{\lambda,\mu}(f)=f,\\
&\mathcal{Q}_{\lambda,\mu}(X)=X^a\nabla_a+\frac{\lambda}{1-\delta}(\nabla_aX^a),\\
&\mathcal{Q}_{\lambda,\mu}(S)=S^{ab}\nabla_a\nabla_b+\beta_1(\nabla_aS^{ab})\nabla_b+\beta_2\mathrm{g}^{ab}(\nabla_a\operatorname{Tr}S)\nabla_b\\
&\qquad+\beta_3\big(\nabla_a\nabla_bS^{ab}\big)+\beta_4\mathrm{g}^{ab}\nabla_a\nabla_b(\operatorname{Tr}S),+\beta_5\operatorname{Ric}_{ab}S^{ab}+\beta_6\operatorname{Sc}(\operatorname{Tr}S),
\end{aligned}\tag{2.11}$$

where f, X, S are symbols of degrees 0, 1, 2 respectively and $\operatorname{Tr}S=\mathrm{g}_{ab}S^{ab}$. Moreover the coefficients β_i entering the last formula are given by

$$\begin{aligned}
&\beta_1=\frac{2(n\lambda+1)}{2+n(1-\delta)},\\
&\beta_2=\frac{n(\lambda+\mu-1)}{(2+n(1-\delta))(2-n\delta)},\\
&\beta_3=\frac{n\lambda(n\lambda+1)}{(1+n(1-\delta))(2+n(1-\delta))},\\
&\beta_4=\frac{n\lambda(n^2\mu(2-\lambda-\mu)+2(n\lambda+1)^2-n(n+1))}{(1+n(1-\delta))(2+n(1-\delta))(2+n(1-2\delta))(2-n\delta)},\\
&\beta_5=\frac{n^2\lambda(\mu-1)}{(n-2)(1+n(1-\delta))},\\
&\beta_6=\frac{n^2\lambda(\mu-1)(n\delta-2)}{(n-1)(n-2)(1+n(1-\delta))(2+n(1-2\delta))}.
\end{aligned}\tag{2.12}$$

In a second step, the proof of the existence of such a quantization, at an arbitrary order and for generic values of λ, μ, was given in [9, 25, 30] in different ways. We provide a slightly refined statement in the next section.

2.5 Adjoint operation and quantization

For all weights $\lambda\in\mathbb{R}$, there exists a non-degenerate symmetric bilinear pairing

$$\begin{aligned}
&\mathcal{F}^c_\lambda\times\mathcal{F}^c_{1-\lambda}\to\mathbb{R},\\
&(\varphi,\psi)\qquad\mapsto\int_M\varphi\psi,
\end{aligned}$$

where $\mathcal{F}^c_\lambda$ is the space of compactly supported λ-densities. On a manifold M, this pairing is $\mathrm{Diff}(M)$-invariant since 1-density is the right object for integration. In consequence, we can define an adjoint operation $* : \mathcal{D}_{\lambda,\mu} \to \mathcal{D}_{1-\mu,1-\lambda}$ by

$$\big(\varphi, D^*\psi\big) = (D\varphi, \psi),$$

for all $\varphi \in \mathcal{F}^c_\lambda$ and $\psi \in \mathcal{F}^c_{1-\mu}$. We introduce the following subset of $\mathbb{R}^2$,

$$I = \left\{(\lambda,\mu) \in \mathbb{R}^2 \,\middle|\, \mu - \lambda \notin \frac{1}{2n}(\mathbb{N}\setminus\{0\})\right\} \cup \{(\lambda_0,\mu_0)\},$$

where $\lambda_0 = \frac{n-2}{2n}$ and $\mu_0 = \frac{n+2}{2n}$ are the weights of the conformal Laplacian (see (2.7)). The set I is stable under the involutive map $(\lambda,\mu) \mapsto (1-\mu, 1-\lambda)$. Note that

$$\sigma_2(\Delta_Y) = |\mathrm{Vol_g}|^{\delta_0} H,$$

where $\delta_0 = \mu_0 - \lambda_0$ and $H = \mathrm{g}^{ij} p_i p_j$ in canonical coordinates. The proof of existence of a natural and conformally invariant quantization $\mathcal{Q}_{\lambda,\mu}$ in [30] leads easily to the following statement.

Theorem 2.5. *There exists a family $(\mathcal{Q}_{\lambda,\mu})_{(\lambda,\mu)\in I}$ of natural and conformally invariant quantizations that satisfies:*

- *the reality condition:*
$$\mathcal{Q}_{\lambda,\mu}(S)^* = (-1)^k \mathcal{Q}_{1-\mu,1-\lambda}(S), \qquad \forall\, S \in \mathcal{S}^k_\delta, \qquad \forall\,(\lambda,\mu) \in I, \tag{2.13}$$
- *the factorization property:*
$$\begin{aligned} \mathcal{Q}_{\lambda_0,\lambda_0}\big(|\mathrm{Vol_g}|^{\delta_0} HS\big) &= \mathcal{Q}_{\mu_0,\lambda_0}(S) \circ \Delta_Y, \qquad \forall\, S \in \mathcal{S}^k_{-\delta_0},\\ \mathcal{Q}_{\mu_0,\mu_0}\big(|\mathrm{Vol_g}|^{\delta_0} HS\big) &= \Delta_Y \circ \mathcal{Q}_{\mu_0,\lambda_0}(S), \qquad \forall\, S \in \mathcal{S}^k_{-\delta_0}, \end{aligned} \tag{2.14}$$
- *the restriction of $\mathcal{Q}_{\lambda,\mu}$ to $\mathcal{S}^{\leq 2}_\delta$ is given by the formulas in (2.11) if $(\lambda,\mu) \in I \setminus \{(\lambda_0,\mu_0)\}$.*

Proof. We prove the theorem in four steps.

In [30, Theorem 4.4], one of us determines that for $(\lambda,\mu) \in I$ there exists a natural and conformally invariant quantization map $\mathcal{Q}'''_{\lambda,\mu}$.

From the above family of quantizations $(\mathcal{Q}'''_{\lambda,\mu})_{(\lambda,\mu)\in I}$, we define $(\mathcal{Q}''_{\lambda,\mu})_{(\lambda,\mu)\in I}$ by

$$\mathcal{Q}''_{\lambda,\mu} : \; S \;\mapsto\; \frac{1}{2}\left(\mathcal{Q}'''_{\lambda,\mu}(S) + (-1)^k \mathcal{Q}'''_{1-\mu,1-\lambda}(S)^*\right), \qquad \forall\, S \in \mathcal{S}^k_\delta, \quad \forall\,(\lambda,\mu) \in I.$$

The maps $\mathcal{Q}''_{\lambda,\mu}$ are again natural and conformally invariant quantizations. Indeed, the adjoint operation $*$ is natural, does not depend of the choice of metric on M and satisfies

$$\sigma_k\big(D^*\big) = (-1)^k \sigma_k(D),$$

for all differential operators D of order k. The newly defined quantization maps clearly satisfy the property (2.13) since $*$ is an involution.

For $(\lambda,\mu) \in I \setminus \{(\lambda_0,\lambda_0),(\mu_0,\mu_0)\}$, we define $\mathcal{Q}'_{\lambda,\mu} := \mathcal{Q}''_{\lambda,\mu}$. On the space of traceless symbols we set $\mathcal{Q}'_{\lambda_0,\lambda_0} := \mathcal{Q}''_{\lambda_0,\lambda_0}$ and $\mathcal{Q}'_{\mu_0,\mu_0} := \mathcal{Q}''_{\mu_0,\mu_0}$. We extend both maps to the whole symbol space by the formulas in (2.14). They are clearly still natural and conformally invariant and satisfy the reality condition (2.13).

For all $(\lambda,\mu) \in I \setminus \{(\lambda_0,\mu_0)\}$, we denote by $\mathcal{Q}_{\lambda,\mu}$ the natural and conformally invariant quantizations restricted to $\mathcal{S}^{\leq 2}_\delta$, given by the formulas (2.11). A direct computation shows that they satisfy the reality condition (2.13) and the factorization property (2.14). For $(\lambda,\mu) = (\lambda_0,\mu_0)$, we set $\mathcal{Q}_{\lambda_0,\mu_0} := \mathcal{Q}'_{\lambda_0,\mu_0}$ on $\mathcal{S}^{\leq 2}_{\delta_0}$. We extend then the quantizations $\mathcal{Q}_{\lambda,\mu}$ (where $(\lambda,\mu) \in I$) to the whole symbol space by setting $\mathcal{Q}_{\lambda,\mu} := \mathcal{Q}'_{\lambda,\mu}$ on $\mathcal{S}^{\geq 3}_\delta$, for all $(\lambda,\mu) \in I$. ∎

In the following, the quantization maps $\mathcal{Q}_{\lambda,\mu}$ that we will use are always taken from a family $(\mathcal{Q}_{\lambda,\mu})_{(\lambda,\mu)\in I}$ provided by Theorem 2.5. In fact, we will need only four of them, namely: $\mathcal{Q}_{\lambda_0,\lambda_0}$, $\mathcal{Q}_{\mu_0,\mu_0}$, $\mathcal{Q}_{\lambda_0,\mu_0}$, $\mathcal{Q}_{\mu_0,\lambda_0}$. With such a convention, it is worth noticing that the conformal Laplacian can be obtained as

$$\Delta_Y = \mathcal{Q}_{\lambda_0,\mu_0}\big(|\mathrm{Vol}_{\mathrm{g}}|^{\delta_0} H\big).$$

The conformal invariance of the symbol $|\mathrm{Vol}_{\mathrm{g}}|^{\delta_0} H$ translates into the conformal invariance of Δ_Y.

3 On particular conformally invariant operators

First, we introduce notation for classical objects of the pseudo-Riemannian and conformal geometries and recall basic facts about natural and conformally invariant operators. Then, we classify the natural conformally invariant operators between particular subspaces of symbols.

3.1 More on pseudo-Riemannian and conformal geometry

We complete here Section 2.2, and use freely the notation introduced there.

First, we work over a pseudo-Riemannian manifold. The Riemann tensor admits the following decomposition

$$\mathrm{R}_{ab}{}^c{}_d = \mathrm{C}_{ab}{}^c{}_d + 2\delta^c_{[a}\mathrm{P}_{b]d} + 2\mathrm{g}_{d[b}\mathrm{P}_{a]}{}^c, \tag{3.1}$$

where $\mathrm{C}_{ab}{}^c{}_d$ is the totally trace-free Weyl curvature, $\mathrm{P}_{ab} = \frac{1}{n-2}\Big(\mathrm{Ric}_{ab} - \frac{1}{2(n-1)}\,\mathrm{Sc}\,\mathrm{g}_{ab}\Big)$ is the Schouten tensor, δ_a^b is the Kronecker delta and square brackets denote antisymmetrization of enclosed indices. The Weyl tensor C_{abcd} is zero for the dimension $n = 3$. Note also that C_{abcd} obeys the same symmetries of indices as R_{abcd} does. Further curvature quantities we shall need are

$$\mathrm{J} = \mathrm{g}^{ab}\mathrm{P}_{ab} \qquad \text{and} \qquad \mathrm{A}_{abc} = 2\nabla_{[b}\mathrm{P}_{c]a},$$

where A_{abc} is the Cotton–York tensor and J is related to the scalar curvature via $\mathrm{J} = \frac{1}{2(n-1)}\,\mathrm{Sc}$. Bianchi identities have the from $\mathrm{R}_{[abc]d} = 0$ and $\nabla_{[a}\mathrm{R}_{bc]de} = 0$ and lead to

$$(n-3)\mathrm{A}_{abc} = \nabla_r \mathrm{C}_{bc}{}^r{}_a \qquad \text{and} \qquad \nabla_b \mathrm{P}^b{}_a = \nabla_a \mathrm{J}.$$

Second, we consider a conformal manifold $(M, [\mathrm{g}])$. The Weyl tensor $\mathrm{C}_{ab}{}^c{}_d$ is a conformal invariant, i.e. it does not depend on the choice of the representative metric from $[\mathrm{g}]$. The same is true for A_{abc} in the dimension 3. Further, a choice of metric provides a canonical trivialization of the bundle of λ-densities F_λ via the global section $|\mathrm{Vol}_{\mathrm{g}}|^\lambda$ (see Section 2.1). According to the transformation rule $|\mathrm{Vol}_{\hat{\mathrm{g}}}|^\lambda = e^{n\lambda\Upsilon}|\mathrm{Vol}_{\mathrm{g}}|$ if $\hat{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$, we have the conformally invariant object $\mathbf{g}_{ab} = \mathrm{g}_{ab} \otimes |\mathrm{Vol}_{\mathrm{g}}|^{-\frac{2}{n}}$, termed *conformal metric*, with the inverse $\mathbf{g}^{ab}$ in $\Gamma(S^2TM)\otimes\mathcal{F}_{2/n}$, see e.g. [1] for details. (Note that the space of densities $\mathcal{E}[w]$ in [1] corresponds to $\mathcal{F}_{-w/n}$ in our notation.) The conformal metric gives a conformally invariant identification $TM \cong T^*M \otimes F_{-2/n}$. In other words, we can raise and lower indices, with expense of the additional density, in a conformally invariant way. For example, we get $\mathrm{C}_{abcd} \in \Gamma(S^2(\Lambda^2 T^*M)) \otimes \mathcal{F}_{-2/n}$. Note also that $\mathbf{g}_{ab}$ and $\mathbf{g}^{ab}$ are parallel for any choice of a Levi-Civita connection from the conformal class.

3.2 Description of natural conformally invariant operators

Now we recall basic facts about natural and conformally invariant operators. Every natural operator on the Riemannian structure (M, g) between natural bundles V_1 and V_2 is a linear combination of terms of the form

$$\underbrace{\mathrm{g}^{-1}\cdots\mathrm{g}^{-1}}_{r_1}\underbrace{\mathrm{g}\cdots\mathrm{g}}_{r_2}\underbrace{(\nabla^{(i_1)}\mathrm{R})\cdots(\nabla^{(i_s)}\mathrm{R})}_{s}\underbrace{\nabla\cdots\nabla}_{t}f \tag{3.2}$$

to which one applies a $\mathrm{GL}(n)$-invariant operation

$$\Gamma\big(\bigotimes{}^{2r_1+s}TM\otimes\bigotimes{}^{2r_2+3s+i+t}T^*M\otimes V_1\big)\longrightarrow\Gamma(V_2). \tag{3.3}$$

Here $f\in\Gamma(V_1)$, g^{-1} stands for the inverse of the metric g, $\nabla^{(i_j)}$ denotes the i_jth iterated covariant derivative where $i_j\geq 0$, abstract indices are omitted, $i=i_1+\cdots+i_s$, and ∇ and R correspond to the choice of the metric g. The existence of a $\mathrm{GL}(n)$-invariant operation (3.3) gives in general constraints on the possible values of r_1, r_2, s, t, i. See [20] for details.

A natural operator on (M, g) is conformally invariant if it does not depend on the choice of metric in the conformal class. Then, it defines a natural operator on the conformal structure $(M, [\mathrm{g}])$. It is convenient to use the conformal metric $\mathbf{g}$ instead of g and the inverse $\mathbf{g}^{-1}$ instead of g^{-1} in (3.2) since they are conformally invariant, namely

$$\begin{aligned}&\underbrace{\mathbf{g}^{-1}\cdots\mathbf{g}^{-1}}_{r_1}\underbrace{\mathbf{g}\cdots\mathbf{g}}_{r_2}\underbrace{(\nabla^{(i_1)}\mathrm{R})\cdots(\nabla^{(i_s)}\mathrm{R})}_{s}\underbrace{\nabla\cdots\nabla}_{t}f\\ &\qquad\in\Gamma\left(\bigotimes{}^{2r_1+s}TM\otimes\bigotimes{}^{2r_2+3s+i+t}T^*M\otimes V_1\otimes F_{\frac{2}{n}(r_1-r_2)}\right)\end{aligned} \tag{3.4}$$

for $f\in\Gamma(V_1)$. It is generally a difficult problem to determine which linear combinations of terms as in (3.2), together with suitable projections as in (3.3), give rise to a conformally invariant operator. We shall need details only in specific cases.

3.3 Conformally invariant operators on the symbol space

This section concerns existence and uniqueness of natural and conformally invariant operators of certain type. The first one is well-known and can be obtained as an easy consequence of [17], or deduced from the general work [10] on curved BGG-sequences. We present a detailed proof to demonstrate the technique which is used (in much more complicated setting) later in the proof of Proposition 3.2.

Recall first that $\Gamma(S^{k,0}TM)$ is the space of trace-less symmetric k-tensors. In terms of the abstract index notation, a section f of $\bigotimes^k TM$ is denoted by $f^{a_1\cdots a_k}$. In the following, we write $f^{[a_1\ldots a_k]}$, $f^{(a_1\ldots a_k)}$ and $f^{(a_1\ldots a_k)_0}$ for the projections of f to $\Gamma(\Lambda^kTM)$, $\Gamma(S^kTM)$ and $\Gamma(S^{k,0}TM)$, respectively. Similar notation will be used for covariant indices.

Proposition 3.1. *Up to multiplication by a scalar, there exists a unique natural conformally invariant operator $\mathcal{S}^k_0\to\mathcal{S}^{k+1}_{2/n}$. It is given by the conformal Killing operator $\mathbf{G}$, such that for all $f\in\mathcal{S}^k_0$,*

$$(\mathbf{G}(f))^{a_0\ldots a_k}=\nabla^{(a_0}f^{a_1\ldots a_k)_0}. \tag{3.5}$$

Proof. Identifying $\mathcal{S}^k_0$ and $\mathcal{S}^{k+1}_{2/n}$ with corresponding spaces of sections of symmetric tensors, we consider natural and conformally invariant operators $\Gamma(S^kTM)\to\Gamma(S^{k+1}TM\otimes F_{2/n})$. By naturality, such operators are linear combinations of terms in (3.2) with $V_1 = S^kTM$, composed with GL_n-invariant maps

$$\bigotimes{}^{2r_1+s}TM \otimes \bigotimes{}^{2r_2+3s+i+t}T^*M \otimes S^kTM \otimes F_{\frac{2}{n}(r_1-r_2)} \longrightarrow S^{k+1}TM \otimes F_{2/n}.$$

Explicitly, those maps may consist of: contracting covariant and contravariant indices, projecting the covariant and contravariant tensors on tensors of prescribed symmetry type (given by a Young diagram) and tensorizing with the density $|\mathrm{Vol}_{\mathrm{g}}|^\delta$ for arbitrary $\delta \in \mathbb{R}$. The conformal invariance does not allow for the last operation, hence $r_1-r_2 = 1$. The difference between the number of covariant and contravariant indices is a constant therefore $(2r_1+s+k)-(2r_2+3s+i+t) = k+1$, i.e. $2s+i+t = 1$. This means $s = i = 0$ and $t = 1$. The sought operators are then first order (gradient) natural operators and using moreover the conformal invariance, the statement follows from the classification in [17]. ■

The next proposition is a crucial technical tool in the following.

Proposition 3.2. *Every natural conformally invariant operator $\mathcal{S}^{k,0}_0 \to \mathcal{S}^{k-1}_{2/n}$ has its target space in $\mathcal{S}^{k-1,0}_{2/n} \subseteq \mathcal{S}^{k-1}_{2/n}$. The space of natural conformally invariant operators $\mathcal{S}^{k,0}_0 \to \mathcal{S}^{k-1,0}_{2/n}$ on a pseudo-Riemannian manifold (M,g) is at most two-dimensional and depends on $k \in \mathbb{N}$ as follows.*

(i) This space is trivial for $k = 1$.

(ii) If $k = 2$ or $n = 3$, this space is one-dimensional and generated by the operator $\mathbf{F}$ such that, for all $f \in \mathcal{S}^{k,0}_0$,

$$(\mathbf{F}(f))^{a_1\dots a_{k-1}} = \mathrm{C}^r{}_{st}{}^{(a_1}\nabla_r f^{a_2\dots a_{k-1})_0 st} - (k+1)\mathrm{A}_{st}{}^{(a_1} f^{a_2\dots a_{k-1})_0 st}.$$

(iii) If $k = 3$ and $n > 3$, this space is two-dimensional and generated by two operators, $\mathbf{F}_1$ and $\mathbf{F}_2$, such that for all $f \in \mathcal{S}^{k,0}_0$,

$$\begin{aligned}(\mathbf{F}_1(f))^{a_1\dots a_{k-1}} &= (\mathbf{F}(f))^{a_1\dots a_{k-1}} + \frac{k-2}{n+2k-2}\mathrm{C}^{(a_1}{}_r{}^{a_2}{}_s\nabla_t f^{a_3\dots a_{k-1})_0 rst},\\ (\mathbf{F}_2(f))^{a_1\dots a_{k-1}} &= 4\mathrm{C}^{(a_1}{}_r{}^{a_2}{}_s\nabla_t f^{a_3\dots a_{k-1})_0 rst} + (n+2k-2)(\nabla_r \mathrm{C}_s{}^{(a_1}{}_t{}^{a_2)}) f^{a_3\dots a_{k-1})_0 rst}\\ &\quad + 2(n+2k-2)\mathrm{A}_{rs}{}^{(a_1} f^{a_2\dots a_{k-1})_0 rs}.\end{aligned}$$

Remark 3.3. Let (x^i, p_i) be a canonical coordinate system on T^*M. We can then write the operators $\mathbf{G}$ and $\mathbf{F}$ as follows on $\mathcal{S}^k_0$

$$\mathbf{G} = \Pi_0 \circ \left(\mathbf{g}^{ij} p_i \nabla_j\right) \qquad \text{and} \qquad \mathbf{F} = \Pi_0 \circ \mathbf{g}^{im} p_i \partial_{p_j} \partial_{p_l} \left(\mathrm{C}^k{}_{jlm}\nabla_k - (k+1)\mathrm{A}_{jlm}\right), \tag{3.6}$$

where $\Pi_0 : \mathcal{S}^{k-1}_0 \to \mathcal{S}^{k-1,0}_{2/n}$ is the canonical projection on trace-less symbols. Actually, we will see in the sequel that the conformal Killing operator $\mathbf{G}$ can be used to define the conformal Killing tensors whereas the operator $\mathbf{F}$ occurs in the computation of the obstruction to the existence of conformal symmetries of Δ_Y.

Let us note that the proof of Proposition 3.2 is long, technical and interesting rather for experts in conformal geometry. The reader interested mainly in results about symmetries can continue the reading in Section 4 (details from the proof will not be needed there).

Proof. We study natural and conformally invariant operators $\Gamma(S^{k,0}TM)\to\Gamma(S^{k-1}TM\otimes F_{2/n})$. In the first part of the proof we consider the naturality and in the second part the conformal invariance.

I. **Naturality.** We start in a similar way as in the proof of Proposition 3.1. By naturality, the considered operators are linear combinations of terms in (3.2) composed with GL_n-invariant maps

$$\bigotimes{}^{2r_1+s}TM\otimes\bigotimes{}^{2r_2+3s+i+t}T^*M\otimes S^kTM\otimes F_{\frac{2}{n}(r_1-r_2)}\longrightarrow S^{k-1}TM\otimes F_{2/n}.$$

The conformal invariance of the discussed operators leads to $r_1-r_2=1$ and the GL_n-invariance of the maps above imposes $(2r_1+s+k)-(2r_2+3s+i+t)=k-1$, i.e. $2s+i+t=3$. This means either $s=i=0$, $t=3$ or $s=i=1$, $t=0$ or $s=t=1$, $i=0$. Hence, omitting abstract indices, the natural operators $\Gamma(S^{k,0}TM)\to\Gamma(S^{k-1}TM\otimes F_{2/n})$ are a linear combination of terms

$$\underbrace{\mathbf{g}^{-1}\cdots\mathbf{g}^{-1}}_{r+1}\underbrace{\mathbf{g}\cdots\mathbf{g}}_{r}\nabla\nabla\nabla f,\qquad \underbrace{\mathbf{g}^{-1}\cdots\mathbf{g}^{-1}}_{r+1}\underbrace{\mathbf{g}\cdots\mathbf{g}}_{r}\mathrm{R}\nabla f,\qquad \underbrace{\mathbf{g}^{-1}\cdots\mathbf{g}^{-1}}_{r+1}\underbrace{\mathbf{g}\cdots\mathbf{g}}_{r}(\nabla\mathrm{R})f, \tag{3.7}$$

where $r\geq 0$ and $f\in\Gamma(S^{k,0}TM)$, each of which is followed by a $\mathrm{GL}(n)$-invariant projection to $\Gamma(S^{k-1}TM\otimes F_{2/n})$. Irreducible components of the target bundle $S^{k-1}TM\otimes F_{2/n}$ are

$$S^{k-1,0}TM\otimes F_{2/n},\qquad S^{k-3,0}TM,\qquad S^{k-5,0}TM\otimes F_{-2/n},\qquad \dots,$$

but since f is trace-free, one easily verifies from (3.7) that only possible target bundles are $S^{k-1,0}TM\otimes F_{2/n}$ and $S^{k-3,0}TM$. In other words, in the expressions (3.7), one can restrict to $r=0$.

It remains to describe possible $\mathrm{GL}(n)$-invariant projections of the terms in (3.7) in details. Using the decomposition (3.1) of R into Weyl and Schouten tensors, they split into five terms: $\mathbf{g}^{-1}\nabla\nabla\nabla f$, $\mathbf{g}^{-1}\mathrm{C}\nabla f$, $\mathbf{g}^{-1}(\nabla\mathrm{C})f$, $\mathbf{g}^{-1}\mathrm{P}\nabla f$ and $\mathbf{g}^{-1}(\nabla\mathrm{P})f$.

We shall start with natural operators $\Gamma(S^{k,0}TM)\to\Gamma(S^{k-1,0}TM\otimes F_{2/n})$. In this situation, at least one of the two indices above $\mathbf{g}^{-1}$ in the expression of the operator has to be contracted with a covariant index. For an operator of type $\mathbf{g}^{-1}\nabla\nabla\nabla f$, the two resulting operators are (up to the order of covariant derivatives) respectively

$$\nabla^r\nabla_r\nabla_s f^{a_1\dots a_{k-1}s},\qquad \nabla^{(a_1}\nabla_s\nabla_t f^{a_2\dots a_{k-1})_0 st}. \tag{3.8}$$

Since the change of the order of covariant derivatives gives rise to curvature operators of the form $\mathbf{g}^{-1}\mathrm{R}\nabla f$ and $\mathbf{g}^{-1}(\nabla\mathrm{R})f$, the previous display is sufficient for operators of type $\mathbf{g}^{-1}\nabla\nabla\nabla f$. Using that C is completely trace-free and $(n-3)\mathrm{A}_{abc}=\nabla_r\mathrm{C}_{bc}{}^r{}_a$, the different possibilities of contraction of indices for the expressions $\mathbf{g}^{-1}\mathrm{C}\nabla f$ and $\mathbf{g}^{-1}(\nabla\mathrm{C})f$ lead to the operators

$$\begin{aligned}&\mathrm{C}^r{}_{st}{}^{(a_1}\nabla_r f^{a_2\dots a_{k-1})_0 st},\qquad \mathrm{C}^{(a_1}{}_r{}^{a_2}{}_s\nabla_t f^{a_3\dots a_{k-1})_0 rst},\\ &\big(\nabla_r\mathrm{C}_s{}^{(a_1}{}_t{}^{a_2})f^{a_3\dots a_{k-1}}\big)_0{}^{rst},\qquad \mathrm{A}_{rs}{}^{(a_1}f^{a_2\dots a_{k-1})_0 rs}.\end{aligned} \tag{3.9}$$

Thanks to the decomposition of P into irreducible components and to the equality $\nabla_a\mathrm{P}^a{}_b=\nabla_b J$, we see that the different configurations of indices in the expressions $\mathbf{g}^{-1}\mathrm{P}\nabla f$ and $\mathbf{g}^{-1}(\nabla\mathrm{P})f$ give rise to the operators

$$\begin{aligned}&\mathrm{P}_{(rs)_0}\nabla^r f^{sa_1\dots a_{k-1}},\qquad \mathrm{P}_{(rt)_0}\mathbf{g}^{t(a_1}\nabla_s f^{a_2\dots a_{k-1})_0 rs},\qquad \mathrm{P}_{rs}\nabla^{(a_1}f^{a_2\dots a_{k-1})_0 rs},\\ &\mathrm{J}\nabla_r f^{a_1\dots a_{k-1}r},\qquad (\nabla_{(r}\mathrm{P}_{st)_0})\mathbf{g}^{t(a_1}f^{a_2\dots a_{k-1})_0 rs},\qquad (\nabla_r\mathrm{J})f^{a_1\dots a_{k-1}r}.\end{aligned} \tag{3.10}$$

Hence all natural operators $\Gamma(S^{k,0}TM)\to\Gamma(S^{k-1,0}TM\otimes F_{2/n})$ are linear combinations of terms in (3.8)–(3.10).

A similar discussion can be applied to natural operators $\Gamma(S^{k,0}TM) \to \Gamma(S^{k-3,0}TM)$. In this situation, none of the two indices of $\mathbf{g}^{-1}$ is contracted in the expressions (3.7). Reasoning as above, using the properties of symmetry of C and the fact that C and f are trace-free, we obtain (since the target bundle is now $S^{k-3,0}TM$) a simpler list of possible terms:

$$\nabla_r\nabla_s\nabla_t f^{a_1\dots a_{k-3}rst}, \qquad \mathrm{P}_{(rs)_0}\nabla_t f^{a_1\dots a_{k-3}rst}, \qquad (\nabla_r\mathrm{P}_{st})f^{a_1\dots a_{k-3}rst}, \tag{3.11}$$

for $k \geq 3$. Hence all natural operators $\Gamma(S^{k,0}TM) \to \Gamma(S^{k-3,0}TM)$ are linear combinations of terms in (3.11).

II. **Conformal invariance.** We shall denote quantities corresponding to the conformally related metric $\hat{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$ and the corresponding Levi-Civita connection $\hat{\nabla}$ by $\hat{\mathrm{R}}_{abcd}$, $\hat{\mathrm{P}}_{ab}$, $\hat{\mathrm{J}}$ and $\hat{\mathrm{A}}_{abc}$. (The Weyl tensor is missing here since $\hat{\mathrm{C}}_{abcd} = \mathrm{C}_{abcd}$.) This transformation is controlled by the one-form $\Upsilon_a = \nabla_a\Upsilon$, see e.g. [1] for details. Explicitly, one can compute that

$$\widehat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon^r\Upsilon_r\mathrm{g}_{ab}, \tag{3.12}$$

$$\widehat{\mathrm{J}} = \mathrm{J} - \nabla^r\Upsilon_r - \frac{n-2}{2}\Upsilon^r\Upsilon_r \qquad \text{and} \qquad \widehat{\mathrm{A}}_{abc} = \mathrm{A}_{abc} + \Upsilon_r\mathrm{C}_{bc}{}^r{}_a \tag{3.13}$$

and also that

$$\widehat{\nabla}_{(a}\widehat{\mathrm{P}}_{bc)_0} = \nabla_{(a}\mathrm{P}_{bc)_0} - \nabla_{(a}\nabla_b\Upsilon_{c)_0} + 4\Upsilon_{(a}\nabla_b\Upsilon_{c)_0} - 4\Upsilon_{(a}\Upsilon_b\Upsilon_{c)_0} - 2\Upsilon_{(a}\mathrm{P}_{bc)_0}, \tag{3.14}$$

$$\widehat{\nabla}_a\widehat{\mathrm{J}} = \nabla_a\mathrm{J} - \nabla_a\nabla^r\Upsilon_r - (n-2)\Upsilon^r\nabla_r\Upsilon_a + 2\Upsilon_a\nabla^r\Upsilon_r - 2\Upsilon_a\mathrm{J} + (n-2)\Upsilon_a\Upsilon^r\Upsilon_r, \tag{3.15}$$

$$\widehat{\nabla}_{\left(a\right.}\mathrm{C}_b{}^d{}_{\left.c\right)_0}{}^e = \nabla_{(a}\mathrm{C}_b{}^d{}_{c)_0}{}^e - 4\Upsilon_{(a}\mathrm{C}_b{}^d{}_{c)_0}{}^e + 2\Upsilon_r\delta_{(a}{}^{(d}\mathrm{C}_b{}^{e)}{}_{c)}{}^r. \tag{3.16}$$

We shall start with operators $\Gamma(S^{k,0}TM) \to \Gamma(S^{k-1,0}TM \otimes F_{2/n})$. First observe that the space of such natural and conformally invariant operators is trivial in the flat case [4, 5] hence the two terms of (3.8) cannot appear. We need to know how remaining terms in (3.9) and (3.10) transform under the conformal rescaling $\hat{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$. First observe that the rescaling of first order expressions we need is

$$\begin{aligned}
\widehat{\nabla}_r f^{a_1\dots a_{k-1}r} &= \nabla_r f^{a_1\dots a_{k-1}r} + (n+2k-2)\Upsilon_r f^{a_1\dots a_{k-1}r},\\
\widehat{\nabla}_{[b}f_{c]}{}^{a_1\dots a_{k-1}} &= \nabla_{[b}f_{c]}{}^{a_1\dots a_{k-1}} + (k+1)\Upsilon_{[b}f_{c]}{}^{a_1\dots a_{k-1}} + (k-1)\Upsilon_r\delta_{[b}^{(a_1}f_{c]}{}^{a_2\dots a_{k-1})r},\\
\widehat{\nabla}^{(a_1}f^{a_2\dots a_{k-1})_0}{}_{bc} &= \nabla^{(a_1}f^{a_2\dots a_{k-1})_0}{}_{bc} + 2\Upsilon^{(a_1}f^{a_2\dots a_{k-1})_0}{}_{bc} - 2\Upsilon_{(b}f_{c)}{}^{a_1\dots a_{k-1}}\\
&\quad + 2\Upsilon_r\delta_{(b}^{(a_1}f_{c)}{}^{a_2\dots a_{k-1})_0 r},\\
\widehat{\nabla}_{(b}f_{c)}{}^{a_1\dots a_{k-1}} &= \nabla_{(b}f_{c)}{}^{a_1\dots a_{k-1}} + (k-1)\Upsilon_{(b}f_{c)}{}^{a_1\dots a_{k-1}} - (k-1)\Upsilon^{(a_1}f^{a_2\dots a_{k-1})}{}_{bc}\\
&\quad + \mathrm{g}_{bc}\Upsilon_r f^{a_1\dots a_{k-1}r} + (k-1)\Upsilon_r\delta_{(b}^{(a_1}f_{c)}{}^{a_2\dots a_{k-1})r}.
\end{aligned} \tag{3.17}$$

We are interested in linear combinations of terms in (3.9) and (3.10) which are independent on the rescaling $\hat{\mathrm{g}} = e^{2\Upsilon}\mathrm{g}$. Considering formulas (3.12)–(3.17), we observe that the term $\nabla_{(a}\nabla_b\Upsilon_{c)_0}$ appears only in (3.14) and the term $\nabla_a\nabla^r\Upsilon_r$ appears only on the right hand side of (3.15). This means, terms $(\nabla_r\mathrm{J})f^{a_1\dots a_{k-1}r}$ and $(\nabla_{(r}\mathrm{P}_{st)_0})\mathbf{g}^{t(a_1}f^{a_2\dots a_{k-1})_0 rs}$ do not appear in the required linear combination.

The Weyl tensor appears in the conformal transformation of the terms in (3.9) but not of the ones in (3.10). Therefore, we look for conformally invariant linear combinations

$$\begin{aligned}
&x_1\mathrm{C}^r{}_{st}{}^{(a_1}\nabla_r f^{a_2\dots a_{k-1})_0 st} + x_2\mathrm{C}^{(a_1}{}_r{}^{a_2}{}_s\nabla_t f^{a_3\dots a_{k-1})_0 rst}\\
&\quad + x_3\big(\nabla_r\mathrm{C}_s{}^{(a_1}{}_t{}^{a_2)}\big)f^{a_3\dots a_{k-1}\big)_0 rst} + x_4\mathrm{A}_{rs}{}^{(a_1}f^{a_2\dots a_{k-1})_0 rs}
\end{aligned} \tag{3.18}$$

and

$$y_1 \mathrm{J}\nabla_r f^{a_1\ldots a_{k-1}r} + y_2 \overline{\mathrm{P}}_{rs}\nabla^{(r} f^{s)a_1\ldots a_{k-1}} \\ + y_3 \overline{\mathrm{P}}_r{}^{(a_1}\nabla_s f^{a_2\ldots a_{k-1})_0 rs} + y_4 \overline{\mathrm{P}}_{rs}\nabla^{(a_1} f^{a_2\ldots a_{k-1})_0 rs}, \tag{3.19}$$

where $\overline{\mathrm{P}}_{rs} = \mathrm{P}_{(rs)_0}$ denotes the trace-free part of P. In other words, we search for scalars $x_i, y_j \in \mathbb{R}$ such that both (3.18) and (3.19) are invariant independently.

First we discuss (3.18) which is possible only for $k \geq 2$ and some terms only for $k \geq 3$. Assuming $k \geq 3$, conformal transformations of these terms are

$$\begin{aligned}
\mathrm{C}^r{}_{st}{}^{(a_1}\widehat{\nabla}_r f^{a_2\ldots a_{k-1})_0 st} &= \mathrm{C}^r{}_{st}{}^{(a_1}\widehat{\nabla}_r f^{a_2\ldots a_{k-1})_0 st} + (k+1)\Upsilon_r \mathrm{C}^r{}_{st}{}^{(a_1} f^{a_2\ldots a_{k-1})_0 st} \\
&\quad + (k-2)\mathrm{C}^{(a_1}{}_{st}{}^{a_2}\Upsilon_r f^{a_3\ldots a_{k-1})_0 str}, \\
\mathrm{C}^{(a_1}{}_r{}^{a_2}{}_s \widehat{\nabla}_t f^{a_3\ldots a_{k-1})_0 rst} &= \mathrm{C}^{(a_1}{}_r{}^{a_2}{}_s \nabla_t f^{a_3\ldots a_{k-1})_0 rst} - (n+2k-2) C^{(a_1}{}_{rs}{}^{a_2}\Upsilon_t f^{a_3\ldots a_{k-1})_0 rst}, \\
\big(\widehat{\nabla}_r \mathrm{C}_s{}^{(a_1}{}_t{}^{a_2)} f^{a_3\ldots a_{k-1}}\big)_0{}^{rst} &= (\nabla_r \mathrm{C}_s{}^{(a_1}{}_t{}^{a_2)}) f^{a_3\ldots a_{k-1})_0 rst} + 4\mathrm{C}^{(a_1}{}_{rs}{}^{a_2}\Upsilon_t f^{a_3\ldots a_{k-1})_0 rst} \\
&\quad - 2\Upsilon_r \mathrm{C}^r{}_{st}{}^{(a_1} f^{a_2\ldots a_{k-1})_0 st}, \\
\widehat{\mathrm{A}}_{rs}{}^{(a_1} f^{a_2\ldots a_{k-1})_0 rs} &= \mathrm{A}_{rs}{}^{(a_1} f^{a_2\ldots a_{k-1})_0 rs} + \Upsilon_r \mathrm{C}^r{}_{st}{}^{(a_1} f^{a_2\ldots a_{k-1})_0 st}
\end{aligned}$$

using (3.17), (3.16) and (3.13). Now, considering where the term $\Upsilon_r \mathrm{C}^r{}_{st}{}^{(a_1} f^{a_2\ldots a_{k-1})_0 st}$ appears in the previous display, we see that $(k+1)x_1 - 2x_3 + x_4 = 0$. Considering the other term $\mathrm{C}^{(a_1}{}_{rs}{}^{a_2}\Upsilon_t f^{a_3\ldots a_{k-1})_0 rst}$, we conclude that $(k-2)x_1 - (n+2k-2)x_2 + 4x_3 = 0$. Solutions of this pair of linear equations are generated by $(x_1, x_2, x_3, x_4) = \big(n+2k-2, k-2, 0, -(k+1)(n+2k-2)\big)$ and $(x_1, x_2, x_3, x_4) = \big(0, 4, n+2k-2, 2(n+2k-2)\big)$, therefore the space of corresponding invariant linear operators is generated by the operators $\mathbf{F}_1$ and $\mathbf{F}_2$ defined in the following way:

$$\begin{aligned}
(\mathbf{F}_1(f))^{a_1\ldots a_{k-1}} &= \mathrm{C}^r{}_{st}{}^{(a_1}\nabla_r f^{a_2\ldots a_{k-1})_0 st} - (k+1)\mathrm{A}_{rs}{}^{(a_1} f^{a_2\ldots a_{k-1})_0 rs} \\
&\quad + \frac{k-2}{n+2k-2}\mathrm{C}^{(a_1}{}_r{}^{a_2}{}_s \nabla_t f^{a_3\ldots a_{k-1})_0 rst}, \\
(\mathbf{F}_2(f))^{a_1\ldots a_{k-1}} &= 4\mathrm{C}^{(a_1}{}_r{}^{a_2}{}_s \nabla_t f^{a_3\ldots a_{k-1})_0 rst} + (n+2k-2)(\nabla_r \mathrm{C}_s{}^{(a_1}{}_t{}^{a_2)}) f^{a_3\ldots a_{k-1})_0 rst} \\
&\quad + 2(n+2k-2)\mathrm{A}_{rs}{}^{(a_1} f^{a_2\ldots a_{k-1})_0 rs}.
\end{aligned}$$

This shows that the operators in the statement of the proposition for $k \geq 3$ are invariant.

In the case $k = 2$ only some terms from (3.18) can appear. Specifically, we study the conformal invariance of the linear combination

$$x_1 \mathrm{C}^r{}_{st}{}^a \nabla_r f^{st} + x_4 \mathrm{A}_{rs}{}^a f^{rs} \tag{3.20}$$

for the section f^{ab} in $\Gamma(S^{2,0}TM)$. Since $\widehat{\nabla}_a f^{bc} = \nabla_a f^{bc} + 2\Upsilon_a f^{bc} - 2\Upsilon^{(b} f^{c)}{}_a + 2\delta_a{}^{(b}\Upsilon_r f^{c)r}$, conformal transformations of terms in the previous display are

$$\begin{aligned}
&\mathrm{C}^r{}_{st}{}^a \widehat{\nabla}_r f^{st} = \mathrm{C}^r{}_{st}{}^a \widehat{\nabla}_r f^{st} + 3\Upsilon_r \mathrm{C}^r{}_{st}{}^a f^{st} \qquad \text{and} \\
&\widehat{\mathrm{A}}_{rs}{}^a f^{rs} = \mathrm{A}_{rs}{}^a f^{rs} + \Upsilon_r \mathrm{C}^r{}_{st}{}^a f^{st}.
\end{aligned}$$

By the same reasoning as in the case $k \geq 3$, we obtain that the operator given in (3.20) is invariant if and only if (x_1, x_4) is a multiple of $(1, -3)$. In the case $k = 2$, the only invariant operators are thus the multiples of the operator $\mathbf{F}$ defined by

$$(\mathbf{F}(f))^a = \mathrm{C}^r{}_{st}{}^a \nabla_r f^{st} - 3\mathrm{A}_{rs}{}^a f^{rs}.$$

Now we shall discuss terms (3.19) and we assume $k \geq 2$ first. Consider an arbitrary but fixed point $x \in M$. We can choose the function Υ such that $\Upsilon_a(x) = 0$, $\nabla_{(a}\Upsilon_{b)_0}(x) = \Phi_{ab}(x)$

and $\nabla^r \Upsilon_r(x) = \Psi(x)$ for any prescribed values of $\Phi_{ab}(x)$ and $\Psi(x)$. Therefore, the conformal transformation of terms in (3.19) is

$$\widehat{\mathrm{J}}\widehat{\nabla}_r f^{a_1\ldots a_{k-1}r} = \mathrm{J}\nabla_r f^{a_1\ldots a_{k-1}r} - \Psi\nabla_r f^{a_1\ldots a_{k-1}r},$$
$$\widehat{\overline{\mathrm{P}}}_{rs}\widehat{\nabla}^{(r} f^{s)a_1\ldots a_{k-1}} = \overline{\mathrm{P}}_{rs}\nabla^{(r} f^{s)a_1\ldots a_{k-1}} - \Phi_{rs}\nabla^{(r} f^{s)a_1\ldots a_{k-1}},$$
$$\widehat{\overline{\mathrm{P}}}_r{}^{(a_1}\widehat{\nabla}_s f^{a_2\ldots a_{k-1})_0 rs} = \overline{\mathrm{P}}_r{}^{(a_1}\nabla_s f^{a_2\ldots a_{k-1})_0 rs} - \Phi_r{}^{(a_1}\nabla_s f^{a_2\ldots a_{k-1})_0 rs},$$
$$\widehat{\overline{\mathrm{P}}}_{rs}\widehat{\nabla}^{(a_1} f^{a_2\ldots a_{k-1})_0 rs} = \overline{\mathrm{P}}_{rs}\nabla^{(a_1} f^{a_2\ldots a_{k-1})_0 rs} - \Phi_{rs}\nabla^{(a_1} f^{a_2\ldots a_{k-1})_0 rs}$$

at the point x (which is for simplicity omitted in the previous display). Choosing $\Psi(x) \neq 0$ and $\Phi_{rs}(x) = 0$, the invariance of (3.19) means that $y_1 = 0$. Henceforth we assume $\Psi(x) = 0$ and $\Phi_{rs}(x) \neq 0$. To determine y_2, y_3 and y_4, we shall test invariance of (3.19) for $f^{a_1\ldots a_k}$ with specific properties at x. First assume that $\nabla^b f^{a_1\ldots a_k}(x) = \nabla^{(b} f^{a_1\ldots a_k)}(x)$, or equivalently that $\nabla^b f^{a_1\ldots a_k}(x) = \nabla^{a_1} f^{ba_2\ldots a_k}(x)$. This in particular implies that $\nabla_r f^{a_1\ldots a_{k-1}r}(x) = 0$ and the invariance of (3.19) then means that $y_2+y_4 = 0$. Second, we assume $\nabla^{(b} f^{a_1\ldots a_k)}(x) = 0$ or equivalently $2\nabla^{(b} f^{c)a_1\ldots a_{k-1}}(x) + (k-1)\nabla^{(a_1} f^{a_2\ldots a_{k-1})bc}(x) = 0$. This also implies $\nabla_r f^{a_1\ldots a_{k-1}r}(x) = 0$ and the invariance of (3.19) now means that $-\frac{k-1}{2}y_2 + y_4 = 0$. Overall, this yields $y_2 = y_4 = 0$, and $y_3 = 0$ follows. All scalars in (3.19) are thus equal to zero. This completes the proof of the part (ii) of the proposition.

If $k = 1$, (3.19) reduces to the linear combination $y_1\mathrm{J}\nabla_r f^{ra} + y_2\overline{\mathrm{P}}_{rs}\nabla^r f^{sa}$. As above, the choice $\Psi(x) \neq 0$ and $\Phi_{rs}(x) = 0$ shows that $y_1 = 0$. Hence $y_2 = 0$ and the part (i) follows.

In order to complete the proof of the part (iii), it remains to describe natural and conformally invariant operators $\Gamma(S^{k,0}TM) \to \Gamma(S^{k-3}TM)$. The space of these operators is also trivial in the flat case [4, 5], hence the first term in (3.11) cannot appear. Thus the required operator is a linear combination of the form

$$x_1\mathrm{P}_{(rs)_0}\nabla_t f^{a_1\ldots a_{k-3}rst} + x_2(\nabla_r\mathrm{P}_{st})f^{a_1\ldots a_{k-3}rst},$$

where $x_1, y_1 \in \mathbb{R}$. Reasoning similarly as above, we observe that $\nabla_{(a}\nabla_b\Upsilon_{c)_0}$ appears only in the conformal transformation of the second term in the previous display. Therefore $x_2 = 0$, hence also $x_1 = 0$ and the proposition follows. ∎

4 Classification of second order symmetries of Δ_Y

We start this section with the definition of the algebra $\mathcal{A}$ of conformal symmetries of the conformal Laplacian. Afterwards, we provide our main result: a complete description of the space $\mathcal{A}^2$ of second order conformal symmetries.

4.1 The algebra of symmetries of the conformal Laplacian

Let $(M, [\mathrm{g}])$ be a conformal manifold of dimension n. Fixing a metric $\mathrm{g} \in [\mathrm{g}]$, we can regard the conformal Laplacian, $\Delta_Y = \nabla_a \mathrm{g}^{ab}\nabla_b - \frac{n-2}{4(n-1)}\,\mathrm{Sc}$, as acting on functions. The symmetries of Δ_Y are defined as differential operators which commute with Δ_Y. Hence, they preserve the eigenspaces of Δ_Y. More generally, conformal symmetries D_1 are defined by the weaker algebraic condition

$$\Delta_Y \circ D_1 = D_2 \circ \Delta_Y, \tag{4.1}$$

for some differential operator D_2, so that they only preserve the kernel of Δ_Y. The operator Δ_Y can be considered in equation (4.1) as acting between different line bundles and in particular as an element of $\mathcal{D}_{\lambda_0,\mu_0}$, where $\lambda_0 = \frac{n-2}{2n}$, $\mu_0 = \frac{n+2}{2n}$. With this choice, Δ_Y is conformally invariant and the space of conformal symmetries depends only on the conformal class of the metric g. It is stable under linear combinations and compositions.

The operators of the form $P\Delta_Y$, i.e. in the left ideal generated by Δ_Y, are obviously conformal symmetries. Since they act trivially on the kernel of Δ_Y, they are considered as trivial. Following [15, 18, 26], this leads to

Definition 4.1. Let $(M, [\mathrm{g}])$ be a conformal manifold with conformal Laplacian $\Delta_Y \in \mathcal{D}_{\lambda_0,\mu_0}$. The algebra of conformal symmetries of Δ_Y is defined as

$$\mathcal{A} := \{D_1 \in \mathcal{D}_{\lambda_0,\lambda_0} \,|\, \exists\, D_2 \in \mathcal{D}_{\mu_0,\mu_0} \text{ s.t. } D_2 \circ \Delta_Y = \Delta_Y \circ D_1\},$$

and the subspace of trivial symmetries as

$$(\Delta_Y) := \{A\Delta_Y \,|\, A \in \mathcal{D}_{\mu_0,\lambda_0}\}.$$

Thus, $\mathcal{A}$ is a subalgebra of $\mathcal{D}_{\lambda_0,\lambda_0}$ and (Δ_Y) is the left ideal generated by Δ_Y in $\mathcal{D}_{\lambda_0,\lambda_0}$. The filtration by the order on $\mathcal{D}_{\lambda_0,\lambda_0}$ induces a filtration on $\mathcal{A}$ and we denote by

$$\mathcal{A}^k := \mathcal{A} \cap \mathcal{D}^k_{\lambda_0,\lambda_0}$$

the algebra of conformal symmetries of order k. Obviously, $\mathcal{A}^0 \simeq \mathbb{R}$ is the space of constant functions, identified with zero order operators on λ_0-densities. Moreover, the invariance of Δ_Y under the action of conformal Killing vector fields, see (2.9), shows that $\mathcal{A}^1$ is the direct sum of $\mathcal{A}^0$ with the space of Lie derivatives $L^{\lambda_0}_X \in \mathcal{D}^1_{\lambda_0,\lambda_0}$ along conformal Killing vector fields X. Since $\mathcal{A}$ is an algebra, $\mathcal{A}^2$ contains in particular $L^{\lambda_0}_X \circ L^{\lambda_0}_Y$ for X, Y conformal Killing vector fields.

4.2 The algebra of symmetries of the null geodesic flow

Let (M, g) be a pseudo-Riemannian manifold and (x^i, p_i) denote a canonical coordinate system on T^*M. The inverse metric g^{-1} pertains to $\Gamma(S^2TM)$ and identifies with $H := \mathrm{g}^{ij}p_ip_j \in \mathcal{S}_0$, where $\mathcal{S}_0 = \mathrm{Pol}(T^*M) \cong \Gamma(STM)$ (see Section 2.1). Along the isomorphism $T^*M \cong TM$ provided by the metric, the Hamiltonian flow of H corresponds to the geodesic flow of g.

The symmetries of the geodesic flow are given by functions $K \in \mathcal{S}_0$ which Poisson commute with H. They coincide with the symmetric Killing tensors. The null geodesic flow, i.e. the geodesic flow restricted to the level set $H = 0$, depends only on the conformal class of g. It admits additional symmetries, namely all the functions $K \in \mathcal{S}_0$ such that

$$\{H, K\} \in (H),$$

where $\{\cdot, \cdot\}$ stands for the canonical Poisson bracket on T^*M, defined in (2.3), and (H) for the ideal spanned by H in $\mathcal{S}_0$. The linearity and Leibniz property of the Poisson bracket ensure that the space of symmetries of the null geodesic flow is a subalgebra of $\mathcal{S}_0$. Besides, remark that all the functions in (H) are symmetries which act trivially on the null geodesic flow.

Definition 4.2. Let (M, g) be a pseudo-Riemannian manifold and $H \in \mathcal{S}_0$ the function associated to g. The algebra of symmetries of the null geodesic flow of g is given by the following subalgebra of $\mathcal{S}_0$,

$$\mathcal{K} := \{K \in \mathcal{S}_0 \,|\, \{H, K\} \in (H)\}.$$

In particular, the algebra $\mathcal{K}$ contains the ideal (H) of trivial symmetries. It inherits the gradation of $\mathcal{S}_0$ by the degree,

$$\mathcal{K}^k := \mathcal{K} \cap \mathcal{S}_0^k.$$

The space $\mathcal{K}^0$ is the space of constant functions on T^*M. The Hamiltonian flows of functions in $\mathcal{K}^1$ coincide with the Hamiltonian lift to T^*M of the conformal Killing vectors on $(M, [\mathrm{g}])$. For higher degrees, the elements in $\mathcal{K}$ are symmetric conformal Killing tensors whose Hamiltonian flows do not preserve the configuration manifold M. They are symmetries of the whole phase space but not of the configuration manifold and often named hidden symmetries by physicists.

Proposition 4.3. *The elements $K \in \mathcal{K}^k$ are symmetric conformal Killing k-tensors. They are characterized equivalently as:*

- *symmetric tensors of order k s.t. $\nabla_{(a_0} K_{a_1 \dots a_k)_0} = 0$,*
- *symbols of degree k satisfying $\{H, K\} \in (H)$,*
- *elements of $\mathcal{S}_0^k$ in the kernel of the conformal Killing operator $\mathbf{G}$ (see (3.5) or (3.6)).*

The proof is both classical and straightforward, we let it to the reader. The next proposition is essential to determine the algebra $\mathcal{A}$ of conformal symmetries.

Proposition 4.4. *If $D_1 \in \mathcal{A}^k$ then $\sigma_k(D_1) \in \mathcal{K}^k$. Under the identification $\mathrm{gr}\mathcal{D}_{\lambda_0,\lambda_0} \cong \mathcal{S}_0$, the associated graded algebra $\mathrm{gr}\mathcal{A}$ becomes a subalgebra of $\mathcal{K}$ and $\mathrm{gr}(\Delta_Y)$ identifies with (H).*

Proof. Suppose that D_1 is a conformal symmetry of order k, i.e. satisfies $\Delta_Y \circ D_1 = D_2 \circ \Delta_Y$ for some D_2. Working in the algebra $\mathcal{D}_{\lambda_0,\lambda_0}$ we deduce that $[\Delta_Y, D_1] \in (\Delta_Y)$ and the property (2.5) leads then to $\{H, \sigma_k(D_1)\} \in (H)$, i.e. $\sigma_k(D_1) \in \mathcal{K}^k$. The inclusion $\mathrm{gr}\mathcal{A} \leq \mathcal{K}$ follows. As $\sigma_2(\Delta_Y) = H$, the property (2.4) of the principal symbol maps implies that $\mathrm{gr}(\Delta_Y) \cong (H)$. ■

4.3 Second order conformal symmetries

We adapt the strategy used in [26], dealing with conformally flat manifolds. Thanks to a natural and conformally invariant quantization, we get a first description of the potential obstruction for a conformal Killing tensor giving rise to a conformal symmetry of Δ_Y.

Theorem 4.5. *Let $\mathcal{Q}_{\lambda,\mu}$ be a family of natural and conformally invariant quantizations as in Theorem 2.5. We get then*

$$\Delta_Y \circ \mathcal{Q}_{\lambda_0,\lambda_0}(S) - \mathcal{Q}_{\mu_0,\mu_0}(S) \circ \Delta_Y = \mathcal{Q}_{\lambda_0,\mu_0}\big(2\mathbf{G}(S) + \mathbf{Obs}(S)\big), \qquad \forall\, S \in \mathcal{S}_0^{\leq 2}. \tag{4.2}$$

The operator **Obs** *is the natural and conformally invariant operator defined by*

$$\mathbf{Obs} = \frac{2(n-2)}{3(n+1)}\mathbf{F},$$

where $(\mathbf{F}(S))^a = \mathrm{C}^r{}_{st}{}^a \nabla_r S^{st} - 3\mathrm{A}_{rs}{}^a S^{rs}$ for $S \in \mathcal{S}_0^2$ and we set $\mathbf{F}(S) = 0$ for $S \in \mathcal{S}_0^{\leq 1}$.

Proof. According to (2.6), we have $\mathcal{S}_0^2 = \mathcal{S}_0^{2,0} \oplus \mathcal{S}_0^{2,1}$ and $S \in \mathcal{S}_0^{2,1}$ is of the following form $S = (|\mathrm{Vol}_{\mathrm{g}}|^{\delta_0} H) S_0$ with $S_0 \in \mathcal{F}_{-\delta_0}$. By Theorem 2.5, we have the identities

$$\mathcal{Q}_{\lambda_0,\lambda_0}(S) = \mathcal{Q}_{\mu_0,\lambda_0}(S_0) \circ \Delta_Y \qquad \text{and} \qquad \mathcal{Q}_{\mu_0,\mu_0}(S) = \Delta_Y \circ \mathcal{Q}_{\mu_0,\lambda_0}(S_0).$$

Besides, from the expressions of the operators $\mathbf{G}$ and $\mathbf{F}$ (see e.g. (3.6)), we deduce

$$\mathbf{G}(S) = 0 \qquad \text{and} \qquad \mathbf{Obs}(S) = 0.$$

Hence the equality (4.2) holds for all $S \in \mathcal{S}_0^{2,1}$.

Next, we define a natural and conformally invariant operator QS on $\mathcal{D}^2_{\lambda_0,\lambda_0}$ by $D \mapsto \Delta_Y \circ D - D \circ \Delta_Y$. Pulling this map back to trace-free symbols via the quantization maps,

$$\begin{array}{ccc} \mathcal{D}^2_{\lambda_0,\lambda_0} & \xrightarrow{\text{QS}} & \mathcal{D}^3_{\lambda_0,\mu_0} \\ \mathcal{Q}_{\lambda_0,\lambda_0}\uparrow & & \uparrow\mathcal{Q}_{\lambda_0,\mu_0} \\ (\mathcal{S}^{\leq 1}_0 \oplus \mathcal{S}^{2,0}_0) & \xrightarrow[\text{CS}]{} & \mathcal{S}^{\leq 3}_{\delta_0} \end{array}$$

this leads to a natural and conformally invariant operator CS on $\mathcal{S}^{\leq 1}_0 \oplus \mathcal{S}^{2,0}_0$. Since Δ_Y is formally self-adjoint and the quantization maps satisfy the reality condition (2.13), we deduce that, for all $S \in \mathcal{S}^{k,0}_0$,

$$\Delta_Y \circ \mathcal{Q}_{\lambda_0,\lambda_0}(S) - \mathcal{Q}_{\mu_0,\mu_0}(S) \circ \Delta_Y$$

is of degree $k+1$ and is formally skew-adjoint (resp. self-adjoint) if k is even (resp. odd). As such, it is of the form $\mathcal{Q}_{\lambda_0,\mu_0}(P)$, with $P \in \mathcal{S}^3_{\delta_0} \oplus \mathcal{S}^1_{\delta_0}$ if S is of degree 2, $P \in \mathcal{S}^2_{\delta_0} \oplus \mathcal{S}^0_{\delta_0}$ if S is of degree 1 and $P \in \mathcal{S}^1_{\delta_0}$ if S is of degree 0. We can reduce accordingly the target space of CS restricted to homogeneous symbols. Applying Proposition 3.1 and Proposition 3.2, we deduce that $\text{CS} = a\mathbf{G} + b\mathbf{F}$ for some real constants a, b. We have then

$$\Delta_Y \circ \mathcal{Q}_{\lambda_0,\lambda_0}(S) - \mathcal{Q}_{\mu_0,\mu_0}(S) \circ \Delta_Y = \mathcal{Q}_{\lambda_0,\mu_0}\big(a\mathbf{G}(S) + b\mathbf{F}(S)\big), \qquad \forall\, S \in \mathcal{S}^{\leq 2}_0. \tag{4.3}$$

It is straightforward to prove that $a = 2$. To prove $b = \frac{2(n-2)}{3(n+1)}$, we study a specific conformal symmetry of Δ_Y.

Lemma 4.6. *Let η be the pseudo-Euclidean flat metric of signature (p,q), h a non-vanishing function on $\mathbb{R}^2$ and $n = p+q+2$. Let (M_0, g) be the pseudo-Riemannian manifold $(\mathbb{R}^2 \times \mathbb{R}^{n-2}, \mathrm{g}_0 \times \eta)$, where the metric on $\mathbb{R}^2$ is determined by $(\mathrm{g}_0)^{-1} = h(x_1,x_2)p_1^2 + p_2^2$ in canonical Cartesian coordinates (x^i, p_i) on $T^*\mathbb{R}^n$. Then, $K = p_3^2$ is a Killing tensor on (M_0,g), and we have the following relation:*

$$\Delta_Y \circ \mathcal{Q}_{\lambda_0,\lambda_0}(K) - \mathcal{Q}_{\mu_0,\mu_0}(K) \circ \Delta_Y = \mathcal{Q}_{\lambda_0,\mu_0}(\mathbf{Obs}(K)) \neq 0.$$

Proof. Using the relation that links the coefficients of g and the Christoffel symbols Γ^i_{jk} of the associated Levi-Civita connection, it is obvious that $\Gamma^i_{jk} = 0$ if at least one of the indices i, j, k is greater than or equal to 3. Thus, the only non-vanishing components of the Riemann tensor and the Ricci tensor associated with g are given by the corresponding components of the Riemann tensor and the Ricci tensor of g_0. In the same way, the scalar curvature of g is equal to the scalar curvature of g_0.

Using these facts and the formula for $\mathcal{Q}_{\lambda_0,\lambda_0}(K)$ presented in the proof of Proposition 4.9, it is easy to see that

$$\mathcal{Q}_{\lambda_0,\lambda_0}(K) = \mathcal{Q}_{\mu_0,\mu_0}(K) = \partial^2_{x^3} + \frac{1}{2(n-1)(n+1)}\,\mathrm{Sc}\,.$$

By a direct computation, we obtain the following relation:

$$\begin{aligned} &\Delta_Y \circ \mathcal{Q}_{\lambda_0,\lambda_0}(K) - \mathcal{Q}_{\mu_0,\mu_0}(K) \circ \Delta_Y \\ &\quad = [\Delta_Y, \mathcal{Q}_{\lambda_0,\lambda_0}(K)] = \frac{1}{(n-1)(n+1)}\mathcal{Q}_{\lambda_0,\mu_0}(\mathrm{g}^{ij}(\partial_i\,\mathrm{Sc})p_j) + f, \end{aligned}$$

with $f \in \mathcal{C}^\infty(M)$. According to (4.3), the function f vanishes.

Besides, we can compute easily the Cotton–York tensor A associated with g. Indeed, if P denotes the Schouten tensor, we have

$$\mathrm{A}_{ijk} = 2\nabla_{[i}\mathrm{P}_{j]k} = \frac{2}{n-2}\nabla_{[i}\left(\mathrm{Ric}_{j]k} - \frac{1}{2(n-1)}\mathrm{g}_{j]k}\,\mathrm{Sc}\right).$$

Using the peculiar form of K and the remark done previously about the Christoffel symbols and the curvature tensors of g, it is obvious that

$$\mathrm{A}_{ijk}K^{jk} = -\frac{1}{2(n-1)(n-2)}\partial_i\,\mathrm{Sc}$$

for all i. The conclusion follows immediately. ■

By naturality of the map CS defined above, the coefficient b in (4.3) depends only on the signature of the metric. As b is equal to $\frac{2(n-2)}{3(n+1)}$ in the example presented in the previous lemma, where the dimension M_0 is of arbitrary dimension n and g of arbitrary signature, we conclude that $b = \frac{2(n-2)}{3(n+1)}$ in (4.3). ■

Obviously, we have $\mathbf{Obs}(S) = 0$ if S is a symbol of degree 0 or 1. Thus, we recover that $\mathcal{A}^1 \cong \mathcal{K}^1 \oplus \mathcal{K}^0$ and the isomorphism is provided by $\mathcal{Q}_{\lambda_0,\lambda_0}$. Since the symmetric conformal Killing tensors K satisfy $\mathbf{G}K = 0$, we deduce the following

Corollary 4.7. *Let (M, g) be a pseudo-Riemannian manifold of dimension n endowed with a symmetric conformal Killing 2-tensor K. The operator*

$$\mathcal{Q}_{\lambda_0,\lambda_0}(K) = K^{ab}\nabla_a\nabla_b + \frac{n}{n+2}(\nabla_a K^{ab})\nabla_b + \frac{n(n-2)}{4(n+2)(n+1)}(\nabla_a\nabla_b K^{ab}) - \frac{n+2}{4(n+1)}\mathrm{Ric}_{ab}\,K^{ab},$$

is a conformal symmetry of Δ_Y if and only if $\mathbf{Obs}(K) = 0$.

Proof. Indeed, the condition is obviously sufficient. Next, the condition is necessary because if $\mathcal{Q}_{\lambda_0,\lambda_0}(K)$ is a conformal symmetry of Δ_Y, there exists a differential operator D such that

$$\Delta_Y \circ \mathcal{Q}_{\lambda_0,\lambda_0}(K) = D \circ \Delta_Y.$$

We have then successively, using Theorem 4.5:

$$\begin{aligned} 0 &= \Delta_Y \circ \mathcal{Q}_{\lambda_0,\lambda_0}(K) - D \circ \Delta_Y \\ &= (\Delta_Y \circ \mathcal{Q}_{\lambda_0,\lambda_0}(K) - \mathcal{Q}_{\mu_0,\mu_0}(K)\circ\Delta_Y) + (\mathcal{Q}_{\mu_0,\mu_0}(K)\circ\Delta_Y - D\circ\Delta_Y) \\ &= \mathcal{Q}_{\lambda_0,\mu_0}(\mathbf{Obs}(K)) + (\mathcal{Q}_{\mu_0,\mu_0}(K) - D)\Delta_Y. \end{aligned}$$

The operator $\mathcal{Q}_{\lambda_0,\mu_0}(\mathbf{Obs}(K))$ is of order one but not the operator $(\mathcal{Q}_{\mu_0,\mu_0}(K) - D)\Delta_Y$, unless it vanishes. Hence, both terms $\mathcal{Q}_{\lambda_0,\mu_0}(\mathbf{Obs}(K))$ and $(\mathcal{Q}_{\mu_0,\mu_0}(K) - D)\Delta_Y$ have to vanish and then $\mathbf{Obs}(K) = 0$. ■

In particular, on a conformally flat manifold, all the conformal Killing 2-tensors give rise to conformal symmetries of Δ_Y after quantization by $\mathcal{Q}_{\lambda_0,\lambda_0}$, as proved in [26]. We are now in position to prove our main theorem, which provides a full description of the conformal symmetries of Δ_Y given by second order differential operators. The isomorphism $\Gamma(TM) \cong \Gamma(T^*M)$ provided by the metric is denoted by $^\flat$.

Theorem 4.8. *The second order conformal symmetries of Δ_Y are classified as follows:*

(i) $\mathcal{A}^1 = \{L_X^{\lambda_0} + c \,|\, c \in \mathbb{R}$ *and* $X \in \mathcal{K}^1\}$,

(ii) $\mathcal{A}^2/\mathcal{A}^1 \cong \{K \in \mathcal{K}^2 \,|\, \mathbf{Obs}(K)^\flat$ *is an exact* 1*-form*$\}$, *and if* $K \in \mathcal{K}^2$ *satisfies* $\mathbf{Obs}(K)^\flat = -2df$, *with* $f \in \mathcal{C}^\infty(M)$, *the corresponding element in* $\mathcal{A}^2/\mathcal{A}^1$ *is given by*

$$\mathcal{Q}_{\lambda_0,\lambda_0}(K) + f.$$

Proof. We deduce from Proposition 4.4 that the principal symbol K of a second-order conformal symmetry D_1 is a symmetric conformal Killing 2-tensor. Since quantization maps are bijective, the operator D_1 reads as

$$D_1 = \mathcal{Q}_{\lambda_0,\lambda_0}(K + X + f),$$

with f and X symbols of degree 0 and 1 respectively. Theorem 4.5 implies that

$$\Delta_Y \circ D_1 - \mathcal{Q}_{\mu_0,\mu_0}(K + X + f) \circ \Delta_Y = \mathcal{Q}_{\lambda_0,\mu_0}(2\mathbf{G}(X) + \mathbf{Obs}(K) + 2\mathbf{G}(f)).$$

Hence $\Delta_Y \circ D_1 \in (\Delta_Y)$ leads to $\mathbf{G}(X) \in (H)$. By definition of $\mathbf{G}$, this means that $\mathbf{G}(X) = 0$, i.e. $X \in \mathcal{K}^1$. As the symbols $\mathbf{Obs}(K)$ and $\mathbf{G}(f)$ are of degree 1, they cannot pertains to (Δ_Y). Therefore, $\Delta_Y \circ D_1 \in (\Delta_Y)$ is equivalent to $X \in \mathcal{K}^1$ and $\mathbf{Obs}(K) + 2\mathbf{G}(f) = 0$.

The items (i) and (ii) in the statement of the theorem are then easily proved. ∎

4.4 Second order symmetries

The general formula (2.11) for the natural and conformally invariant quantization on symbols of degree 2 leads to the following result.

Proposition 4.9. *Let* (M, g) *be a pseudo-Riemannian manifold of dimension* n *endowed with a symmetric Killing* 2*-tensor* K. *The operator*

$$\begin{aligned}\mathcal{Q}_{\lambda_0,\lambda_0}(K) = \mathcal{Q}_{\mu_0,\mu_0}(K) &= K^{ab}\nabla_a\nabla_b + (\nabla_a K^{ab})\nabla_b - \frac{n-2}{4(n+1)}(\nabla_a\nabla_b K^{ab}) \\ &\quad - \frac{n+2}{4(n+1)}\operatorname{Ric}_{ab} K^{ab} + \frac{1}{2(n-1)(n+1)}\operatorname{Sc}\left(\mathrm{g}_{ab}K^{ab}\right),\end{aligned} \tag{4.4}$$

is a symmetry of Δ_Y, *i.e.* $[\Delta_Y, \mathcal{Q}_{\lambda_0,\lambda_0}(K)] = 0$, *if and only if* $\mathbf{Obs}(K) = 0$.

Proof. Let (x^i, p_i) be a canonical coordinate system on T^*M. The Killing equation satisfied by K reads as $\mathrm{g}^{ij}p_i\nabla_j K = 0$. Applying the trace operator $\operatorname{Tr} = \mathrm{g}_{ij}\partial_{p_i}\partial_{p_j}$ we deduce that

$$\mathrm{g}^{kl}(\nabla_k \operatorname{Tr} K)\nabla_l = -2(\nabla_i K^{il})\nabla_l \qquad \text{and} \qquad \mathrm{g}^{kl}(\nabla_k\nabla_l \operatorname{Tr} K) = -2\nabla_i\nabla_l K^{il}.$$

Moreover, if $\lambda = \mu$ and $\delta = 0$, we have $\beta_1 - 2\beta_2 = 1$ and $\beta_3 - 2\beta_4 = \frac{n^2\lambda(1-\lambda)}{(n+1)(n+2)}$, where the β_i are defined in (2.12). The formula for the quantization $\mathcal{Q}_{\lambda,\lambda}$ reduces then, for K a Killing tensor, to

$$\begin{aligned}\mathcal{Q}_{\lambda,\lambda}(K) &= K^{ab}\nabla_a\nabla_b + \left(\nabla_a K^{ab}\right)\nabla_b - \frac{n^2\lambda(1-\lambda)}{(n+1)(n+2)}\left(\nabla_a\nabla_b K^{ab}\right) \\ &\quad - \frac{n^2\lambda(\lambda-1)}{(n-2)(n+1)}\operatorname{Ric}_{ab} K^{ab} + \frac{2n^2\lambda(1-\lambda)}{(n-2)(n-1)(n+1)(n+2)}\operatorname{Sc}\left(\mathrm{g}_{ab}K^{ab}\right).\end{aligned}$$

Since $\lambda_0 + \mu_0 = 1$ we deduce that $\mathcal{Q}_{\lambda_0,\lambda_0}(K) = \mathcal{Q}_{\mu_0,\mu_0}(K)$. In consequence, the equality $[\Delta_Y, \mathcal{Q}_{\lambda_0,\lambda_0}(K)] = 0$ is equivalent to the fact that $\mathcal{Q}_{\lambda_0,\lambda_0}(K)$ is a conformal symmetry of Δ_Y. By Corollary 4.7, this means that $\mathbf{Obs}(K) = 0$. ∎

As a straightforward consequence, we get

Corollary 4.10. *Let (M, g) be a conformally flat manifold and K be a Killing 2-tensor. Then, we have $[\mathcal{Q}_{\lambda_0,\lambda_0}(K), \Delta_Y] = 0$.*

This corollary enlights some of the results obtained in [2]. As for conformal symmetries, we provide a full description of the symmetries of Δ_Y given by second order differential operators.

Theorem 4.11. *The second order symmetries of Δ_Y are exactly the operators*

$$\mathcal{Q}_{\lambda_0,\lambda_0}(K + X) + f,$$

where X is a Killing vector field, K is a Killing 2-tensor such that $\mathbf{Obs}(K)^\flat$ is an exact one-form and $f \in \mathcal{C}^\infty(M)$ is defined up to a constant by $\mathbf{Obs}(K)^\flat = -2df$.

Proof. Let D_1 be a second order symmetry of Δ_Y. In view of (2.5), we can deduce from $[\Delta_Y, D_1] = 0$ that $\{H, \sigma_2(D_1)\} = 0$. This means that $K = \sigma_2(D_1)$ has to be a symmetric Killing 2-tensor. Since quantization maps are bijective, the operator D_1 reads as

$$D_1 = \mathcal{Q}_{\lambda_0,\lambda_0}(K + X + f),$$

with f and X symbols of degree 0 and 1 respectively. Theorem 4.5 implies that

$$\begin{aligned}[\Delta_Y, D_1] = {}& \mathcal{Q}_{\lambda_0,\mu_0}(2\mathbf{G}(X) + \mathbf{Obs}(K) + 2\mathbf{G}(f)) \\ &+ \big(\mathcal{Q}_{\mu_0,\mu_0}(K + X + f) - \mathcal{Q}_{\lambda_0,\lambda_0}(K + X + f)\big) \circ \Delta_Y.\end{aligned}$$

We have shown that $\mathcal{Q}_{\mu_0,\mu_0}(K) = \mathcal{Q}_{\lambda_0,\lambda_0}(K)$ in Proposition 4.9. Moreover, the general formulas in Theorem 2.4 prove that $\mathcal{Q}_{\mu_0,\mu_0}(f) = \mathcal{Q}_{\lambda_0,\lambda_0}(f)$ and $\mathcal{Q}_{\mu_0,\mu_0}(X) - \mathcal{Q}_{\lambda_0,\lambda_0}(X) = \frac{2}{n}\nabla_a X^a$. Hence, we get

$$[\Delta_Y, D_1] = \mathcal{Q}_{\lambda_0,\mu_0}(2\mathbf{G}(X)) + \frac{2}{n}(\nabla_a X^a)\Delta_Y + \mathcal{Q}_{\lambda_0,\mu_0}(\mathbf{Obs}(K) + 2\mathbf{G}(f))$$

and

$$\sigma_2([\Delta_Y, D_1]) = 2\mathbf{G}(X) + \frac{2}{n}(\nabla_a X^a)H.$$

As $S^2TM = S^{2,0}TM \oplus S^{2,1}TM$, each of the two terms in the right hand side of the second equation are independent. Therefore, $[\Delta_Y, D_1] = 0$ is equivalent to $\mathbf{G}(X) = 0$, $\nabla_a X^a = 0$ and $\mathbf{Obs}(K) + 2\mathbf{G}(f) = 0$. The equations $\mathbf{G}(X) = 0$ and $\nabla_a X^a = 0$ mean that X is a conformal Killing vector field with vanishing divergence, i.e. X is a Killing vector field. Applying the metric, the equation $\mathbf{Obs}(K) + 2\mathbf{G}(f) = 0$ translates into $\mathbf{Obs}(K)^\flat = -2df$. The result follows. ■

For comparison, we recall the alternative classification obtained in [3].

Theorem 4.12 ([3])**.** *Let K be a Killing 2-tensor and put $\mathbf{I}(K)^{ab} = K^{ac}\,\mathrm{Ric}_c^b - \mathrm{Ric}^{ac}\,K_c^b$. Then, we have*

$$\big[\nabla_a K^{ab}\nabla_b + f, \Delta + V\big] = 0 \iff K^{ab}(\nabla_a V) - \frac{1}{3}\big(\nabla_b \mathbf{I}(K)^{ab}\big) = \nabla^a f,$$

where $\Delta = \nabla_a \mathrm{g}^{ab}\nabla_b$ and $f, V \in \mathcal{C}^\infty(M)$.

As an advantage of our method, the obtained condition to get a symmetry (namely $\mathbf{Obs}(K)^\flat$ exact one-form) is conformally invariant and obviously vanishes on conformally flat manifolds. As an advantage of the approach used in [3] and initiated by Carter [11], one recovers easily that

$$[\Delta_Y, \nabla_a K^{ab}\nabla_b] = 0,$$

for all Killing 2-tensors K on an Einstein manifold.

4.5 Higher order conformal symmetries

Up to now we discussed symbols of order ≤ 2. The more general version (which we shall state without proof) of Theorem 4.5 is as follows. Assume that $\mathcal{Q}_{\lambda,\mu}$ is a family of natural and conformally invariant quantizations as in Theorem 2.5 and let S be a trace-free symbol $S \in \mathcal{S}_0^{k,0}$. Then we get

$$\Delta_Y \circ \mathcal{Q}_{\lambda_0,\lambda_0}(S) - \mathcal{Q}_{\mu_0,\mu_0}(S) \circ \Delta_Y = \mathcal{Q}_{\lambda_0,\mu_0}\big(2\mathbf{G}(S) + x\mathbf{F}_1(S) + y\mathbf{F}_2(S) + \Phi(S)\big), \qquad (4.5)$$

where operators $\mathbf{F}_1, \mathbf{F}_2 : \mathcal{S}_0^{k,0} \to \mathcal{S}_{\delta_0}^{k-1,0}$ are defined in Proposition 3.2, scalars x and y have the value

$$x = \frac{k(k-1)(n+2k-6)}{3(n+2k-2)(n+2k-3)} \qquad \text{and} \qquad y = \frac{k(k-1)(k-2)(n+2k)}{12(n+2k-2)(n+2k-3)},$$

and Φ is a natural and conformally invariant operator $\Phi : \mathcal{S}_0^{k,0} \to \mathcal{S}_{\delta_0}^{\leq k-3}$. For $k \leq 2$ this recovers Theorem 4.5, the general case $k \geq 3$ can be shown by a direct (but tedius) computation.

Using (4.5) we can formulate a higher order version of Corollary 4.7: If $K \in \mathcal{K}^k$ is a conformal Killing k-tensor such that the operator $\mathcal{Q}_{\lambda_0,\lambda_0}(K)$ is a conformal symmetry of Δ_Y, then $x\mathbf{F}_1(K) + y\mathbf{F}_2(K) = 0$. Moreover, the same reasoning as in the proof of Theorem 4.8 yields a higher order analogue of this theorem, i.e.

$$\mathcal{A}^k/\mathcal{A}^{k-1} \subseteq \big\{K \in \mathcal{K}^k \,|\, x\mathbf{F}_1(K) + y\mathbf{F}_2(K) = \mathbf{G}(\overline{K}) \text{ for some } \overline{K} \in \mathcal{S}_0^{k-1,0}\big\}.$$

5 Examples in dimension 3

In this section, we consider the space $\mathbb{R}^3$ endowed successively with two types of metrics: the conformal Stäckel metrics and the Di Pirro metrics.

The conformal Stäckel metrics are those for which the Hamilton–Jacobi equation

$$\mathrm{g}^{ij}(\partial_i W)(\partial_j W) = E$$

admits additive separation in an orthogonal coordinate system for $E = 0$ (see [8] and references therein). They are conformally related to the Stäckel metrics, for which the additive separation of the Hamilton–Jacobi equation holds for all $E \in \mathbb{R}$. Moreover, the separating coordinates, called (conformal) Stäckel coordinates are characterized by two commuting (conformal) Killing 2-tensors.

Except for the Stäckel metrics, every diagonal metric on $\mathbb{R}^3$ admitting a diagonal Killing tensor is a Di Pirro metric g (see [28, p. 113]), whose corresponding Hamiltonian is (see e.g. [14])

$$H = \mathrm{g}^{-1} = \frac{1}{2(\gamma(x_1,x_2) + c(x_3))}\left(a(x_1,x_2)p_1^2 + b(x_1,x_2)p_2^2 + p_3^2\right), \qquad (5.1)$$

where a, b, c and γ are arbitrary functions and (x^i, p_i) are canonical coordinates on $T^*\mathbb{R}^3$.

5.1 An example of second order symmetry

The Di Pirro metrics defined via equation (5.1) admit diagonal Killing tensors K given by

$$K = \frac{1}{\gamma(x_1,x_2) + c(x_3)}\left(c(x_3)a(x_1,x_2)p_1^2 + c(x_3)b(x_1,x_2)p_2^2 - \gamma(x_1,x_2)p_3^2\right).$$

For generic functions a, b, c and γ, the vector space of Killing 2-tensors is generated by H and K. However, for some choices of functions, this metric can admit other Killing tensors. For example, if (r,θ) denote the polar coordinates in the plane with coordinates (x_1,x_2), if the functions a, b, γ depend only on r and if $a = b$, then the metric is Stäckel and admits p_θ^2 as additional Killing tensor.

Proposition 5.1. *On the space $\mathbb{R}^3$, endowed with the metric* g *defined by* (5.1)*, there exists a symmetry D of Δ_Y whose principal symbol is equal to the Killing tensor K. In terms of the conformally related metric*

$$\hat{\mathrm{g}} := \frac{1}{2(\gamma(x_1,x_2)+c(x_3))}\mathrm{g},$$

this symmetry is given by: $D = \mathcal{Q}_{\lambda_0,\lambda_0}(K) + \frac{1}{16}(3\widehat{\mathrm{Ric}}_{ab} - \widehat{\mathrm{Sc}}\hat{\mathrm{g}}_{ab})K^{ab}$, i.e. by:

$$D = \widehat{\nabla}_a K^{ab}\widehat{\nabla}_b - \frac{1}{16}(\widehat{\nabla}_a\widehat{\nabla}_b K^{ab}) - \frac{1}{8}\widehat{\mathrm{Ric}}_{ab}K^{ab},$$

where $\widehat{\nabla}$, $\widehat{\mathrm{Ric}}$ and $\widehat{\mathrm{Sc}}$ represent respectively the Levi-Civita connection, the Ricci tensor and the scalar curvature associated with the metric $\hat{\mathrm{g}}$.

Proof. We use Theorem 4.11. In order to compute the obstruction $\mathbf{Obs}(K)^\flat$, we used a Mathematica package called “Riemannian Geometry and Tensor Calculus”, by Bonanos [6].

This obstruction turns out to be an exact one-form equal to $d(-\frac{1}{8}(3\widehat{\mathrm{Ric}}_{ab} - \widehat{\mathrm{Sc}}\hat{\mathrm{g}}_{ab})K^{ab})$. The first expression of the symmetry D follows, the second one is deduced from (4.4), giving $\mathcal{Q}_{\lambda_0,\lambda_0}(K)$. ■

5.2 An example of obstructions to symmetries

If written in conformal Stäckel coordinates, the conformal Stäckel metrics g on $\mathbb{R}^3$ admit four possible normal forms, depending on the numbers of ignorable coordinates (see [8]). A coordinate x is ignorable if ∂_x is a conformal Killing vector field of the metric.

Thus, if x_1 is an ignorable coordinate, the conformal Stäckel metrics g read as

$$\mathrm{g} = Q\left((dx_1)^2 + \big(u(x_2)+v(x_3)\big)\big((dx_2)^2+(dx_3)^2\big)\right), \tag{5.2}$$

where $Q \in \mathcal{C}^\infty(\mathbb{R}^3)$ is the conformal factor and where u and v are functions depending respectively on the coordinates x_2 and x_3. Such metrics admit ∂_{x_1} as conformal Killing vector field and

$$K = (u(x_2)+v(x_3))^{-1}\big(v(x_3)p_2^2 - u(x_2)p_3^2\big) \tag{5.3}$$

as conformal Killing 2-tensor.

Proposition 5.2. *On $\mathbb{R}^3$, there exist metrics* g *as in* (5.2) *whose conformal Laplacian Δ_Y admits no conformal symmetry with principal symbol K.*

Proof. Indeed, the obstruction associated with K, $\mathbf{Obs}(K)^\flat$, is generally not closed. Thanks to the Mathematica package “Riemannian Geometry and Tensor Calculus”, by Bonanos [6], we can actually compute that

$$d\mathbf{Obs}(K)^\flat = -\frac{1}{4}\big(\partial_{x_2}^2+\partial_{x_3}^2\big)\partial_{x_2}\partial_{x_3}\log(u(x_2)+v(x_3))dx_2\wedge dx_3,$$

where the symbol $'$ denotes the derivatives with respect to the coordinates x_2 and x_3. This expression does not vanish e.g. for the functions $u(x_2)=x_2$ and $v(x_3)=x_3$.

We conclude then using Theorem 4.8. ■

An example of a metric of the form (5.2) is provided by the Minkowski metric on $\mathbb{R}^4$ reduced along the Killing vector field $X = x_3\partial_t + t\partial_{x_3} + a(x_1\partial_{x_2} - x_2\partial_{x_1})$, $a \in \mathbb{R}$ (see [21]). In the time-like region of X and in appropriate coordinates (r, ϕ, z), the reduced metric is equal to

$$\mathrm{g} = dr^2 + \frac{r^2 z^2}{z^2 - a^2 r^2} d\phi^2 + dz^2$$

and admits ∂_ϕ as Killing vector field. Moreover, after reduction, the Killing tensor $p_{x_1}^2 + p_{x_2}^2$ is equal to

$$K = p_r^2 + \frac{1}{r^2} p_\phi^2.$$

Notice that the metric g is a Stäckel metric with one ignorable coordinate. Indeed, the metric takes the form (5.2), with $Q(r,z) = \frac{r^2 z^2}{z^2 - a^2 r^2}$, $u(r) = 1/r^2$ and $v(z) = -a^2/z^2$, whereas the conformal Killing tensor $K - \frac{z^2}{z^2 - a^2 r^2} H$ can be written as in (5.3). Here, $H = \mathrm{g}^{-1}$ is the metric Hamiltonian.

In this situation, there is no conformal symmetry of Δ_Y with principal symbol K if $a \neq 0$. Indeed, the one-form $\mathbf{Obs}(K)^\flat$ is then non-exact, as shown by Mathematica computations

$$d\mathbf{Obs}(K)^\flat = \frac{3}{2}(a + a^3)\left(\frac{1}{(z + ar)^4} - \frac{1}{(z - ar)^4}\right) dr \wedge dz.$$

Remark 5.3. Extending the metric (5.2) to $\mathbb{R}^n$ as

$$\mathrm{g} = Q\left((dx_1)^2 + (u(x_2) + v(x_3))\big((dx_2)^2 + (dx_3)^2\big) + (dx^4)^2 + \cdots + (dx^n)^2\right),$$

one can check that K, given in (5.3), is again a conformal Killing tensor and that the one-form $\mathbf{Obs}(K)^\flat$ is in general non-exact. Thus, there is no conformal symmetry of Δ_Y with principal symbol K.

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On a new normalization for tractor covariant derivatives

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Abstract. A regular normal parabolic geometry of type G/P on a manifold M gives rise to sequences D_i of invariant differential operators, known as the curved version of the BGG resolution. These sequences are constructed from the normal covariant derivative ∇^ω on the corresponding tractor bundle V, where ω is the normal Cartan connection. The first operator D_0 in the sequence is overdetermined and it is well known that ∇^ω yields the prolongation of this operator in the homogeneous case $M = G/P$. Our first main result is the curved version of such a prolongation. This requires a new normalization of the tractor covariant derivative on V. Moreover, we obtain an analogue for higher operators D_i. In that case one needs to modify the exterior covariant derivative d^{∇^ω} by differential terms. Finally we illustrate these results with simple examples in projective, conformal and Grassmannian geometry. Our approach is based on standard BGG techniques.

Keywords. Parabolic geometry, prolongation of invariant overdetermined PDE's, BGG sequence, tractor covariant derivatives

1. Introduction

The problem how to find a prolongation of an overdetermined system of PDE's acting between sections of vector bundles is classical and has been studied for a long time. A systematic procedure to solve such problems was developed by D. C. Spencer (see [32]) and his coworkers. One of the tools employed by him was the Spencer resolution of the system, which is useful for description of many properties of solutions of the system. In particular, there is a class of systems of finite type whose solutions are determined by a finite jet at a chosen point. Spencer found a suitable characterization of such systems in his studies. His general results are quite useful but in specific examples, in particular for equations arising in a geometric context, a more efficient analysis can be obtained by employing techniques more adapted to the geometric structure.

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Important examples of overdetermined systems can be found in many areas of geometry. Examples in conformal geometry include, e.g., Killing vectors, conformal Killing vectors, Killing–Yano forms, equations for Einstein scales, etc. (see [30, 19, 26]). In such cases, it was possible to get much better results, because the relevant manifolds were equipped with a rich geometric structure making it possible to use very efficient tools coming from representation theory. To illustrate this in more detail, let us recall that the most famous example of a resolution of an overdetermined system is the de Rham sequence for differential forms on a general manifold. The overdetermined system in this case is the gradient of a function. The de Rham differentials d forming the resolution are distinguished by their invariance with respect to the group of diffeomorphisms acting on the manifold.

Many more explicit examples of overdetermined systems and their resolutions can be described in cases where the manifold is equipped with a richer geometric structure. Typical examples are manifolds with a given projective, conformal, quaternionic, or CR structure. Homogeneous models of such structures are given by homogeneous spaces G/P, where G is a semisimple Lie group and P a parabolic subgroup. On such spaces there exist infinite sequences of resolutions (analogues to the de Rham resolutions), one for each irreducible G-module. The de Rham resolution is the resolution for the trivial G-module. A feature of such resolutions is that operators forming the sequence are (typically) higher order operators (with orders rising with the complexity of the G-module). They are dual versions of the famous Bernstein–Gel'fand–Gel'fand resolutions of irreducible G-modules by Verma modules found in the 70's in representation theory. Following ideas of É. Cartan, it is possible to introduce 'curved versions' of such homogeneous models known under the name of parabolic geometries (see [8]). Curved versions of such resolutions were constructed recently in complete generality in [9, 5]. They are again formed by invariant differential operators, but their composition is now nontrivial due to nontrivial curvature of general curved structures.

To be more specific, let us now recall more details on parabolic geometries. Let G be a (real) semisimple Lie group and P its parabolic subgroup. Following ideas of É. Cartan, the homogeneous space G/P is a flat model for a curved parabolic geometry of type (G, P), which is specified by a couple $(\mathcal{G}, \omega)$, where $\mathcal{G} \to M$ is a principal P-bundle and ω is a Cartan connection. It is well known that such a geometry can be characterized by an underlying geometric structure on the manifold M, together with suitable conditions applied to the Cartan connection needed to remove ambiguities in its definition. A key condition is a normalization condition expressed using the language of cohomology of Lie algebras. Cartan connections satisfying this normalization condition are called normal Cartan connections. To have an equivalence of categories between the category of parabolic structures $(\mathcal{G}, \omega)$ on M and the underlying geometric structure on M, it is necessary to add an additional technical condition on ω (called regularity). Full details on this correspondence can be found in [8].

Distinguished examples of this procedure are the normal Cartan connections constructed for a conformal structure by É. Cartan and for a CR structure by Chern and Moser ([10, 12]). Let us consider a regular normal parabolic geometry $(\mathcal{G}, \omega)$ of type (G, P). For any G-module $\mathbb{V}$, the *tractor bundle* V over M is (by definition) the vector bundle associated to $\mathcal{G}$ and the representation $\mathbb{V}$ (restricted to P). The normal Cartan connection ω on $\mathcal{G}$ then induces the tractor covariant derivative ∇^ω on V, which is then used in various problems in analysis and/or geometry on M (e.g., to construct differential invariants on the corresponding parabolic geometry). For example, it plays the key role in the construction of Bernstein–Gel'fand–Gel'fand (BGG) sequences of invariant differential operators (see [9, 5]) and prolongation procedures for first operators in BGG sequences (see e.g. [4]).

In particular, there is a lot of interest in the study of properties of the first operators in the BGG sequences, or their semilinear versions. Ideas behind the construction of these operators by the BGG machinery can be helpful in such problems. The construction uses tractor covariant derivatives acting on tractor bundles and suitable splitting operators (for details, see Sect. 3). In some simple cases there is a one-to-one correspondence between solutions of the first BGG equation and the kernel of the corresponding tractor covariant derivative. In other words, the tractor covariant derivative is the prolongation of the first BGG operator. But such a simple correspondence is not valid in general.

A general scheme for prolongation of the first BGG operator for parabolic geometries with commutative nilpotent radical was introduced in [4]. The authors not only treat the prolongation for linear overdetermined systems with a particular behavior of the symbol but they also allow semilinear systems having the same symbol as in the linear case and allowing general nonlinear behavior of the lower order part of the operator. A generalization to contact cases can be found in [16] and an extension to general parabolic geometries is discussed in [28]. The procedure used in [4] is efficient but not invariant. In quite a few special cases (see [6, 13, 17, 15, 21, 19]), several authors found an invariant way to compute a deformation of the normal tractor covariant derivative having the property that its kernel can be identified with solutions of the first BGG sequence.

The new normalization of tractor covariant derivatives developed in this paper is motivated by a wish to extend these examples to a general scheme. We shall study the problem of a suitable normalization for tractor covariant derivatives for a general parabolic geometry in a systematic way and show that there is a distinguished alternative of the usual normalization of tractor covariant derivatives on tractor bundles giving directly a canonical prolongation of the first BGG operator in an invariant way.

The normal tractor covariant derivative is induced from the normal Cartan connection on the principal bundle $\mathcal{G}$. An important observation is that if we want to find a covariant derivative on tractor bundles giving an invariant prolongation of the first BGG operator, it is necessary to adapt (in contrast to ∇^ω) the normalization condition to the choice of the tractor bundle under consideration.

The main results of the paper can be described as follows. Let us consider a regular normal parabolic geometry of type (G, P) given by the couple $(\mathcal{G}, \omega)$. To any irreducible G-module $\mathbb{V}$, there is associated the covariant derivative ∇^ω on the associated vector bundle V. The space of all covariant derivatives on V is the affine space modeled on the vector space $\mathcal{E}^1(\operatorname{End} V)$. We want to find a deformation of ∇^ω by $\Phi \in \mathcal{E}^1(\operatorname{End} V)$ satisfying a new normalization condition (adapted to the choice of $\mathbb{V}$) in such a way that the resulting covariant derivative will have suitable properties.

The deformation Φ cannot be chosen arbitrarily. Firstly, the construction of the BGG sequence leads to the requirement of preserving the lowest homogeneous component of ∇^ω (having homogeneity zero), hence we shall restrict to $\Phi \in \mathcal{E}^1(\operatorname{End} V)^1$, where the superscript 1 indicates that Φ should have (total) homogeneity at least one. The desire to have good properties of the new covariant derivative in the prolongation procedure for the first BGG operator imposes further restrictions on the choice of Φ. They will be expressed by properties of values of $\Phi(s) \in \mathcal{E}^1(V)$, where s is a section of V. This leads to the following class of covariant derivatives on the tractor bundle V.

Definition 1.1. Let ω be the regular normal Cartan connection on the principal bundle $\mathcal{G}$ and let ∇^ω be the associated covariant derivative on the associated vector bundle V. The class $\mathcal{C}$ of admissible covariant derivatives on V is defined by

$$\mathcal{C} = \{\nabla = \nabla^\omega + \Phi \mid \Phi \in \operatorname{Im}(\partial_V^* \otimes \mathrm{Id}_{V^*}),\ \Phi \in \mathcal{E}^1(\operatorname{End} V)^1\},$$

where ∂_V^* is the Kostant differential corresponding to homology of $\mathfrak{g}_-$ with values in $\mathbb{V}$ (cf. [25]).

The condition $\Phi \in \operatorname{Im}(\partial_V^* \otimes \mathrm{Id}_{V^*})$ is equivalent to $\Phi(s) \in \operatorname{Im} \partial_V^* \subset \mathcal{E}^1(V)$ for all $s \in \Gamma(V)$, where $\Gamma(V)$ denotes the space of sections of V.

The main theorem of the paper is then

Theorem 1.2. *There exists a unique covariant derivative $\nabla \in \mathcal{C}$ such that*

$$(\partial_V^* \otimes \mathrm{Id}_{V^*})(R^\nabla) = 0,$$

where $R^\nabla \in \mathcal{E}^2(\operatorname{End} V)$ is the curvature of ∇. Again, the condition $(\partial_V^* \otimes \mathrm{Id}_{V^*})(R^\nabla) = 0$ can be equivalently expressed as $\partial_V^*(R^\nabla(s)) = 0$ for all sections s of V.

The new covariant derivative ∇ constructed in Theorem 1.2 gives a prolongation of the first BGG operator, hence we shall call the covariant derivative satisfying this new normalization condition the *prolongation covariant derivative.* The next main result is the theorem stating this property.

Theorem 1.3. *Let us consider a parabolic geometry $(\mathcal{G}, \omega)$ modeled on a couple (G, P). There is a one-to-one correspondence between the kernel of the first BGG operator for a G-module $\mathbb{V}$ and the kernel of the prolongation covariant derivative on the associated bundle V over M.*

In Section 4, we extend the previous construction to other operators in the BGG sequence. In these cases, we have to consider a more general deformation of the exterior derivative d^∇ by adding a differential term (instead of just an algebraic one, which was sufficient for the first operator in the BGG sequence).

Finally, we compare the general procedure developed in this paper with particular results obtained in some special cases and compute some other examples of prolongation covariant derivatives. They come from projective and Grassmannian geometry.

2. Normalization of tractor covariant derivatives

2.1. The double filtration on $\operatorname{End}\mathbb{V}$

Let G be a semisimple Lie group (real or complex) and P a parabolic subgroup of G. The choice of P induces a grading $\mathfrak{g} = \bigoplus_{i=-k}^{k} \mathfrak{g}_i$ on the Lie algebra of G: there is a grading element E in $\mathfrak{g}_0$ acting by i on $\mathfrak{g}_i$.

Every irreducible module $\mathbb{V}$ for G is also graded by the action of E as follows:

$$\mathbb{V} = \bigoplus_{a\in A} \mathbb{V}_a, \qquad \mathbb{V}^* = \bigoplus_{b\in A} \mathbb{V}^*_{-b},$$

where A is the set of all eigenvalues of E on $\mathbb{V}$. A similar decomposition of $\mathfrak{g}_+$ is given by $\mathfrak{g}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$.

The representation $\operatorname{End}\mathbb{V} \simeq \mathbb{V}\otimes\mathbb{V}^*$ has the standard 'diagonal' grading induced by the action of E, given by

$$\operatorname{End}\mathbb{V} = \bigoplus_{\ell} (\operatorname{End}\mathbb{V})_\ell, \qquad (\operatorname{End}\mathbb{V})_\ell := \bigoplus_{a-b=\ell;\ a,b\in A} \mathbb{V}_a \otimes \mathbb{V}^*_{-b}.$$

The key point for the iterative process below is to consider the second 'vertical' grading on the product $\mathbb{V}\otimes\mathbb{V}^*$ by keeping the grading on $\mathbb{V}$ and using the trivial grading on $\mathbb{V}^*$. Hence the vertical grading is given by

$$\operatorname{End}\mathbb{V} = \bigoplus_{a\in A} (\operatorname{End}\mathbb{V})_a, \qquad (\operatorname{End}\mathbb{V})_a := \mathbb{V}_a \otimes \mathbb{V}^*.$$

The gradings are not P-invariant. We shall hence consider filtrations induced by the gradings above. For the diagonal grading, we shall define the filtration by

$$(\operatorname{End}\mathbb{V})^\ell = \bigoplus_{k\geq \ell} (\operatorname{End}\mathbb{V})_k.$$

In particular, $(\operatorname{End}\mathbb{V})^1$ always denotes the corresponding component with respect to the diagonal filtration.

For the vertical grading, the filtration is defined by

$$(\operatorname{End}\mathbb{V})^a = \bigoplus_{b\geq a} (\operatorname{End}\mathbb{V})_b.$$

The grading of $\mathfrak{g}_+$ also gives the standard filtration $\mathfrak{g}^k \subset \cdots \subset \mathfrak{g}^1 = \mathfrak{g}_+$.

These filtrations (together with the filtration on $\mathfrak{g}_+$) also induce the filtrations on the chain spaces $\Lambda^j(\mathfrak{g}_+)\otimes\operatorname{End}\mathbb{V}$ for the Lie algebra homology and cohomology complexes. The differentials in the Lie algebra (co)homology of $\mathfrak{g}_-$ with values in $\mathfrak{g}$-modules $\mathbb{W}$ are the maps $\partial_{\mathbb{W}} : \Lambda^j(\mathfrak{g}_+)\otimes\mathbb{W} \to \Lambda^{j+1}(\mathfrak{g}_+)\otimes\mathbb{W}$, resp. $\partial^*_{\mathbb{W}} : \Lambda^j(\mathfrak{g}_+)\otimes\mathbb{W} \to \Lambda^{j-1}(\mathfrak{g}_+)\otimes\mathbb{W}$. If $\mathbb{W} = \operatorname{End}\mathbb{V} \simeq \mathbb{V}\otimes\mathbb{V}^*$ for a $\mathfrak{g}$-module $\mathbb{V}$, we shall denote the operators $\partial_{\mathbb{V}}\otimes\mathrm{Id}_{\mathbb{V}^*}$, resp. $\partial^*_{\mathbb{V}}\otimes\mathrm{Id}_{\mathbb{V}^*}$, simply by $\partial_{\mathbb{V}}$, resp. $\partial^*_{\mathbb{V}}$. This should not lead to any confusion.

The definition of $\partial_{\mathbb{V}}$ and $\partial^*_{\mathbb{V}}$ implies immediately that they preserve both the vertical and diagonal gradings on $\Lambda^j(\mathfrak{g}_+) \otimes \operatorname{End}\mathbb{V}$. Hence they respect both vertical and diagonal filtrations on $\Lambda^j(\mathfrak{g}_+) \otimes \operatorname{End}\mathbb{V}$. Below we shall use the induced operators between the graded bundles associated to the vertical filtration and we shall denote them by $\operatorname{gr}\partial_{\mathbb{V}}$, resp. $\operatorname{gr}\partial^*_{\mathbb{V}}$.

2.2. *Induced operators on associated graded bundles*

The spaces of j-forms on M with values in a bundle W will be denoted by $\mathcal{E}^j(W)$. They are isomorphic to the bundle induced by the P-module $\Lambda^j(\mathfrak{g}_+) \otimes \mathbb{W}$. Similarly, the tangent bundle is isomorphic to the bundle associated to the P-module $\mathfrak{g}/\mathfrak{p}$. All filtrations mentioned above are P-invariant and so they induce the corresponding filtrations on $\mathcal{E}^j(\operatorname{End}V)$. We shall need, in particular, the diagonal filtrations $\mathcal{E}^j(\operatorname{End}V)^{\ell}$, resp. the vertical filtration $\mathcal{E}^j(\operatorname{End}V)^a$, induced on $\mathcal{E}^j(\operatorname{End}V)$. We shall denote by $\operatorname{gr}_{\ell}(\mathcal{E}^j(\operatorname{End}V))$, resp. $\operatorname{gr}_a(\mathcal{E}^j(\operatorname{End}V))$ the associated graded bundles.

The operators $\operatorname{gr}\partial^*_{\mathbb{V}}$ and $\operatorname{gr}\partial_{\mathbb{V}}$ are P-equivariant, hence they induce well defined maps ∂^*_V, resp. ∂_V, between the corresponding associated graded bundles.

We shall denote by $\operatorname{gr}\partial_V$, resp. $\operatorname{gr}\partial^*_V$, the direct sum of all maps $\operatorname{gr}_a\partial_V$, resp. $\operatorname{gr}_a\partial^*_V$, acting on the direct sum $\operatorname{gr}\mathcal{E}^j(\operatorname{End}V) := \bigoplus_a \operatorname{gr}_a(\mathcal{E}^j(\operatorname{End}V))$. The operators $\operatorname{gr}\partial_V$ and $\operatorname{gr}\partial^*_V$ then have the usual properties of the Kostant differentials. In particular, they are dual to each other (with respect to a suitable scalar product), which implies the usual properties of their kernels and images (Hodge decomposition).

Note also that $\mathcal{E}^j(V) \otimes V^* = \mathcal{E}^j(\operatorname{End}V)$. Hence the standard filtration on $\mathcal{E}^j(V)$ is transferred (by the tensor product with V^*) to the vertical grading on $\mathcal{E}^j(\operatorname{End}V)$. As an immediate corollary, $\varphi \in \mathcal{E}^j(\operatorname{End}V)^a$ if and only if $\varphi s \in \mathcal{E}^j(V)^a$ for all sections $s \in \mathcal{E}^0(V)$.

2.3. *A choice of normalization*

Let us consider a regular parabolic geometry $(\mathcal{G}, \omega)$ over M with the homogeneous model given by a couple (G, P). For an irreducible G-module $\mathbb{V}$, we shall consider the associated tractor bundle V on M. The curvature κ of the Cartan connection ω is a two-form with values in the adjoint tractor bundle $\mathcal{A} \simeq \mathcal{G} \times_P \mathfrak{g}$. The usual normalization condition for ω, expressed in terms of the Kostant differential ∂^* corresponding to homology of $\mathfrak{g}_-$ with values in $\mathfrak{g}$, requires the curvature κ to be ∂^*-closed. In terms of the associated covariant derivative ∇^{ω} on V, the curvature $R^{\nabla^{\omega}}$ of ∇^{ω} is a two-form with values in $\operatorname{End}V$ and the normalization condition can be expressed using the Kostant differential ∂^* for $\operatorname{End}V$ as

$$\partial^*(R^{\nabla^{\omega}}) = 0.$$

Given a choice of the bundle V, we are going to change the normalization condition for a covariant derivative ∇ on V. Let Id_{V^*} denote the identity map on V^*. As above in the algebraic version, we shall consider the operators

$$\partial_V \otimes \operatorname{Id}_{V^*}, \quad \partial^*_V \otimes \operatorname{Id}_{V^*}$$

acting on $\mathcal{E}^j(\mathrm{End}\,V)$, forms with values in $\mathrm{End}\,V \simeq V \otimes V^*$. Abusing the notation, we shall denote them by ∂_V, resp. ∂_V^*. It will always be clear whether the differentials act on forms with values in V or forms with values in $\mathrm{End}\,V$.

We shall now introduce a new normalization for covariant derivatives on V.

Definition 2.1. We shall call a covariant derivative $\nabla \in \mathcal{C}$ the *prolongation covariant derivative* if

$$\partial_V^*(R^\nabla) = 0,$$

where $R^\nabla \in \mathcal{E}^2(\mathrm{End}\,V)$ is the curvature of ∇.

The choice of the name should suggest that the new normalization condition gives better properties to ∇ in the prolongation procedure for the first operator in the BGG sequence corresponding to the representation $\mathbb{V}$.

We shall need the following property.

Lemma 2.2. *If $\varphi \in \mathcal{E}^1(\mathrm{End}\,V)^a$ and $\tau \in \mathcal{E}^1(V)$, then*

$$\varphi \wedge \tau \in \mathcal{E}^2(V)^{a+1}.$$

Proof. Indeed, we can decompose φ into homogeneous components

$$\varphi = \sum_j \alpha_j \otimes v_j \otimes w_j, \qquad \alpha_j \in \mathcal{E}^1,\ \ v_j \in V,\ w_j \in V^*,$$

where the sum of the homogeneities of α_j and v_j is greater than or equal to a. If we also decompose τ as

$$\tau = \sum_k \beta_k \otimes u_k, \qquad \beta_k \in \mathcal{E}^1,\ u_k \in V,$$

then the expression

$$\varphi \wedge \tau = \sum_{j,k} w_j(u_k)\alpha_j \wedge \beta_k \otimes v_j$$

clearly has summands of homogeneity greater than or equal to $a + 1$. □

2.4. The main lemma

The key information for the normalization procedure is the following fact concerning the induced change of curvature.

Lemma 2.3. *Let ∇_1, resp. ∇_2, be two covariant derivatives from $\mathcal{C}$ related to each other by the deformation $\Phi = \nabla_2 - \nabla_1 \in \mathcal{E}^1(\mathrm{End}\,V)^1$ and let R_1, resp. R_2, be the corresponding curvatures. If $\Phi \in \mathcal{E}^1(\mathrm{End}\,V)^a$, then $R_2 - R_1 \in \mathcal{E}^2(\mathrm{End}\,V)^a$ and*

$$\mathrm{gr}_a(R_2 - R_1) = (\mathrm{gr}_a\ \partial_V)(\mathrm{gr}_a\ \Phi).$$

Proof. Let ω be the normal Cartan connection for the chosen parabolic geometry and ∇ the associated covariant derivative. It is well known that ∇ and d^∇ preserve the standard filtration on $\mathcal{E}^j(V)$ and that the corresponding graded version of ∇, resp. d^∇, is equal to $\operatorname{gr}\partial_V$. A shift of ∇ by $\Phi \in \mathcal{E}^1(\operatorname{End} V)^1$ does not change this property, the same being true for $d^{\nabla+\Phi}$.

The change in curvature is then

$$R_2 - R_1 = d^\nabla\Phi + [\Phi, \Phi].$$

The result clearly belongs to $\mathcal{E}^2(\operatorname{End} V)^a$, because the operator d^∇ preserves the filtrations and we can use Lemma 2.2 for the second term.

Then we get, for any $s \in \mathcal{E}^0(V)$,

$$\begin{aligned} \operatorname{gr}_a((d^\nabla\Phi + [\Phi, \Phi])s) &= \operatorname{gr}_a((d^\nabla\Phi)s) = \operatorname{gr}_a(d^\nabla(\Phi s) - \Phi \wedge (\nabla s)) \\ &= \operatorname{gr}_a(\partial_V(\Phi(s))) = (\operatorname{gr}_a \partial_V)(\operatorname{gr}_a(\Phi(s))). \end{aligned}$$ □

2.5. Existence and uniqueness of the prolongation covariant derivative

We now show the main theorem of this article:

Theorem 2.4. *For each irreducible G-module $\mathbb{V}$, there exists a unique prolongation covariant derivative $\nabla \in \mathcal{C}$, i.e., a unique $\nabla \in \mathcal{C}$ such that*

$$\partial_V^*(R^\nabla) = 0. \tag{1}$$

Proof. The curvature function of the regular normal connection ω for the corresponding parabolic geometry belongs (by definition of regularity) to $\mathcal{E}^2(\mathcal{A})^1$, so $R^{\nabla^\omega} \in \mathcal{E}^2(\operatorname{End} V)^1$, and $\partial_V^*(R^{\nabla^\omega}) \in \mathcal{E}^1(\operatorname{End} V)^1$. Lemma 2.5 below now shows that we can start with ∇^ω and obtain by induction a unique $\nabla \in \mathcal{C}$ satisfying (1). □

Lemma 2.5. *Suppose that there is a tractor covariant derivative $\nabla \in \mathcal{C}$ with*

$$\partial_V^*(R^\nabla) \in \mathcal{E}^1(\operatorname{End} V)^1 \cap \mathcal{E}^1(\operatorname{End} V)^a,$$

where $a \in A$ is such that $a + 1$ belongs to A. Then there exists

$$\Phi \in \mathcal{E}^1(\operatorname{End} V)^1 \cap \mathcal{E}^1(\operatorname{End} V)^a \cap \operatorname{Im}(\partial_V^* \otimes \operatorname{Id}_{V^*})$$

such that for $\tilde\nabla = \nabla + \Phi$, one has

$$\partial_V^*(R^{\tilde\nabla}) \in \mathcal{E}^1(\operatorname{End} V)^1 \cap \mathcal{E}^1(\operatorname{End} V)^{a+1}. \tag{2}$$

Moreover, Φ is unique up to terms of homogeneity $a + 1$. In particular, $\tilde\nabla \in \mathcal{C}$ satisfying (2) is unique up to modifications by elements in

$$\mathcal{E}^1(\operatorname{End} V)^1 \cap \operatorname{Im}(\partial_V^* \otimes \operatorname{Id}_{V^*}) \cap \mathcal{E}^1(\operatorname{End} V)^{a+1}.$$

Proof. The spaces $\{\mathcal{E}^1(\operatorname{End} V)^1 \cap \mathcal{E}^1(\operatorname{End} V)^a\}_{a=0}^r$ give a descending filtration of $\mathcal{E}^1(\operatorname{End} V)^1$. The filtration is preserved by the maps ∂_V and ∂_V^*, hence they induce maps on the associated graded bundle (for simplicity we denote them by the same symbols as for the full filtration of $\mathcal{E}^1(\operatorname{End} V)$). The standard Kostant decomposition says that $\operatorname{Ker} \operatorname{gr} \partial_V^*$ and $\operatorname{Im} \operatorname{gr} \partial_V$ are complementary subspaces of the graded bundle $\operatorname{gr} \mathcal{E}^1(\operatorname{End} V)^1$. In particular, $\operatorname{gr} \partial_V^*$ restricts to an isomorphism of $\operatorname{Im} \operatorname{gr} \partial_V$ and $\operatorname{Im} \operatorname{gr} \partial_V^*$.

Hence we can define $\varphi \in \operatorname{gr}_a(\mathcal{E}^1(\operatorname{End} V)^1)$ by

$$\varphi := \Box^{-1} \partial_V^*(R^{\tilde{\nabla}}),$$

which then has the property that

$$(\operatorname{gr} \partial_V^*)((\operatorname{gr} \partial_V)(\varphi)) = \operatorname{gr}_a(\partial_V^*(R^{\nabla})).$$

Let $\Phi \in \mathcal{E}^1(\operatorname{End} V)^1 \cap \mathcal{E}^1(\operatorname{End} V)^a \cap \operatorname{Im}(\partial_V^* \otimes \operatorname{Id}_{V^*})$ be a preimage of φ. Then we define a corrected covariant derivative by $\tilde{\nabla} := \nabla - \Phi$.

Due to Lemma 2.3, we get

$$\begin{aligned} \operatorname{gr}_a(\partial_V^*(R^{\tilde{\nabla}})) &= \operatorname{gr}_a(\partial_V^*(R^{\nabla})) - (\operatorname{gr} \partial_V^*)(\operatorname{gr}_a(R^{\nabla} - R^{\tilde{\nabla}})) \\ &= \operatorname{gr}_a(\partial_V^*(R^{\nabla})) - (\operatorname{gr} \partial_V^*)((\operatorname{gr} \partial_V)(\operatorname{gr}_a(\Phi))) = 0. \end{aligned}$$

Hence $\tilde{\nabla}$ has the required properties.

For the uniqueness up to terms of homogeneity higher than a, assume that we have another $\Phi' \in \mathcal{E}^1(\operatorname{End} V)^1 \cap \mathcal{E}^1(\operatorname{End} V)^a \cap \operatorname{Im}(\partial_V^* \otimes \operatorname{Id}_{V^*})$ such that $\tilde{\nabla}' = \nabla + \Phi'$ satisfies $\partial_V^*(R^{\tilde{\nabla}'}) \in \mathcal{E}^1(\operatorname{End} V)^{a+1}$. Then $\Phi' - \Phi$ belongs to $\mathcal{E}^1(\operatorname{End} V)^1 \cap \operatorname{Im} \partial_V^*$, and by assumption $\operatorname{gr}_a(R^{\tilde{\nabla}'} - R^{\tilde{\nabla}})$ lies in the kernel of $\operatorname{gr} \partial_V^*$. By Lemma 2.3, we have

$$\operatorname{gr}_a(R^{\tilde{\nabla}'} - R^{\tilde{\nabla}}) = (\operatorname{gr} \partial_V)(\operatorname{gr}_a(\Phi' - \Phi)).$$

But $\operatorname{Ker} \operatorname{gr} \partial_V^* \cap \operatorname{Im} \operatorname{gr} \partial_V$ is trivial, hence $\operatorname{gr}_a(R^{\tilde{\nabla}'} - R^{\tilde{\nabla}}) = 0$. Thus $\operatorname{gr}_a(\Phi' - \Phi)$ lies in the kernel of $\operatorname{gr} \partial_V$, and also in the image of $\operatorname{gr} \partial_V^*$, by assumption. Hence $\operatorname{gr}_a(\Phi' - \Phi) = 0$, and thus

$$\tilde{\nabla}' - \tilde{\nabla}' = \Phi' - \Phi \in \mathcal{E}^1(\operatorname{End} V)^{a+1}. \qquad \square$$

Remark. The construction of ∇ as outlined above depends at first on some choices (e.g., the choice of the preimage Φ of φ). However, the uniqueness of the prolongation covariant derivative shows that the result of the construction is independent of all choices. Hence the prolongation covariant derivative is invariant—it only depends on the data of the chosen parabolic structure and the bundle V.

3. Prolongation of the first BGG operator

The BGG complexes are sequences of invariant differential operators on a homogeneous model for a given parabolic geometry. A curved version of them, i.e., an extension of operators in the sequence to invariant differential operators on general (nonflat) manifolds with a given parabolic structure, was first constructed in [9] and the construction was simplified and extended in [5]. The first operator in such a sequence always gives an overdetermined system of invariant differential equations. A prolongation of this operator for the case of 1-graded parabolic geometries was constructed in [4]. However, the methods used there needed a choice of a Weyl structure, hence the resulting covariant derivative was not invariant. We are now going to show that the normalization of tractor covariant derivatives described in this paper can be used to obtain invariant (natural) prolongations.

We begin by introducing the setting and basic operators of the BGG machinery in a generalized version needed for the next section. Let V be a tractor bundle over M with a covariant derivative ∇ and the exterior covariant derivative $d^\nabla : \mathcal{E}^k(V) \to \mathcal{E}^{k+1}(V)$. Recall from the above that we have a well defined differential $\partial^* = \partial^*_V : \mathcal{E}^{k+1}(V) \to \mathcal{E}^k(V)$. The property $\partial^* \circ \partial^* = 0$ allows us to define the cohomology H_k as the vector bundle quotient $H_k = \operatorname{Ker}\partial^*/\operatorname{Im}\partial^*$, where $\operatorname{Ker}\partial^* \subset \mathcal{E}^k(V)$ is the space of cycles and $\operatorname{Im}\partial^* \subset \mathcal{E}^k(V)$ is the space of boundaries. The canonical surjection $\operatorname{Ker}\partial^*_V \subset \mathcal{E}^k(V) \to H_k$ will be denoted by Π_k.

Due to regularity of the parabolic geometry under consideration, the operators d^∇ are homogeneous of degree zero with respect to the natural filtration of the spaces $\mathcal{E}^k(V)$ and they induce the algebraic differential $\operatorname{gr}\partial_V : \operatorname{gr}(\mathcal{E}^k(V)) \to \operatorname{gr}(\mathcal{E}^{k+1}(V))$ on the associated graded spaces. Thus it is possible to regard d^∇ as a natural lift of $\operatorname{gr}\partial_V$ to a differential operator from $\mathcal{E}^k(V)$ to $\mathcal{E}^{k+1}(V)$.

The main ingredients in the BGG machinery are the differential splitting operators $L_k : H_k \to \operatorname{Ker}\partial^*_V \subset \mathcal{E}^k(V)$ with the property $\partial^* \circ d^\nabla \circ L_k = 0$. This allows one to define the BGG operators $D_k : H_k \to H_{k+1}$ in the obvious way: $D_k := \Pi_k \circ d^\nabla \circ L_k$. The definition is encoded in the diagram

$$\begin{tikzcd}
\mathcal{E}^k(V) \arrow[r, "d^\nabla"] & \mathcal{E}^{k+1}(V) \\
\operatorname{Ker}\partial^* \arrow[u, "i"] & \operatorname{Ker}\partial^* \arrow[u, "i"'] \arrow[d, "\Pi_{k+1}"] \\
H_k \arrow[u, "L_k"] \arrow[r, "D_k"] & H_{k+1}
\end{tikzcd} \tag{3}$$

where i denotes inclusion.

We shall introduce the construction of the splitting operators in a more general situation, where the exterior covariant derivatives d^∇ on $\mathcal{E}^k(V)$ will be replaced by general differential operators E_k with suitable properties (see the theorem below). The operators D_k are defined by the same construction as the BGG operators and they depend, in general, on the choice of E_k. The theorem below shows that for certain classes of operators E_k, the resulting operators L_k and D_k do not change.

Theorem 3.1. *Let $(\mathcal{E}^k(V))^j$ denote the filtration on $\mathcal{E}^k(V)$ and let $\mathrm{gr}(\mathcal{E}^k(V))$ denote the associated graded bundle, and similarly for $\mathcal{E}^{k+1}(V)$. Let E_k be a filtration preserving differential operator from $\mathcal{E}^k(V)$ to $\mathcal{E}^{k+1}(V)$ with the property that the associated graded map coincides with $\mathrm{gr}\,\partial$. Then for every $\sigma \in H_k$, there exists a unique element $s \in \operatorname{Ker}\partial^*$ with the following properties:*

(1) $\Pi_k(s) = \sigma$,
(2) $E_k(s) \in \operatorname{Ker}\partial^*$.

Moreover, the mapping L_k defined by $\sigma \mapsto L_k(\sigma) := s$ is given by a differential operator. The corresponding operator D_k is then defined by

$$D_k := \Pi_{k+1} \circ E_k \circ L_k : H_k \to H_{k+1}.$$

Suppose that we change the operator E_k to $\tilde{E}_k = E_k + \Phi_k$, where $\Phi_k : \mathcal{E}_k(V) \to \mathcal{E}_{k+1}(V)$ is a differential operator with values in $\operatorname{Im}\partial^*$ and preserving the filtration, with the property that the associated graded map is trivial. Then the construction does not change the splitting operator L_k and the operator D_k.

Proof. The first part of the proof follows the standard line of argument. The operator $\partial^* \circ E_k$ acts on $\mathcal{E}^k(V)$ and it preserves $\operatorname{Im}\partial^*$. It preserves the filtration and its graded version is, by assumption, given by $\mathrm{gr}(\partial^*)\circ\mathrm{gr}(\partial)$, which is invertible on $\operatorname{Im}\partial^*$. Hence also $\partial^* \circ E_k$ is invertible on $\operatorname{Im}\partial^*$ and it is possible to show that its inverse Q is a differential operator.

We can then define a differential operator $\hat{L}_k := \mathrm{Id} - Q \circ \partial^* \circ E_k$, which restricts to zero on $\operatorname{Im}\partial^*$. Hence it induces a well-defined differential operator L_k from H_k to $\operatorname{Ker}\partial^* \subset \mathcal{E}^k(V)$. It is easy to check that the operator L_k has the properties

$$\operatorname{Im} L_k \subset \operatorname{Ker}\partial^*, \quad \Pi_k \circ L_k = \mathrm{Id}, \quad \partial^* \circ E_k \circ L_k = 0.$$

To show that L_k is uniquely characterized by these properties, let us consider $s_1, s_2 \in \operatorname{Ker}\partial^*$ such that $E_k(s_i) \in \operatorname{Ker}\partial^*$, $i = 1, 2$, and $\Pi_k(s_1) = \Pi_k(s_2)$. Then the difference $s = s_1 - s_2$ belongs to $\operatorname{Im}\partial^*$. By definition of $\hat{L}_k$, the relation $\partial^* \circ E_k(s) = 0$ implies $\hat{L}_k(s) = s$. On the other hand, $\hat{L}_k$ is trivial on $\operatorname{Im}\partial^*$. Hence $\hat{L}_k(s) = 0$.

To prove the last statement of the theorem, we shall consider a section s of $\mathcal{E}^k(V)$. The new operator $\tilde{E}_k$ preserves the filtration and the induced graded map is still $\mathrm{gr}\,\partial$. Since $(\tilde{E}_k - E_k)s$ belongs to $\operatorname{Im}\partial^*_V$, one has $\tilde{E}_k(s) \in \operatorname{Ker}\partial^*_V$ iff $E_k(s) \in \operatorname{Ker}\partial^*_V$, which shows that $\tilde{L}_k = L_k$. Thus, for $\sigma \in H_k$, one has $(\tilde{E}_k\tilde{L}_k - E_k L_k)\sigma \in \operatorname{Im}\partial^*_V$, but this lies in the kernel of the projection $\Pi_{k+1} : \operatorname{Ker}\partial^* \to H_{k+1}$. □

Now we want to discuss the relation between $\operatorname{Ker} E_k$ and $\operatorname{Ker} D_k$. For that, we have to consider two consecutive operators E_k and E_{k+1} at the same time. They define two splitting operators L_k and L_{k+1}. We get in this way the diagram

$$\begin{tikzcd} \mathcal{E}^k(V) \arrow[r, "E_k"] & \mathcal{E}^{k+1}(V) \\ H_k \arrow[u, "L_k"] \arrow[r, "D_k"] & H_{k+1} \arrow[u, "L_{k+1}"'] \end{tikzcd} \tag{4}$$

which, in general, does not commute but there is a convenient criterion for its commutativity.

Theorem 3.2. *The diagram* (4) *commutes if and only if* $\partial^* \circ E_{k+1} \circ E_k(s) = 0$ *for all sections* $s \in \operatorname{Im} L_k \subset \mathcal{E}^k(V)$.

Proof. The values of L_k are uniquely characterized by the conditions $L_k(\sigma) \in \operatorname{Ker} \partial^*$ and $E_k \circ L_k(\sigma) \in \operatorname{Ker} \partial^*$. Similarly, the values of L_{k+1} are characterized by $L_{k+1}(\tau) \in \operatorname{Ker} \partial^*$ and $E_{k+1} \circ L_{k+1}(\tau) \in \operatorname{Ker} \partial^*$. Hence $E_k \circ L_k(\sigma) = L_{k+1} \circ D_k(\sigma)$ iff $E_{k+1} \circ E_k \circ L_k(\sigma) \in \operatorname{Ker} \partial^*$ for all $\sigma \in H_k$. □

If the diagram above is commutative, we immediately get a one-to-one correspondence between $\operatorname{Ker} E_k \cap \operatorname{Ker} \partial^*$ and $\operatorname{Ker} D_k$.

Theorem 3.3. *Suppose that the diagram* (4) *commutes. Then* Π_k *and* L_k *restrict to inverse isomorphisms between* $\operatorname{Ker} E_k \cap \operatorname{Ker} \partial^*$ *and* $\operatorname{Ker} D_k$.

Proof. Let s be in $\operatorname{Ker} E_k \cap \operatorname{Ker} \partial^*$. Then $s = L_k(\Pi_k(s))$ by definition of L_k, and $\Pi_k(s) \in \operatorname{Ker} D_k$ by definition of D_k.

On the other hand, if $D_k(\sigma) = 0$, then commutativity of the diagram implies that also

$$L_{k+1} \circ D_k(\sigma) = E_k \circ L_k(\sigma) = 0,$$

hence $L_k(\sigma) \in \operatorname{Ker} E_k \cap \operatorname{Ker} \partial^*$.

And by definition of L_k, we have $\Pi_k \circ L_k = \mathrm{Id}$. □

Now we can return to the properties of the prolongation covariant derivative ∇ on V. Using the above claims in the special case of the first square and the operators $E_0 = \nabla$ and $E_1 = d^\nabla$, we see immediately that $E_1 \circ E_0 = R^\nabla$. Hence we get the following corollary.

Corollary 3.1. *Consider a tractor bundle* V *and the corresponding prolongation covariant derivative* ∇. *Set* $E_0 = \nabla$ *and* $E_1 = d^\nabla$. *Then the square constructed using these two operators commutes and the covariant derivative* ∇ *gives a prolongation of the first BGG operator* D_0. *In particular, the splitting operator* L_0 *induces a one-to-one correspondence between the space of parallel sections of* V *with respect to* ∇ *and the kernel of the first BGG operator* D_0.

Remark. In the case of a 1-graded geometry, it was shown in [4] that the map $L_0 : H_0 \to V$ induces an isomorphism of $J^k H_0$ with $V_{\leq k}$ for every k such that the homology of $H_1(\mathfrak{g}_-, \mathbb{V})$ sits in homogeneity $> k$. Thus, for every operator $\tilde{D}_0 : H_0 \to H_1$ which differs from the standard BGG operator D_0 by a linear differential operator of order $\leq k$, there is a map $\Psi \in \mathcal{E}^1(\operatorname{End} V)$ with values in $\operatorname{Ker} \partial_V^*$ whose induced first BGG operator coincides with $\tilde{D}_0$. The mapping Ψ is unique up to maps in $\mathcal{E}^1(\operatorname{End} V)$ with values in $\operatorname{Im} \partial_V^*$, and it is thus easy to see that the resulting normalized connection $\tilde{\nabla} = \nabla + \Psi + \Phi$ does not depend on the choice of Ψ. Thus, natural deformations of D_0 of low enough order can be prolonged naturally as well. We remark that a similar procedure works in the case of general graded parabolic geometries, where one has to use the filtration of the manifold for a suitable version of jet bundles (cf. [27, 28]).

4. Prolongation covariant derivatives for the whole BGG sequence

In this section we shall treat the problem considered above in the case of other squares of the BGG sequence. We want to deform the exterior covariant derivative d^∇ on k-forms in such a way that all squares in the generalized BGG construction will commute, and, at the same time, the BGG operators D_k will not change. In fact, we shall succeed in keeping both the BGG operators D_k and the splitting operators L_k unchanged. The deformation of d^∇ on $\mathcal{E}^k(V)$ will have, however, a different character. It will be of the form $E_k := d^\nabla + \Phi_k$, where Φ_k is a linear differential operator mapping $\mathcal{E}^k(V)$ to $\mathcal{E}^{k+1}(V)$. Hence the deformation Φ_k will not, in general, be algebraic. Necessary tools were already prepared in the previous section (Theorems 3.1–3.3). Methods described in this section can also be applied to the first square but they give a different answer (and also in this case the deformation Φ_0 will not be algebraic in general).

To describe allowed deformations of the exterior derivative d^∇, we shall introduce the following notation. There are two different filtrations on the space $A := \operatorname{Hom}(\mathcal{E}^k(V), \mathcal{E}^{k+1}(V))$. The diagonal filtration A^j is induced by the standard filtration on $\mathcal{E}^k(V)$, which is defined by the condition $\Phi(s) \in \mathcal{E}^{k+1}(V)^{a+j}$ for all $s \in \mathcal{E}^k(V)^a$. The other (vertical) filtration A^a is defined by the condition $\Phi(s) \in \mathcal{E}^{k+1}(V)^a$ for all $s \in \mathcal{E}^k(V)$. In this section, we shall use the symbols ∂ and ∂^* for the Kostant differentials associated to the spaces $\mathcal{E}^k(V)$. Recall that the class $\mathcal{C}$ of admissible covariant derivatives on V was defined by

$$\mathcal{C} = \{\nabla = \nabla^\omega + \Phi \mid \Phi \in \operatorname{Im}(\partial_V^* \otimes \operatorname{Id}_{V^*}),\ \Phi \in \mathcal{E}^1(\operatorname{End} V)^1\}.$$

We shall consider the following spaces $\mathcal{C}_k$ of deformations.

Definition 4.1. The space of allowed deformations will be defined by

$$\mathcal{C}_k := \{E_k \in \operatorname{Hom}(\mathcal{E}^k(V), \mathcal{E}^{k+1}(V)) \mid E_k = d^\nabla + \Phi,\ \Phi \in A^1,\ \operatorname{Im}\Phi \subset \operatorname{Im}\partial^*\}.$$

Theorem 4.2. (1) *Let ∇ be any covariant derivative from $\mathcal{C}$. Let us consider the BGG sequence with the splitting operators L_k and the BGG operators D_k induced* (*via Theorem 3.1*) *by the operators $E_k = d^\nabla$,*

$$\begin{array}{ccc} \mathcal{E}^k(V) & \xrightarrow{d^\nabla} & \mathcal{E}^{k+1}(V) \\ \uparrow{\scriptstyle L_k} & & \uparrow{\scriptstyle L_{k+1}} \\ H_k & \xrightarrow{D_k} & H_{k+1} \end{array} \tag{5}$$

Then there exists a collection of differential operators $\Phi_k \in \mathcal{C}_k$ such that $\partial^* \circ d^\nabla \circ (d^\nabla + \Phi_k) = 0$. Moreover, the collection Φ_k with these properties is unique.

(2) *As a consequence, the diagrams*

$$\begin{array}{ccc} \mathcal{E}^k(V) & \xrightarrow{d^\nabla + \Phi_k} & \mathcal{E}^{k+1}(V) \\ {\scriptstyle L_k}\uparrow & & \uparrow{\scriptstyle L_{k+1}} \\ H_k(V) & \xrightarrow{D_k} & H_{k+1}(V) \end{array} \tag{6}$$

commute for all $k = 0, 1, \ldots, n-1$. Moreover, if ∇ depends only on data of the chosen parabolic geometry, the same is true for the operators $E_k = d^\nabla + \Phi_k$.

Proof. Let us choose $k = 0, \ldots, n-1$ and consider the square (6) in the generalized BGG sequence constructed using operators d^∇, where ∇ is any covariant derivative from $\mathcal{C}$. We shall first prove the first assertion of the theorem.

The spaces $\{A^1 \cap A^a\}_{a=0}^r$ form a decreasing filtration of the space A^1 with $a = 0, \ldots, r$. The filtration is preserved by the maps ∂_V and ∂_V^*, hence they induce maps on the associated graded bundle (we denote them for simplicity by the same symbols as for the full filtration of A). We can consider the Kostant Laplacian $\Box = \operatorname{gr}\partial_V^* \operatorname{gr}\partial_V + \operatorname{gr}\partial_V \operatorname{gr}\partial_V^*$. The standard Kostant decomposition says that $\operatorname{Ker}\Box$, $\operatorname{Im}\operatorname{gr}\partial_V^*$ and $\operatorname{Im}\operatorname{gr}\partial_V$ are complementary subspaces of the graded bundle $\operatorname{gr}\mathcal{E}^i(V)^1$. In particular, $\Box$ is invertible on $\operatorname{Im}\operatorname{gr}\partial_V^*$.

Let us consider two consecutive squares with operators $E_k = d^\nabla$ and $E_{k+1} = d^\nabla$. We know that the operator $G := \partial^* \circ E_{k+1} \circ E_k$ belongs to A^1 and that the k-th square is commutative iff $G = 0$. If it is not the case, we shall consider the maximal index $a = 0$ with the property that $G \in A^a$.

The map $\Phi^{(1)} = -\Box^{-1}\operatorname{gr}(G)$ can be lifted to a linear algebraic map $\Phi^{(1)} : \mathcal{E}^k(V) \to \mathcal{E}^{k+1}(V)$ (e.g., by choosing a Weyl structure) and we shall define the first iteration $E_k^{(1)} = d^\nabla + \Phi^{(1)}$. Note that the lowest homogeneous component of $E_k^{(1)}$ remains to be ∂_V and that the image of $E_k^{(1)}$ is a subset of $\operatorname{Im}\partial^*$.

Since

$$E_{k+1} \circ E_k^{(1)} - E_{k+1} \circ E_k = d^\nabla \circ \Phi^{(1)},$$

we get

$$\begin{aligned} \operatorname{gr}_a(\partial^* \circ E_{k+1} \circ E_k^{(1)})) &= \operatorname{gr}_a(G + \partial^* \circ d^\nabla \circ \Phi^{(1)}) \\ &= \operatorname{gr}_a(G) - (\operatorname{gr}\partial^*)(\operatorname{gr}\partial_V)(\Box^{-1}(\operatorname{gr}_a(G))) = 0. \end{aligned}$$

Hence the first order differential operator $G^{(1)} := \partial^* \circ E_{k+1} \circ E_k^{(1)}$ belongs to A^{a+1}.

The same procedure will be repeated inductively. If we define

$$\Phi^{(2)} = -(\operatorname{gr}\partial^*)\Box^{-1}\operatorname{gr}_{a+1}(G^{(1)})$$

we can again lift this first order differential operator to a first order differential operator $\Phi^{(2)} : \mathcal{E}^k(V) \to \mathcal{E}^{k+1}(V)$ and we can define the next iteration by

$$E_k^{(2)} := E_k^{(1)} + \Phi^{(2)}.$$

Then we get

$$\begin{aligned}\operatorname{gr}_a(\partial_V^* \circ d^\nabla \circ E_k^{(2)})) &= \operatorname{gr}_a(G^{(1)} + \partial_V^* \circ d^\nabla \circ \Phi^{(2)}) \\ &= \operatorname{gr}_a(G^{(1)}) - (\operatorname{gr}\partial_V^*)(\operatorname{gr}\partial_V)(\Box^{-1}(\operatorname{gr}_a(G^{(1)}))) = 0.\end{aligned}$$

Hence the first order differential operator $G^{(2)} := \partial_V^* \circ d^\nabla \circ E_k^{(2)}$ belongs to A^{a+2}.

It is clear that after a finite number of iterations, we shall get the existence part of the theorem.

The proof of the uniqueness part is similar to the procedure employed in Lemma 2.5. Suppose that we have two differential operators Φ_k' and Φ_k'' satisfying the conditions of the theorem. Their difference $\Phi = \Phi_k' - \Phi_k''$ satisfies $\partial_V^*(d^\nabla \circ \Phi) = 0$. To show that $\Phi = 0$, suppose that Φ is nontrivial and consider the largest a such that Φ^a is nontrivial. Then we know that $\operatorname{gr}_a(d^\nabla \circ \Phi) = (\operatorname{gr}\partial_V)(\operatorname{gr}_a \Phi)$, hence $(\operatorname{gr}\partial_V)(\operatorname{gr}_a \Phi)$ is at the same time in $\operatorname{Im}\operatorname{gr}\partial_V$ and $\operatorname{Ker}\operatorname{gr}\partial_V^*$, so it is 0. By definition, $\operatorname{gr}_a \Phi$ also belongs to $\operatorname{Im}\partial_V^*$, hence $\operatorname{gr}_a \Phi$ is trivial and we have a contradiction.

As for the second part of the theorem, let us consider two consecutive squares in the BGG construction induced by $E_k = d^\nabla$, containing the operators D_k and D_{k+1}. If Φ_k is the deformation constructed above, then the replacement of $E_k = d^\nabla$ by $\tilde{E}_k = d^\nabla + \Phi_k$ leads to the same splitting operator L_k. Hence by the first part of the theorem, the k-th diagram commutes. Note that changing the next operator E_{k+1} will not change the splitting operator L_{k+1}, hence the commutativity of the k-th diagram is preserved.

Finally, during the construction there were several choices made but due to the uniqueness of the result, the construction depends only on the data of the chosen parabolic geometry. The same is true for the covariant derivative ∇. $\square$

5. Examples

In this section we want to illustrate the general results presented above by explicit examples showing the form of the prolongation covariant derivative in some simple situations. A more comprehensive set of examples is given in [24].

To calculate the prolongation covariant derivative of the first BGG operator D_0 for some tractor bundle $V = \mathcal{G} \times_P \mathbb{V}$, we employ the theory of Weyl structures [7], [8]. All of our examples below will be $|1|$-graded parabolic geometries, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Modding out $P_+ \cong \mathfrak{g}_1$ of the parabolic structure bundle $\mathcal{G}$, one obtains $\mathcal{G}_0 := \mathcal{G}/P_+$, which is a G_0-principal bundle over M. A splitting $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the canonical projection $\mathcal{G} \to \mathcal{G}_0$ is called a *Weyl structure*, and for our geometric structures below this can be identified with the choice of a *Weyl connection*, which is a linear connection D compatible with the geometry. Under such a choice, all P-associated bundles reduce to G_0-associated bundles, and in particular one gets a decomposition of every tractor bundle V which depends on the choice of the Weyl structure [8]. In particular, the adjoint tractor bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$ decomposes into $\mathcal{A}_-M \oplus \mathcal{A}_0M \oplus \mathcal{A}_1M$, with $\mathcal{A}_-M \cong TM$ and $A_+M \cong T^*M$. The Lie algebraic action of $\mathfrak{g}$ on $\mathbb{V}$ gives rise to an action $\bullet$ of $\mathcal{A}M$ on V, which we can restrict to TM and T^*M. The tractor covariant derivative ∇^ω can be written as $\nabla^\omega = \partial + D + \mathsf{P}\bullet$: the map $\partial : V \to \Omega^1(M, V)$ is obtained by the action of $TM \hookrightarrow \mathcal{A}M$ on V, and $\mathsf{P}\bullet : V \to \Omega^1(M, V)$ is induced by the action of the second slot of the (generalized) Schouten tensor $\mathsf{P} \in \mathcal{E}_{ab}$ of D, which will be symmetric for our choices of D. Recall that this decomposition of ∇^ω depends on the choice of the Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$ resp. Weyl connection D.

In our explicit formulas, we employ abstract index notation [29]: $\mathcal{E}_a = \Omega^1(M)$, $\mathcal{E}^a = \mathfrak{X}(M)$, and multiple indices indicate tensor products. Round brackets denote symmetrizations of the indices enclosed, and square brackets denote skew symmetrizations. A subscript zero indicates taking the trace-free part.

We now prolong an interesting equation in projective geometry which has already been treated in [17] by different methods. Next we consider three well known overdetermined equations in conformal geometry, which govern Einstein rescalings, conformal Killing forms and twistor spinors. Finally we analyse an equation for Grassmannian structures of type $(2, q), q > 2$. For a more detailed exposition of explicit calculations cf. [22]–[24].

5.1. *An example in projective geometry*

Let M be an orientable manifold of dimension n endowed with a projective class of linear, torsion-free connections $[D]$; here D and D' are *projectively equivalent* if there is a $\Upsilon \in \mathcal{E}^1$ such that

$$D'_a\omega_b = D_a\omega - \Upsilon_a\omega_b - \Upsilon_b\omega_a$$

(see e.g. [14]). For simplicity, we will assume that our chosen representatives $D \in [D]$ preserve a volume form on TM.

To define projectively invariant operators we need to employ densities, which are sections of line bundles $\mathcal{E}[w]$, $w \in \mathbb{R}$, associated to the full $\mathrm{GL}(n)$-frame bundle of TM via the 1-dimensional representation

$$C \in \mathrm{GL}(n) \mapsto |\det C|^{w(n+1)/n} \in \mathbb{R}_+.$$

With this parametrization, sections of the bundles $\mathcal{E}[w]$ are often called projective densities. Assume $n \geq 2$. We are going to prolong the following projectively invariant operator, which is written down with respect to a $D \in [D]$, but does not depend on this choice:

$$D_0 : \mathcal{E}^{(ab)}[-2] \to \mathcal{E}_c^{(ab)}{}_0[-2], \qquad \sigma^{ab} \mapsto D_c\sigma^{ab} - \frac{1}{n+1}\delta_c^{(a} D_p\sigma^{b)p}. \tag{7}$$

D_0 projects the Levi-Civita derivative of a symmetric two-tensor σ to its trace-free part. This operator was discussed in [17], where M. Eastwood and V. Matveev showed that this equation governs the metrizability of a projective class of covariant derivatives.

5.1.1. The projective structure as a parabolic geometry. It is a classical result that $(M, [D])$ is equivalent to a unique Cartan geometry (Γ, ω) of type $(G, P) = (\mathrm{SL}(n+1), P)$ with P the stabilizer of a ray in $\mathbb{R}^{n+1}$ (cf. [11, 31, 8]).

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1)$ is 1-graded, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathbb{R}^n \oplus \mathfrak{gl}(n) \oplus (\mathbb{R}^n)^*$, where an element $X \oplus (\alpha \,\mathrm{id} + A) \oplus \varphi \in \mathfrak{g}$ for $\alpha \in \mathbb{R}$, $A \in \mathfrak{sl}(n)$ corresponds to the matrix

$$\begin{pmatrix} -\alpha \frac{n}{n+1} & -\varphi \\ X & \frac{1}{n+1}\alpha \mathbb{I}_n + A \end{pmatrix}.$$

The actions of $\mathfrak{g}_0 = \mathfrak{gl}(n) \subset \mathfrak{g}$ on $\mathfrak{g}_{-1} = \mathbb{R}^n$ and $\mathfrak{g}_1 = (\mathbb{R}^n)^*$ are the standard representation and its dual.

The curvature of the Cartan connection form ω can be regarded as an element of $\mathcal{E}^2(\mathcal{A}M)$, with $\mathcal{A}M = \Gamma \times_P \mathfrak{g}$ the adjoint tractor bundle, and is written

$$K = \begin{pmatrix} 0 & -A_{ac_1c_2} \\ 0 & C_{c_1c_2}{}^a{}_b \end{pmatrix}$$

with A the Cotton–York tensor and C the (projectively invariant) Weyl curvature (cf. [14]).

1-forms and vector fields include into $\mathcal{A}M$ as

$$\eta_a \in T^*M \mapsto \begin{pmatrix} 0 & -\eta_a \\ 0 & 0 \end{pmatrix} \in \mathcal{A}M, \qquad \xi^a \in TM \mapsto \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} \in \mathcal{A}M.$$

5.1.2. The operator D_0 as the first BGG operator. Let $V := \Gamma \times_P S^2\mathbb{R}^{n+1}$. With respect to a choice of a Weyl connection $D \in [D]$, a section s of V can be written

$$[s]_D = \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} \in \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix} := \begin{pmatrix} \mathcal{E}[-2] \\ \mathcal{E}^a[-2] \\ \mathcal{E}^{(ab)}[-2] \end{pmatrix}. \tag{8}$$

We will need that on the first chain spaces the Lie algebra differentials ∂ and ∂^* are explicitly given by

$$\partial \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab}. \end{pmatrix} = \begin{pmatrix} 0 \\ \rho \delta_c{}^a \\ \delta_c{}^{(a}\mu^{b)} \end{pmatrix}, \qquad \partial \begin{pmatrix} \rho_c \\ \mu_c{}^a \\ \sigma_c{}^{ab}. \end{pmatrix} = \begin{pmatrix} 0 \\ 2\delta_{[c_1}{}^a \rho_{c_2]} \\ 2\delta_{[c_1}{}^{(a_1}\mu_{c_2]}{}^{a_2)} \end{pmatrix},$$

$$\partial^* \begin{pmatrix} \rho_c \\ \mu_c{}^a \\ \sigma_c{}^{ab}. \end{pmatrix} = \begin{pmatrix} -2\mu_p{}^p \\ -2\sigma_p{}^{pa} \\ 0 \end{pmatrix}, \qquad \partial^* \begin{pmatrix} \rho_{c_1c_2} \\ \mu_{c_1c_2}{}^a \\ \sigma_{c_1c_2}{}^{ab}. \end{pmatrix} = \begin{pmatrix} 2\mu_{cp}{}^p \\ 2\sigma_{cp}{}^{pa} \\ 0 \end{pmatrix}.$$

As bundles with structure group G_0, V_2, V_1 and $T^*M \otimes V_2$ are irreducible and are contained in the image of ∂^*; $T^*M \otimes V_1$ decomposes into the trace-free part $\operatorname{Im} \partial^* \cap T^*M \otimes V_1$ and the trace part, which lies in the image of ∂. The Kostant Laplacian $\Box$ acts by

$$\Box \begin{pmatrix} \rho_{c_1c_2} \\ \mu_{c_1c_2}{}^a \\ \sigma_{c_1c_2}{}^{ab} \end{pmatrix} = \begin{pmatrix} -2n\rho_{c_1c_2} \\ -(n+1)\mu_{c_1c_2}{}^a \\ 0 \end{pmatrix}$$

on V, by multiplication with $-2(n-1)$ on $T^*M \otimes V_2$ and by multiplication with $-n$ on the trace-free part of $T^*M \otimes V_1$. This is all the algebraic information we need to calculate the splitting operators and the prolongation.

The tractor covariant derivative ∇^ω on V is easily calculated with the above actions of $\mathcal{E}_a$ and $\mathcal{E}^a$ on V together with the formula $\nabla^\omega = \partial + D + \mathsf{P}\bullet$:

$$\nabla^\omega \begin{pmatrix} \rho \\ \mu^a \\ \sigma^{ab} \end{pmatrix} = \begin{pmatrix} D_c\rho - 2\mathsf{P}_{ca}\mu^a \\ D_c\mu^a - 2\mathsf{P}_{cb}\sigma^{ab} + \rho\delta_c{}^a \\ D_c\sigma^{ab} + \delta_c{}^{(a}\mu^{b)} \end{pmatrix}.$$

One calculates that the first splitting operator $L_0 : \Gamma(H_0) \to \Gamma(V)$ is given by

$$\sigma^{(ab)} \mapsto \begin{pmatrix} \frac{1}{n(n+1)} D_p D_q \sigma^{pq} + \frac{1}{2n}\mathsf{P}_{pq}\sigma^{pq} \\ -\frac{1}{n+1} D_p\sigma^{pa} \\ \sigma^{ab} \end{pmatrix},$$

and the composition of $\nabla^\omega \circ L_0$ with the projection to the lowest slot is seen to yield the operator D_0 of (7).

5.1.3. Prolongation of D_0. We calculate the action of the curvature $K \in \Omega^2(M, \mathcal{A}M)$:

$$K_{c_1c_2} \bullet \begin{pmatrix} 0 \\ 0 \\ \sigma^{ab} \end{pmatrix} = \begin{pmatrix} -2A_{pc_1c_2}\mu^p \\ -2A_{pc_1c_2}\sigma^{pa} + C_{c_1c_2}{}^a{}_p\mu^p \\ 2C_{c_1c_2}{}^{(a_1}{}_p\sigma^{a_2)p} \end{pmatrix}. \tag{9}$$

Therefore we define

$$\Phi_1\left(\begin{pmatrix} 0 \\ 0 \\ \sigma^{ab} \end{pmatrix}\right) := \begin{pmatrix} 0 \\ \bar{\Phi}_1\sigma \\ 0 \end{pmatrix} := -\Box^{-1}\left(\partial^*\left(K \bullet \begin{pmatrix} 0 \\ 0 \\ \sigma^{ab} \end{pmatrix}\right)\right) = \begin{pmatrix} 0 \\ \frac{2}{n}C_{cp}{}^a{}_q\sigma^{pq} \\ 0 \end{pmatrix}.$$

Now the curvature of the modified connection $\nabla^\omega + \Phi_1$ is $R = K\bullet + d^\nabla\Phi_1$ since $(\Phi_1 \wedge \Phi_1)(\xi, \eta)$ vanishes. For $\xi_1, \xi_2 \in \mathfrak{X}(M)$ and $s \in V$,

$$\begin{aligned}(d^\nabla\Phi_1)s(\xi_1, \xi_2) = {} & \nabla_{\xi_1}(\Phi_1(\xi_2)s) - \Phi_1(\xi_2)(\nabla_{\xi_1}s) - \nabla_{\xi_2}(\Phi_1(\xi_1)s) + \Phi_1(\xi_1)(\nabla_{\xi_2}s) \\ & - \Phi_1([\xi_1, \xi_2])s.\end{aligned} \tag{10}$$

We may expand (10) and write $(d^\nabla\Phi_1)s$ as

$$\begin{pmatrix} * \\ \begin{pmatrix} D_{\xi_1}(\bar{\Phi}_1(\xi_2)\sigma) - \bar{\Phi}_1(\xi_2)(D_{\xi_1}\sigma) - D_{\xi_2}(\bar{\Phi}_1(\xi_1)\sigma) + \bar{\Phi}_1(\xi_1)(D_{\xi_2}\sigma) \\ -\bar{\Phi}_1([\xi_1, \xi_2])\sigma \\ -\bar{\Phi}_1(\xi_2)\partial_{\xi_1}\varphi + \bar{\Phi}_1(\xi_1)\partial_{\xi_2}\varphi - \bar{\Phi}_1(\xi_2)\partial_{\xi_1}\mu + \bar{\Phi}_1(\xi_1)\partial_{\xi_2}\mu \end{pmatrix} \\ \partial_{\xi_1}\bar{\Phi}_1(\xi_2)\sigma - \partial_{\xi_2}\bar{\Phi}_1(\xi_1)\sigma \end{pmatrix}, \tag{11}$$

where we do not take care about the top component since it will vanish after an application of ∂^*. The lowest component is simply $\partial(\bar{\Phi}_1\sigma) = -\partial\Box^{-1}\partial^*(K \bullet \sigma)$. Thus $\partial^*(Rs)$ lies in the top slot (i.e., in homogeneity 1). So our first adjustment had the effect of moving the expression $\partial^*(Rs)$ one slot up.

The new connection $\nabla^\omega + \Phi_1$ has the following terms in the middle slot of the curvature R_{Φ_1}: From (11) we obtain the terms $2D_{[c_1}\overline{\Phi}_{1c_2]}$ and (via an application of the algebraic Bianchi identity for C), $C_{c_1c_2}{}^a{}_p\mu^p$. By (9), the contribution of $K \bullet s$ to the middle slot is $-2A_{pc_1c_2}\sigma^{pa} + C_{c_1c_2}{}^a{}_p\mu^p$. In total, we find that the action of the curvature R_{Φ_1} is

$$\begin{pmatrix}\rho\\ \mu^a\\ \sigma^{ab}\end{pmatrix} \mapsto \begin{pmatrix} * \\ \frac{2}{n}(D_{[c_1}C_{c_2]p}{}^a{}_q)\sigma^{pq} - 2A_{pc_1c_2}\sigma^{pa} + 2C_{c_1c_2}{}^a{}_p\mu^p \\ * \end{pmatrix}.$$

The entries $(*)$ are irrelevant: the lowest slot is by construction already in the kernel of ∂^* and the highest slot always lies in $\operatorname{Ker}\partial^*$. Now define

$$\Phi_2\left(\begin{pmatrix}\rho\\ \mu^a\\ \sigma^{ab}\end{pmatrix}\right) := -\Box^{-1}\partial^*\left(R_{\Phi_1}\left(\begin{pmatrix}\rho\\ \mu^a\\ \sigma^{ab}\end{pmatrix}\right)\right).$$

Using $D_pC_{c_1c_2}{}^p{}_a = (n-2)A_{ac_1c_2}$ and trace-freeness of C, we calculate

$$\Phi_2\left(\begin{pmatrix}\rho\\ \mu^a\\ \sigma^{ab}\end{pmatrix}\right) = \begin{pmatrix}-\frac{4}{n}A_{pcq}\sigma^{pq}\\ 0\\ 0\end{pmatrix}$$

and find that $\Phi := \Phi_1 + \Phi_2 \in \Gamma(T^*M \otimes \operatorname{End}(V))$ is

$$\begin{pmatrix}\rho\\ \mu^a\\ \sigma^{ab}\end{pmatrix} \mapsto \frac{2}{n}\begin{pmatrix}-2A_{pcq}\sigma^{pq}\\ C_{cp}{}^a{}_q\sigma^{pq}\\ 0\end{pmatrix}. \tag{12}$$

Now, with R_Φ the curvature of $\tilde{\nabla} = \nabla^\omega + \Phi$, one has by construction $\partial^* \circ R_\Phi = 0$. Thus $\tilde{\nabla}$ is the prolongation covariant derivative for $(D_c\sigma^{ab})_0 = 0$.

5.2. *Examples in conformal geometry*

Let M be an n-manifold endowed with a conformal class $[g]$ of (pseudo-)Riemannian signature (p,q) metrics. The conformal structure $(M,[g])$ is equivalent to a reduction of the structure group of the full frame bundle of TM to a $\mathrm{CO}(p,q) = \mathbb{R}_+ \times O(p,q)$-bundle $\mathcal{G}_0 \to M$. To write down conformally invariant differential operators we will employ *conformal density bundles* $\mathcal{E}[w]$, which are associated to the 1-dimensional $\mathrm{CO}(p,q)$-representation $(\alpha, C) \in \mathrm{CO}(p,q) \mapsto \alpha^w \in \mathbb{R}_+$.

The conformal structure can be equivalently encoded as a parabolic geometry $(\mathcal{G},\omega)$ of type $(\mathrm{SO}(p+1,q+1),P)$, with $P \subset \mathrm{SO}(p+1,q+1)$ the stabilizer of an isotropic ray in $\mathbb{R}^{p+1,q+1}$ (cf. [10, 8]); the curvature of ω is an element $\kappa \in \mathcal{E}^2(\mathcal{A}M)$, with $\mathcal{A}M = \mathcal{G}\times_P \mathfrak{so}(p+1,q+1)$, and has to satisfy the normalization condition $\partial^*\kappa = 0$. Choosing a metric $g \in [g]$ yields its Levi-Civita connection D on TM, which serves as a Weyl connection, and we make use of this to get explicit formulas for BGG operators in the following.

Since the necessary explicit tractor calculations for the next three equations are available in [23], and in the case of the conformal Killing form equations rather long, we just summarize the important properties here and relate the results to our general theory.

5.2.1. Almost Einstein scales. Let now $\mathcal{T} := \mathcal{G} \times_P \mathbb{R}^{p+1,q+1}$ be the *standard tractor bundle* of the conformal structure, which is endowed with its normal tractor covariant derivative ∇^ω. It was already observed in [1] that parallel sections of ∇^ω are in one-to-one correspondence with solutions of D_0,

$$D_0 : \mathcal{E}[1] \to \mathcal{E}_{(ab)_0}[1], \qquad \sigma \mapsto \mathbf{tf}(D_a D_b \sigma + \mathrm{P}_{ab}\sigma), \tag{13}$$

where $\mathbf{tf}$ takes the trace-free part. This is a conformally invariant 2^{nd} order PDE, and its solutions σ are *Einstein rescalings* or *almost Einstein scales*, [18]: σ is nonvanishing on an open dense subset, and $\sigma^{-2}g$ is Einstein there. In particular, ∇^ω is already the prolongation covariant derivative of this problem: it is also easy to see directly that its curvature R^{∇^ω} satisfies $\partial_V^*(R^{\nabla^\omega}) = 0$ [23].

5.2.2. Conformal Killing forms. Let now $\mathcal{V} := \Lambda^{k+1}\mathcal{T}$ be an exterior power of the conformal standard tractor bundle, which is again endowed with the normal tractor connection ∇^ω. An explicit tractor computation yields

$$D_0 : \mathcal{E}_{[a_1\cdots a_k]}[k+1] \to \mathcal{E}_{c[a_1\cdots a_k]}[k+1],$$
$$\sigma \mapsto D_c\sigma_{a_1\cdots a_k} - D_{[a_0}\sigma_{a_1\cdots a_k]} - \frac{k}{n-k+1} g_{c[a_1} g^{pq} D_{|p}\sigma_{q|a_2\cdots a_k]},$$

which is the projection of $D\sigma$ to the highest weight component in $\mathcal{E}_{c[a_1\cdots a_k]}[k+1]$. Solutions of $D_0\sigma = 0$ are the *conformal Killing forms* on M. The equations governing conformal Killing forms were first prolonged by U. Semmelmann [30]. In [19] an invariant prolongation was calculated directly. The prolongation covariant derivative ∇ for this equation is already fairly complicated to compute explicitly, and we refer to [23, 22] for this.

5.2.3. Twistor spinors. In the case where one has a reduction of the $CO(p, q)$-bundle of $(M, [g])$ to a $CSpin(p, q) = \mathbb{R}_+ \times Spin(p, q)$-bundle $\mathcal{G}_0$ one knows that $(M, [g])$ is a conformal spin structure. This structure is then equivalently described as a Cartan geometry of type $(Spin(p+1, q+1), P)$, with $P \subset Spin(p+1, q+1)$ again the stabilizer of an isotropic ray in $\mathbb{R}^{p+1,q+1}$. Let $\Delta^{p+1,q+1}$ be the $Spin(p+1, q+1)$ representation, which is decomposable in the case where $p+q$ is even. The corresponding associated tractor bundle is $\mathcal{S} = \mathcal{G} \times_P \Delta^{p+1,q+1}$. Now let $\Delta^{p,q}$ be the spin representation of $Spin(p, q)$, which we extend trivially to $CSpin(p, q)$, and define $S := \mathcal{G}_0 \times_{CSpin(p,q)} \Delta^{p,q}$. Then the first BGG operator of $\mathcal{S}$ is

$$D_0 : \Gamma(S[1/2]) \to \Gamma(\mathcal{E}_c \otimes S[1/2]), \qquad \sigma \to \mathrm{proj}(D_c\sigma).$$

This is the *twistor operator*: it is the composition of $D : \Gamma(S) \to \mathcal{E}_c(S)$ with proj, the projection to the kernel of Clifford multiplication. It is again well known [2, 3, 23] that solutions of D_0 are already in one-to-one correspondence with parallel sections of the normal tractor covariant derivative ∇^ω on the spin tractor bundle $\mathcal{S}$.

5.3. An example in Grassmannian geometry

Let $q \in \mathbb{N}$, $q > 2$, and M be an oriented $2q$-dimensional manifold together with a rank 2 bundle $\mathcal{E}_\alpha$ and a rank q bundle $\mathcal{E}^{\alpha'}$. Assume there is an isomorphism of TM with $\mathcal{E}_\alpha \otimes \mathcal{E}^{\beta'}$, which will be fixed. We say that M together with the identification $TM = \mathcal{E}_\alpha^{\beta'}$ is a *Grassmannian geometry* of type $(2, q)$ if there exists a torsion-free linear connection D on TM which is the product of linear connections (again denoted by D) on $\mathcal{E}_\alpha$ and $\mathcal{E}^{\beta'}$ (see [20], [8]). The class of all such connections are the Weyl connections of $(M, TM \cong \mathcal{E}_\alpha^{\beta'})$.

We are going to prolong the operator

$$D_0 : \mathcal{E}^{[\alpha'\beta']} \to (\mathcal{E}_{\gamma'}^{\gamma[\alpha'\beta']})_0, \qquad u^{\alpha'\beta'} \mapsto D_{\gamma'}^{\gamma} u^{\alpha'\beta'} + \frac{2}{1-q}\delta_{\gamma'}^{[\alpha'} D_{\tau'}^{|\gamma} u^{\tau'|\beta']}. \tag{14}$$

Thus, $D_0(u)$ is the projection of Du to its trace-free part.

5.3.1. Grassmannian structures as parabolic geometries. Let $G = \mathrm{SL}(n)$, $n = 2+q$, and define P as the stabilizer of a two-plane in $(\mathbb{R}^n)^*$. Regular, normal and torsion-free parabolic geometries $(\mathcal{G}, \omega)$ of type (G, P) are Grassmannian structures. In the Cartan picture, $\mathcal{E}_\alpha$ and $\mathcal{E}^{\alpha'}$ are associated to the P-representations $(\mathbb{R}^p)^*$, resp. $\mathbb{R}^q$.

Let $\mathcal{S}$ be the standard tractor bundle of $(\mathcal{G}, \omega)$, i.e., the associated bundle to the standard representation of $\mathrm{SL}(n)$. Via any Weyl structure D, $\mathcal{S}$ decomposes into $\mathcal{E}^\alpha \oplus \mathcal{E}^{\alpha'}$.

The curvature $K \in \mathcal{E}^2(\mathcal{A}M) = \mathcal{E}^2(\mathcal{S})$ of the Cartan connection is of the form

$$K = \begin{pmatrix} C_{c_1 c_2 \eta}{}^{\varphi} & -A_{p c_1 c_2} \\ 0 & C'{}_{c_1 c_2 \eta'}{}^{\varphi'} \end{pmatrix};$$

this employs the (generalized) Weyl curvature components $C \in \Omega^2(M, \mathfrak{sl}(\mathcal{E}^\alpha))$ and $C' \in \mathcal{E}^2(\mathfrak{sl}(\mathcal{E}^{\alpha'}))$ and the generalized *Cotton–York tensor* $A \in \mathcal{E}^2(\mathcal{E}^1)$ (cf. [20]). Normality of the geometry and torsion-freeness imply that any possible trace of $C_{\gamma_1'\gamma_2'\eta}^{\gamma_1\gamma_2\varphi}$, $C'{}_{\gamma_1'\gamma_2'\eta'}^{\gamma_1\gamma_2\varphi'}$ and $A_{\varphi'\gamma_1'\gamma_2'}^{\varphi\gamma_1\gamma_2}$ vanishes.

5.3.2. Description of D_0 as the first BGG operator. We consider the tractor bundle $V = \Lambda^2\mathcal{S}$, which under a choice of a Weyl connection D decomposes according to

$$[V]_D = \Lambda^2(\mathcal{E}^\alpha \oplus \mathcal{E}_{\alpha'}) = \begin{pmatrix} \mathcal{E}^{[\alpha\beta]} \\ \mathcal{E}^{\alpha\beta'} \\ \mathcal{E}^{[\alpha'\beta']} \end{pmatrix}.$$

On the first chain spaces the Lie algebra differentials ∂ and ∂^* are given as follows (indices within vertical bars are not included in the skew symmetrization):

$$\partial \begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix} = \begin{pmatrix} 0 \\ -\delta_{\alpha'}^{\beta'} v^{\alpha\beta} \\ 2\delta_{\alpha'}^{[\beta_1'} w^{|\alpha|\beta_2']} \end{pmatrix}, \qquad \partial \begin{pmatrix} v_{\gamma'}^{\gamma\alpha\beta} \\ w_{\gamma'}^{\gamma\alpha\beta'} \\ u_{\gamma'}^{\gamma\alpha'\beta'} \end{pmatrix} = \begin{pmatrix} 0 \\ 2\delta_{\gamma_2'}^{\beta'} v_{\gamma_1'}^{\gamma_1\gamma_2\beta} - 2\delta_{\gamma_1'}^{\beta'} v_{\gamma_2'}^{\gamma_2\gamma_1\beta} \\ 2\delta_{\gamma_1'}^{[\beta_1'} w_{\gamma_2'}^{|\gamma_2\gamma_1|\beta_2']} - 2\delta_{\gamma_2'}^{[\beta_1'} w_{\gamma_1'}^{|\gamma_1\gamma_2|\beta_2']} \end{pmatrix}$$

$$\partial^*\begin{pmatrix} v^{\gamma\alpha\beta}_{\gamma'} \\ w^{\gamma\alpha\beta'}_{\gamma'} \\ u^{\gamma\alpha'\beta'}_{\gamma'} \end{pmatrix} = \begin{pmatrix} -2w^{[\alpha_1\alpha_2]\tau'}_{\tau'} \\ u^{\alpha\tau'\beta'}_{\tau'} \\ 0 \end{pmatrix}, \qquad \partial^*\begin{pmatrix} v^{\gamma_1\gamma_2\alpha\beta}_{\gamma_1'\gamma_2'} \\ w^{\gamma_1\gamma_2\alpha\beta'}_{\gamma_1'\gamma_2'} \\ u^{\gamma_1\gamma_2\alpha'\beta'}_{\gamma_1'\gamma_2'} \end{pmatrix} = \begin{pmatrix} 2w^{\gamma_1[\alpha\beta]\tau'}_{\gamma_1'\tau'} \\ -u^{\gamma_1\alpha\tau'\beta'}_{\gamma_1'\tau'} \\ 0 \end{pmatrix}.$$

The Kostant Laplacian $\square = \partial \circ \partial^* + \partial^* \circ \partial$ acts on $[V]_D$ via

$$\square\begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix} = \begin{pmatrix} (2q)v^{\alpha\beta} \\ (q-1)w^{\alpha\beta'} \\ 0 \end{pmatrix}.$$

The top slot of $\mathcal{E}^1(V)$ is $\mathcal{E}_c^{[\alpha\beta]} = \mathcal{E}_{\gamma'}^{\gamma[\alpha\beta]}$ and coincides with the image of ∂^*. It is irreducible and the Kostant Laplacian acts by multiplication with $2(2q-1)$. The middle slot of $\mathcal{E}^1(V)$, which is $\mathcal{E}_c^{\alpha\beta'}$, decomposes into $\operatorname{Im}\partial$, which are traces, and the trace-free part $\operatorname{Im}\partial^* = \mathcal{E}_{0c}^{\alpha\beta'}$. One finds that $\mathcal{E}_0{}_{\gamma'}^{\gamma\alpha\beta'} = \mathcal{E}_0{}_{\gamma'}^{[\gamma\alpha]\beta'} \oplus \mathcal{E}_0{}_{\gamma'}^{(\gamma\alpha)\beta'}$ and $\square$ acts by q on the alternating part and by $q-2$ on the symmetric part.

The tractor covariant derivative on V is

$$(\nabla^\omega)^\gamma_{\gamma'}\begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix} = \begin{pmatrix} D^\gamma_{\gamma'}v^{\alpha\beta} + 2\mathsf{P}^{\gamma[\alpha}_{\gamma'\tau'}w^{\beta]\tau'} \\ D^\gamma_{\gamma'}w^{\alpha\beta'} - \delta^\alpha_{\gamma'}v^{\gamma\beta'} + \mathsf{P}^{\gamma\alpha}_{\gamma'\tau'}u^{\beta'\tau'} \\ D^\gamma_{\gamma'}u^{\alpha'\beta'} + 2\delta^{[\alpha'}_{\gamma'}w^{|\gamma|\beta']} \end{pmatrix}.$$

The first BGG splitting operator $L_0 : \mathcal{E}^{(\alpha'\beta')} \to \Gamma(V)$ is computed to be

$$L_0(u^{\alpha'\beta'}) = \begin{pmatrix} \frac{1}{2q}\mathsf{P}^{\alpha\beta}_{\tau_1'\tau_2'}u^{\tau_1'\tau_2'} - \frac{1}{1-q}D^{[\alpha}_{\tau_1'}D^{\beta]}_{\tau_2'}u^{\tau_1'\tau_2'} \\ \frac{1}{1-q}D^\alpha_{\tau'}u^{\tau'\beta'} \\ u^{\alpha'\beta'} \end{pmatrix},$$

and the composition of $\nabla^\omega \circ L_0$ with the projection to the lowest slot is seen to yield our operator (14).

5.3.3. Prolongation of D_0. For a section s of V one first computes $K \bullet s \in \mathcal{E}^2(V)$, which is then mapped by ∂^* into $\mathcal{E}^1(V)$,

$$\partial^*\left(K \bullet \begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix}\right) = \begin{pmatrix} 2C^{\gamma_1[\alpha\beta]}_{\gamma_1'\varphi'\eta}w^{\eta\varphi'} + 2A^{[\alpha|\gamma_1|\beta]}_{\eta'\varphi'}u^{\eta'\varphi'} \\ -2C'^{\gamma_1\alpha\beta'}_{\gamma_1'\varphi'\eta'}u^{\varphi'\eta'} \\ 0 \end{pmatrix}. \tag{15}$$

The first deformation map Φ_1 is defined by $\Phi_1 = -\square^{-1} \circ \partial^* \circ K\bullet$,

$$\Phi_1\left(\begin{pmatrix} 0 \\ 0 \\ u^{\alpha'\beta'} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \frac{2}{q}C'^{[\gamma_1\alpha]\beta'}_{\gamma_1'\varphi'\eta'}u^{\varphi'\eta'} + \frac{2}{q-2}C'^{(\gamma_1\alpha)\beta'}_{\gamma_1' ph'\eta'}u^{\varphi'\eta'} \\ 0 \end{pmatrix}.$$

Now we need to calculate ∂^* of the change in curvature resulting from Φ_1, which is just $\partial^* \circ d^\nabla \Phi_1$, since one quickly sees that $\partial^* \circ \Phi_{1[c_1}\partial_{c_2]} = 0$. Both indices of a section $w^{\alpha\beta'}$ are contracted into C and the trace taken by ∂^* vanishes by trace-freeness of C, C'. Therefore we are only interested in the differential components of $d^\nabla \Phi_1$ given by

$$\begin{pmatrix} 0 \\ 2\big(\tfrac{1}{q} + \tfrac{1}{q-2}\big) D^{\gamma_1}_{\gamma_1'} C'^{\gamma_2\alpha\beta'}_{\gamma_2'\varphi'\eta'} u^{\varphi'\eta'} - 2\big(\tfrac{1}{q} - \tfrac{1}{q-2}\big) D^{\gamma_1}_{\gamma_1'} C'^{\alpha\gamma_2\beta'}_{\gamma_2'\varphi'\eta'} u^{\varphi'\eta'} \\ \qquad - 2\big(\tfrac{1}{q} + \tfrac{1}{q-2}\big) D^{\gamma_2}_{\gamma_2'} C'^{\gamma_1\alpha\beta'}_{\gamma_1'\varphi'\eta'} u^{\varphi'\eta'} + 2\big(\tfrac{1}{q} - \tfrac{1}{q-2}\big) D^{\gamma_2}_{\gamma_2'} C^{\alpha\gamma_1\beta'}_{\gamma_1'\varphi'\eta'} u^{\varphi'\eta'} \\ 0 \end{pmatrix}.$$

Applying ∂^* we obtain the top slot contribution

$$-4\left(\frac{1}{q} + \frac{1}{q-2}\right) D^{[\alpha}_{\tau'} C'^{|\gamma_1|\beta]\tau'}_{\gamma_1'\varphi'\eta'} u^{\varphi'\eta'} + 4\left(\frac{1}{q} - \frac{1}{q-2}\right) D^{[\alpha}_{\tau'} C^{\beta]\gamma_1\tau'}_{\gamma_1'\varphi'\eta'} u^{\varphi'\eta'}. \tag{16}$$

Adding the contributions of the top slot of (15) and (16) (after multiplication by $-\frac{1}{2(2q-1)}$) to the modification map Φ_1, we obtain the full modification map

$$\Phi \begin{pmatrix} v^{\alpha\beta} \\ w^{\alpha\beta'} \\ u^{\alpha'\beta'} \end{pmatrix} = \begin{pmatrix} \frac{1}{2q-1} \Big(2\big(\tfrac{1}{q} + \tfrac{1}{q-2}\big) D^{[\alpha}_{\tau'} C'^{|\gamma_1|\beta]\tau'}_{\gamma_1'\varphi'\eta'} u^{\varphi'\eta'} - 2\big(\tfrac{1}{q} - \tfrac{1}{q-2}\big) D^{[\alpha}_{\tau'} C^{\beta]\gamma_1\tau'}_{\gamma_1'\varphi'\eta'} u^{\varphi'\eta'} \\ \qquad - C^{\gamma_1[\alpha\beta]}_{\gamma_1'\varphi'\eta} w^{\eta\varphi'} + A^{[\alpha|\gamma_1|\beta]}_{\eta'\varphi'} u^{\eta'\varphi'}\Big) \\ \frac{2}{q} C'^{[\gamma_1\alpha]\beta'}_{\gamma_1'\varphi'\eta'} u^{\varphi'\eta'} + \frac{2}{q-2} C'^{(\gamma_1\alpha)\beta'}_{\gamma_1'\varphi'\eta'} u^{\varphi'\eta'} \\ 0 \end{pmatrix}.$$

$\tilde{\nabla} = \nabla^\omega + \Phi$ is then the prolongation covariant derivative of the system $(D^\gamma_{\gamma'} u^{\alpha'\beta'})_0 = 0$.

5.4. *The case of infinitesimal automorphisms*

Let $\mathcal{A}M$ be the adjoint tractor bundle of a regular parabolic geometry $(\mathcal{G}, \omega)$ over M and ∇^ω the adjoint tractor covariant derivative. In [6] it was shown that parallel sections of the connection

$$\tilde{\nabla} s = \nabla^\omega s + \kappa(\Pi(s), \cdot) \tag{17}$$

are in one-to-one correspondence with infinitesimal automorphisms of $(\mathcal{G}, \omega)$, where Π is the natural projection $\Pi : \mathcal{A}M \to TM$. This shows that it is of interest to consider the first BGG operator $\tilde{D}_0$ obtained from $\tilde{\nabla}$. If the parabolic geometry $(\mathcal{G}, \omega)$ is normal, the curvature of $\tilde{\nabla}$ lies in the kernel of $\partial^*_{\mathcal{A}M}$. Therefore, exactly as in Corollary 3.1, one sees that $\Pi_0 : \mathcal{A}M \to H_0$ and $\tilde{L}_0 : H_0 \to \mathcal{A}M$ are inverse isomorphisms between the space of parallel sections of $\tilde{\nabla}$ and the kernel of $\tilde{D}_0$. Thus, the operator $\tilde{D}_0$ describes the infinitesimal automorphisms of $(\mathcal{G}, \omega)$ and is automatically prolonged by $\tilde{\nabla}$.

It is shown that if the parabolic geometry is also torsion-free or 1-graded, one has $\partial^*_{\mathcal{A}M}\kappa = 0$, i.e., for every $s \in \mathcal{A}M$ one has $\partial^*_{\mathcal{A}M}\kappa(\Pi(s), \cdot) = 0$. But in the torsion-free case, the map $\xi \mapsto \kappa(\Pi(s), \xi)$ is evidently homogeneous of degree ≥ 1. Therefore, if we know that $H_1(\mathfrak{g}_-, \mathfrak{g})$ sits in homogeneity ≤ 0, we see that $\xi \mapsto \kappa(\Pi(s), \xi)$ lies in $\operatorname{Im} \partial^*_{\mathcal{A}M}$.

Thus we have:

Theorem 5.1. *Let $(\mathcal{G} \to M, \omega)$ be a torsion-free, normal parabolic geometry with $H_1(\mathfrak{g}_-, \mathfrak{g})$ concentrated in homogeneity ≤ 0. Then $\tilde{\nabla}$ from (17) coincides with the prolongation covariant derivative on $\mathcal{A}M$. In particular, the usual first BGG operator D_0 coincides with $\tilde{D}_0$ and thus describes infinitesimal automorphisms.*

We note that the homogeneity condition on $H_1(\mathfrak{g}_-, \mathfrak{g})$ is satisfied for all parabolic geometries of type (G, P) with $\mathfrak{g}$ simple and (G, P) not corresponding to projective structures or contact projective structures.

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Invariant prolongation of overdetermined PDEs in projective, conformal, and Grassmannian geometry

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Abstract This is the second in a series of articles on a natural modification of the normal tractor connection on parabolic geometries, which naturally prolongs an underlying overdetermined system of invariant differential equations. We give a short review of the general procedure developed in Hammerl et al. (preprint) and then compute the prolongation covariant derivatives for a number of interesting examples in projective, conformal, and Grassmannian geometries.

Keywords Parabolic geometry · Prolongation of invariant PDE's · BGG sequence · Tractor covariant derivatives · Projective geometry · Conformal geometry · Grassmannian geometry

Mathematics Subject Classification (2000) 53A55 · 58A32 · 53A20 · 53A30

1 Introduction

In this article, we study certain overdetermined linear systems of PDEs that have a geometric origin and satisfy strong invariance properties. The goal is to rewrite these systems in a closed form by prolongation. This is achieved by constructing an extended first order system that is described by a covariant derivative and which has the property that parallel sections of that covariant derivative are in one to one correspondence with solutions of the original equation. We call this derivative the *prolongation covariant derivative* of the given equation. A universal construction of this prolongation for a big class of geometric equations that will be introduced below was obtained in [17].

The equations studied here appear naturally for parabolic geometries-like projective, conformal, or Grassmannian structures and include as special instances the equations describing

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the infinitesimal symmetries of geometric structures. Special examples of overdetermined linear systems of invariant equations coming from parabolic geometries are discussed in, e.g., [2,10,4,8,14].

The results of [2,17] provide a priori bounds for the solution spaces of the respective equations. To obtain more subtle information, for instance by analyzing the curvature, one needs to have formulas for the resulting prolongation covariant derivative. While the universal procedure presented in [17] is constructive, the explicit form of the resulting prolongation was so far only known in a small number of cases. The examples in the panorama presented here were selected according to several criteria: they should be useful and non-elementary, going beyond the examples scattered in the references; they should be computable by hand, and at the same time demonstrating the powerful machine developed in [17]. The examples treated in this article should give the interested reader the ability to recognize the complexity of the necessary computations both in general and in specific situations of interest.

The invariant overdetermined operators that give rise to the equations studied here appear in the Bernstein–Gelfand–Gelfand (BGG for short) sequences of natural differential operators on parabolic geometries that was constructed in [7] and later simplified in [3]. The prolongation results of [17] make extensive use of tractor calculus for parabolic geometries, which is also central to the description of the BGG-machinery. In the next sections we briefly recall the basic technical facts and notations necessary for these constructions. For more details we refer to the preceding article [17] and references therein.

Throughout the article we work in the smooth category, i.e., all manifolds, vector bundles, and their sheaves of sections are assumed to be smooth.

1.1 The BGG sequence

Let G be a semi-simple Lie group and $P \subset G$ a parabolic subgroup. A *parabolic geometry* on a manifold M consists of a P-principal bundle $\mathcal{G} \to M$ together with a *Cartan connection* 1*-form* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, [6]. Here $\mathfrak{g}$ denotes the Lie algebra of G. A major development in the construction of differential invariants of parabolic structure was achieved in [3,7].

Let $\mathbb{V}$ be a finite dimensional G-representation. It is well known that the associated *tractor bundle* $V = \mathcal{G} \times_P \mathbb{V}$ carries the canonical *tractor covariant derivative* ∇ induced by the Cartan connection form ω, see, e.g., [1]. The connection extends uniquely to an exterior covariant derivative on the spaces $\mathcal{E}^k(V) := \Omega^k(M, V)$ of k-forms with values in the vector bundle V, denoted by $d^\nabla : \mathcal{E}^k(V) \to \mathcal{E}^{k+1}(V)$. All associated vector bundles are graded with respect to the action of the grading element in the Levi factor G_0 of P, see [17, Section 2.1], leading to the decomposition into homogeneous parts. The lowest homogeneous part of d^∇ is the G_0-equivariant Lie algebraic differential $\partial_k : \mathcal{E}^k(V) \to \mathcal{E}^{k+1}(V)$, termed the *Kostant differential*, [18]. Its adjoint, the *Kostant codifferential* ∂_k^* is P-equivariant and gives rise to a complex

$$\mathcal{E}^{k+1}(V) \xrightarrow{\partial_{k+1}^*} \mathcal{E}^k(V), \quad \partial_k^* \circ \partial_{k+1}^* = 0.$$

There are Lie algebra cohomology bundles $H_k = \ker \partial_k^* / \operatorname{im} \partial_{k+1}^*$ due to the P-equivariant projection

$$\Pi_k : \ker \partial_k^* \to H_k.$$

The basic ingredient of the BGG-machinery are the differential *BGG-splitting operators*

$$L_k : H_k \to \ker \partial_k^*,$$

uniquely defined by the property that for every smooth section $\sigma \in \Gamma(H_k)$ one has

$$\partial^*_{k+1}(d^\nabla(L_k(\sigma))) = 0. \tag{1}$$

In particular, one can form the *BGG-operators*

$$D_k : H_k \to H_{k+1},\ D_k := \Pi_{k+1} \circ d^\nabla \circ L_k.$$

It will be usually clear from the context what is the appropriate value for homogeneity k of the form which is acted upon by any of the operators, i.e., we usually omit this subscript from the notation.

Let us briefly review the invariant prolongation procedure obtained in [17]:

1.2 Prolongation of the first BGG-operator D_0

The *first BGG-operator* D_0 associated to $\mathbb{V}$ is overdetermined, and our aim is the construction of invariant prolongation of the corresponding systems $D_0\sigma = 0$ on $\sigma \in \Gamma(H_0)$. Let us recall that the approach of [17] starts by introducing certain class of linear connections $\nabla^V + \Phi$ on V, which are modifications of the tractor covariant derivative ∇^V. The first condition on a modification map $\Phi \in \mathcal{E}^1(\mathrm{End}\, V)$ is that it is homogeneous of degree ≥ 1 with respect to the natural filterations on TM and V, for which we write $\Phi \in (\mathcal{E}^1(\mathrm{End}\, V))^1$. This insures that basic constructions of the BGG-machinery still work. The next condition is that for any section $s \in \Gamma(V)$ we have that $\Phi s \in \mathcal{E}^1(V)$ has values in $\mathrm{im}\, \partial^*$. As a consequence, the condition (1) is preserved under the modification map Φ. The latter condition can be rewritten as $\Phi \in \mathrm{Im}(\partial^*_V \otimes \mathrm{id}_{V^*})$, thus we arrive at a class of *admissible* covariant derivatives

$$\mathcal{C} = \left\{\tilde{\nabla} = \nabla + \Phi | \Phi \in Im\ (\partial^*_V \otimes \mathrm{id}_{V^*}) \cap (\mathcal{E}^1(\mathrm{End}\, V))^1\right\}.$$

Here ∂^*_V denotes ∂^* acting on $\mathcal{E}^1(V)$ (and *not* on $\mathcal{E}^1(\mathrm{End}\, V)$), and the same applies for ∂^*_V acting on $\mathcal{E}^k(V)$.

The main theorem of [17] is then

Theorem 1.1 *There exists a unique covariant derivative $\tilde{\nabla} \in \mathcal{C}$ characterized by the property*

$$(\partial^*_V \otimes \mathrm{id}_{V^*})(\tilde{\Omega}) = 0,$$

where $\tilde{\Omega} \in \mathcal{E}^2(V \otimes V^\star)$ is the curvature of $\tilde{\nabla}$.

This implies $\tilde{\nabla} \circ L_0 = L_1 \circ D_0$, which in turn yields

Corollary 1.2 *Consider a tractor bundle V and the covariant derivative $\tilde{\nabla}$ in Theorem 1.1. Then $\tilde{\nabla}$ gives a prolongation of the first BGG-operator D_0 in the sense that there is an isomorphism between $\tilde{\nabla}$-parallel sections of V and the kernel of D_0 acting on $\Gamma(H_0)$. This isomorphism is given by the projection $\Pi_0 : \Gamma(V) \to \Gamma(H_0)$ and inverted by the differential splitting operator $L_0 : \Gamma(H_0) \to \Gamma(V)$.*

We therefore say that $\tilde{\nabla}$ is the *prolongation covariant derivative*.

The prolongation of the first operator in the BGG sequence obtained by this theorem can be understood as the construction of a certain commutative square related to the first BGG-operator D_0, cf. [17]. We are constructing also examples of commutative squares for the operators D_k:

1.3 Commutativity for all D_k

In [17] the authors also obtained the analog of $\tilde{\tilde{\nabla}}$ on $\mathcal{E}^k(V)$. Here d^∇ gives rise to the class

$$\mathcal{C}_k := \{\tilde{d}_k = d^\nabla + \Phi \mid \Phi \in A^1, \operatorname{Im}\Phi \subset \operatorname{Im}\partial^*\}$$

where $A := Hom(\mathcal{E}^k(V), \mathcal{E}^{k+1}(V))$ and A^1 denotes homomorphisms homogeneous of the degree ≥ 1. Then it turns out that there is a unique $\tilde{d}_k \in \mathcal{C}_k$ such that $\partial_V^* \circ d^\nabla \circ \tilde{d}_k = 0$. This then implies

$$\tilde{d}_k \circ L_k = L_{k+1} \circ D_k,$$

and Π_k and L_k restrict to inverse isomorphisms between $\operatorname{Ker}\tilde{d}_k \cap \operatorname{Ker}\partial^*$ and $\operatorname{Ker} D_k$.

1.4 The guideline for computing examples

Here is the manual for treating particular examples, which can be used to derive the explicit form of the prolongation covariant derivative. In practice, the normalization procedure for the canonical tractor covariant derivative can be summarized as an algorithm based on the following list of steps:

(1) Choose a parabolic geometry $(\mathcal{G}, P, M, \omega)$, where $\mathcal{G} \to M$ is a principal P-bundle on M and $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Choose also a finite dimensional G-module $\mathbb{V}$ and its associated vector bundle V, termed tractor bundle. Let us fix the two consecutive vector bundles of k, respectively, $(k+1)$-forms twisted by V.

(2) Choose a Weyl structure, so that there is a well-defined splitting of the filtered bundle V into a direct sum of homogeneous components. Decompose both spaces of k, respectively, $(k+1)$-forms twisted by V with respect to G_0. Then compute the value of the Laplace-Kostant algebraic operator $\Box$ associated to ∂^* on each irreducible G_0-summand (i.e., G_0-graded components associated to P-equivariant filtration) either by evaluating the action of Casimir operator or from the definition $\Box = \partial^*\partial + \partial\partial^*$.

(3) Now the procedure splits into two cases:

- The computation of the prolongation covariant derivative.
  Check if $(\partial_V^* \otimes \mathrm{id}_{V^*})(\Omega)$ is trivial, where Ω is the curvature of ∇. If this is the case the procedure ends, and we have computed the prolongation covariant derivative.
  If $\alpha := (\partial_V^* \otimes \mathrm{id}_{V^*})(\Omega) \neq 0$, take the lowest nontrivial homogeneous part α_j of α and define

$$\Phi = -\Box^{-1}\alpha_j; \quad \nabla' = \nabla + \Phi.$$

  Then repeat the procedure with ∇ replaced by ∇'. By construction, the lowest nontrivial component of α in the next step will have higher degree than in the previous step, hence the procedure will terminate in a finite number of steps (bounded by the length of the grading of $\mathbb{V}$).
- The case of the whole sequence of commuting squares.
  Here we use another procedure based on the following algorithm: Consider two consecutive squares containing the exterior covariant derivatives $d_k^\nabla : \mathcal{E}^k(V) \mapsto \mathcal{E}^{k+1}(V)$ and $d_{k+1}^\nabla : \mathcal{E}^{k+1}(V) \mapsto \mathcal{E}^{k+2}(V)$. First check if

$$(\partial_V^* \otimes \mathrm{id}_{V^*})(d_{k+1}^\nabla \circ d_k^\nabla)$$

is trivial. If not, the first step is the same as for the construction of prolongation covariant derivative above. Consider $\alpha := (\partial^* \otimes \mathrm{id}_{V^*})(d^\nabla_{k+1} \circ d^\nabla_k) \neq 0$, take the lowest non-trivial homogeneous part α_j of α and define

$$\Phi = -\Box^{-1}\alpha_j; \quad d'_k = d^\nabla_k + \Phi.$$

If $\alpha' := (\partial^* \otimes \mathrm{id}_{V^*})(d^\nabla_k \circ d'_k)$ is trivial, the procedure terminates and we define $\tilde{d}_k = d'_k$. If not, take the lowest non-trivial homogeneous part $\alpha'_{j'}$ of α' and define

$$\Phi' = -\Box^{-1}\alpha'_{j'}; \quad d''_k = d'_k + \Phi'.$$

By construction, the degree j' will be bigger than j, hence the procedure will terminate in a finite number of steps (bounded again by the length of the grading of V). Note that the operators Φ that occur in this iteration are (in general) differential operators, and their order rises by one with each iteration.

2 Notation

In this section we review the basic notation and conventions related to the results of our article.

2.1 Forms, tensors, and tensorial actions

In order to be explicit and efficient in calculations involving bundles of possibly high rank it is necessary to introduce some further abstract index notation. In the usual abstract index convention one would write $\mathcal{E}_{[ab\cdots c]}$, where there are implicitly k-indices skewed over, for the space $\mathcal{E}^k$. To simplify subsequent expressions we use the following conventions: First, indices labeled with sequential superscripts that are at the same level (i.e., all contravariant or all covariant) indicate a completely skew set of indices. Formally we set $a^1 \cdots a^k = [a^1 \cdots a^k]$ and so, for example, $\mathcal{E}_{a^1 \ldots a^k}$ is an alternative notation for $\mathcal{E}^k$, while $\mathcal{E}_{a^1 \ldots a^{k-1}}$ and $\mathcal{E}_{a^2 \ldots a^k}$ both denote $\mathcal{E}^{k-1}$. Next we abbreviate this notation via multi-indices: We will use the form indices

$$\begin{aligned}
\mathbf{a}^k &:= a^1 \cdots a^k = [a^1 \cdots a^k], \quad k \geq 0,\\
\dot{\mathbf{a}}^k &:= a^2 \cdots a^k = [a^2 \cdots a^k], \quad k \geq 1,\\
\ddot{\mathbf{a}}^k &:= a^3 \cdots a^k = [a^3 \cdots a^k], \quad k \geq 2,\\
\dddot{\mathbf{a}}^k &:= a^4 \cdots a^k = [a^4 \cdots a^k], \quad k \geq 3.
\end{aligned}$$

If, for example, $k = 1$, then $\dot{\mathbf{a}}^k$ simply means the index is absent, whereas if $k = 1$, then $\ddot{\mathbf{a}}$ means the term containing the index $\ddot{\mathbf{a}}$ is absent. For example, a 3-form φ can have the following possible equivalent structures of indices:

$$\varphi_{a^1a^2a^3} = \varphi_{[a^1a^2a^3]} = \varphi_{\mathbf{a}^3} = \varphi_{a^1\dot{\mathbf{a}}^3} = \varphi_{[a^1\dot{\mathbf{a}}^3]} = \varphi_{a^1a^2\ddot{\mathbf{a}}^3} \in \mathcal{E}_{\mathbf{a}^3} = \mathcal{E}^3.$$

Note that the exterior derivative d on a k-form $f_{\mathbf{a}}$ can be written as $(df)_{a^0\mathbf{a}} = \nabla_{a^0} f_{\mathbf{a}}$ for any torsion-free affine connection ∇.

Later on we define the standard tractor bundle denoted by $\mathcal{E}^A$ and its dual $\mathcal{E}_B$. The form index notation developed above will also be used for skew symmetric powers of these bundles. For example, the bundle of *tractor k-forms* $\mathcal{E}_{[A^1 \ldots A^k]}$ will be denoted by $\mathcal{E}_{A^1 \ldots A^k}$ or $\mathcal{E}_{\mathbf{A}^k}$.

The bundle of endomorphisms of $\mathcal{E}^A$ (or $\mathcal{E}_A$), $\mathcal{E}^E{}_F$, clearly injects $\mathcal{E}^E{}_F \subseteq \mathrm{End}(\mathcal{T})$ for any tractor bundle $\mathcal{T} \subseteq (\bigotimes \mathcal{E}^A) \otimes (\bigotimes \mathcal{E}_B)$. Consider $\gamma^E{}_F \in \mathcal{E}^E{}_F$ and $f \in \mathcal{T}$. The endomorphism γ acts on $\mathcal{T}$ and we denote this action by $\sharp$. That is, $\gamma \sharp f \in \mathcal{T}$. Using the abstract tractor indices, $\sharp$ is given by the usual tensorial action, i.e., $(\gamma \sharp f)^A = \gamma^A{}_P f^P$ for $f^A \in \mathcal{E}^A$ and $(\gamma \sharp f)_A = -\gamma^P{}_A f_P$ for $f_A \in \mathcal{E}_A$. One then computes $\sharp$ on the tensor products of $\mathcal{E}^A$ and $\mathcal{E}_B$ using the Leibniz rule. We further put $\gamma\sharp$ to be zero on $\mathcal{E}^a$, $\mathcal{E}_b$ and density bundles (which we introduce later) and, using the Leibniz rule, extend $\gamma\sharp$ to the tensor products of $\mathcal{T}$ with latter three bundles. Finally we remark the action $\sharp$ is denoted $\bullet$ in [6].

2.2 The adjoint tractor bundle and the Laplace-Kostant operator

The bundle $\mathcal{A} = \mathcal{G} \times_P \mathfrak{g}$ is called the *adjoint* tractor bundle. By definition, $\mathcal{A} \subseteq \mathcal{E}^A{}_B$, and more generally $\mathcal{A} \hookrightarrow \mathrm{End}(\mathcal{T})$ for any tractor bundle $\mathcal{T}$. We shall use $\sharp$ to denote the action of sections of $\mathcal{A}$ on $\mathcal{T}$ as introduced above. Note that the curvature of the normal tractor covariant derivative ∇ is the section of $\mathcal{E}_{a^0a^1} \otimes \mathcal{A}$, and the curvature action is $2(d^\nabla \nabla f)_{a^0a^1} = 2\nabla_{a^0}\nabla_{a^1} f = (\Omega \sharp f)_{a^0a^1} \in \mathcal{E}_{[ab]} \otimes \mathcal{T}$ for each $f \in \mathcal{T}$.

We have identifications $\mathcal{E}_a \cong \mathcal{G} \times_P \mathfrak{p}_+$ and $\mathcal{E}^a \cong \mathcal{A}/\mathcal{A}'$, $\mathcal{A}' := \mathcal{G} \times_P \mathfrak{p}$, which allow us to define inclusions $\iota : \mathcal{E}_a \hookrightarrow \mathcal{A}$ and $\bar{\iota} : \mathcal{E}^a \hookrightarrow \mathcal{A}/\mathcal{A}'$, where the latter is just the identity. We extend these inclusions to

$$\iota : \mathcal{E}_{\mathbf{a}} \hookrightarrow \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{A} \quad \text{and} \quad \bar{\iota} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\delta_{a^0}{}^b} \mathcal{E}_{a^0\mathbf{a}}{}^b \hookrightarrow \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{A}/\mathcal{A}'.$$

Recall that here and below we use a chosen Weyl structure and the corresponding splittings.

Our aim is to use these tools to express Kostant's differential ∂, co-differential ∂^*, and in particular the Laplace-Kostant operator $\Box$ (see [18]) in a form suitable for computations in abstract indices. For any tractor bundle $\mathcal{T}$ these operators act on associated vector bundles of forms twisted by $\mathcal{T}$, $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$, $\mathbf{a} = \mathbf{a}^k$. Using $\sharp$ they are given by

$$\begin{aligned}
&\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \xrightarrow{\bar{\iota}} \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{A}/\mathcal{A}' \otimes \mathcal{T} \xrightarrow{\sharp} \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{T}, \\
&\partial^* : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \xrightarrow{\iota} \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{A} \otimes \mathcal{T} \xrightarrow{\sharp} \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{T} \quad \text{and} \\
&\Box_k = \partial\partial^* + \partial^*\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \longrightarrow \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}.
\end{aligned}$$

Note ∂^* is invariant but ∂ (and consequently also $\Box_k$) depends on the choice of splitting of the tractor bundles in question, i.e., on the Weyl structure. The Weyl structure allows us to identify the quotient $\mathcal{A}/\mathcal{A}'$ with $\mathcal{G} \times_P \mathfrak{p}_+ \subset \mathcal{A}$. However, $\Box_k$ is invariant on completely reducible subquotients of $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$ and acts by a scalar multiple on each irreducible component of such subquotients. That is, we choose a splitting of the tractor bundle $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$ to compute $\Box_k$ but the value of $\Box_k$ on a given completely reducible subquotient alone is independent of this choice.

The symbol $\in\!\!\!+$ denotes the composition P-module structure of representations or vector bundles.

Finally, note that one can compute $\Box_k$ from highest weight of bundles concerned, see [18]. We shall use this less explicit approach in cases where the abstract index computation is getting too complicated.

Now we are ready to discuss specific geometries. In each case we first summarize the tractor calculus. In particular, we shall need formulas for the normal tractor covariant derivative ∇ and Kostant's differential and co-differential ∂ and ∂^*. Using these we compute the prolongation covariant derivative $\widetilde{\nabla}$ and/or $\tilde{d}$ on certain bundles.

3 Projective geometry

We follow the notation from [1] here. The projective structure on a smooth manifold M is given by a class $[\nabla]$ of projectively equivalent torsion-free connections. That is, connections $\hat{\nabla} \in [\nabla]$ are parametrized by one forms $\Upsilon_a \in \mathcal{E}_a \cong \Gamma(T^*M)$ and have the form

$$\begin{aligned} \hat{\nabla}_a \varphi &= \nabla_a \varphi + w \Upsilon_a \varphi, \quad \varphi \in \mathcal{E}(w), \\ \hat{\nabla}_a f^b &= \nabla_a f^b + \Upsilon_a f^b + \Upsilon_c f^c \delta_a^b, \quad f^b \in \mathcal{E}^b \\ \hat{\nabla}_a \omega_b &= \nabla_a \omega_b - \Upsilon_a \omega_b - \Upsilon_b \omega_a, \quad \omega_a \in \mathcal{E}_a. \end{aligned} \tag{2}$$

The curvature tensor $R_{ab}{}^c{}_d$ of a torsion-free ∇ is defined by $(\nabla_a \nabla_b - \nabla_b \nabla_a) f^c = R_{ab}{}^c{}_p f^p$ and it decomposes according to

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + 2\delta_{[a}{}^c \mathsf{P}_{b]d} + \beta_{ab} \delta^c{}_d, \quad \beta_{ab} = -2\mathsf{P}_{[ab]}.$$

Here $W_{ab}{}^c{}_d$ is the projectively invariant (and irreducible) Weyl tensor, P is the Schouten tensor, $\hat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b$, and $\hat{\beta}_{ab} = \beta_{ab} + 2\nabla_{[a} \Upsilon_{b]}$. We put $A_{abc} := 2\nabla_{[a} \mathsf{P}_{b]c}$. Then the Bianchi identity $\nabla_{[a} R_{bc]}{}^d{}_e = 0$ implies

$$\nabla_c W_{ab}{}^c{}_d = (n-2) A_{abd} \quad \text{and} \quad \nabla_{[a} \beta_{cd]} = 0.$$

The cohomology class $[\beta] \in H^2(M, \mathbb{R})$ is a global invariant of the projective structure. Moreover, $(\nabla_a \nabla_b - \nabla_b \nabla_a)\varphi = w \beta_{ab} \varphi$ for $\varphi \in \mathcal{E}(w)$.

3.1 Projective tractors

We shall write sections of the standard projective tractor bundle $\mathcal{E}^A = \mathcal{E}^a[-1] \oplus\!\!\!\!+\, \mathcal{E}[-1]$, respectively, its dual $\mathcal{E}_A = \mathcal{E}[1] \oplus\!\!\!\!+\, \mathcal{E}_a[1]$ using the injectors Y^A and X^A, respectively, Y_A and X_A as

$$\begin{pmatrix} \sigma^a \\ \rho \end{pmatrix} = Y_a^A \sigma^a + X^A \rho \in \mathcal{E}^A, \quad \text{respectively,} \quad \begin{pmatrix} \nu \\ \mu_a \end{pmatrix} = Y_A \nu + X_A^a \mu_a \in \mathcal{E}_A.$$

Such splittings of $\mathcal{E}^A$ and $\mathcal{E}_A$ are parametrized by choices of projective connections and we call them *projective splittings*. The change of the splitting under a change of the connection parametrized by $\Upsilon_a \in \mathcal{E}_a$ is

$$\widehat{\begin{pmatrix} \sigma^a \\ \rho \end{pmatrix}} = \begin{pmatrix} \sigma^a \\ \rho - \Upsilon_a \sigma^a \end{pmatrix}, \text{ i.e., } \hat{Y}_a^A = Y_a^A + X^A \Upsilon_a, \ \hat{X}^A = X^A \quad \text{and}$$
$$\widehat{\begin{pmatrix} \nu \\ \mu_a \end{pmatrix}} = \begin{pmatrix} \nu \\ \mu_a + \Upsilon_a \nu \end{pmatrix}, \text{ i.e., } \hat{Y}_A = Y_A - X_A^a \Upsilon_a, \ \hat{X}_A^a = X_A^a.$$

That is, $X^A \in \mathcal{E}^A[1]$, $X_A^a \in \mathcal{E}_A^a[-1]$ are invariant and $Y_a^A \in \mathcal{E}_a^A[1]$, $Y_A \in \mathcal{E}_A[-1]$ depend on the choice of the projective scale. We assume the normalization of these is such that $Y_A X^B + X_A^c Y_c^B = \delta_A{}^B$, i.e., $Y_C X^C = 1$ and $X_C^a Y_b^C = \delta^a{}_b$.

The normal covariant derivative is given by

$$\nabla_c \begin{pmatrix} \sigma^a \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_c \sigma^a + \rho \delta_c{}^a \\ \nabla_c \rho - P_{cp} \sigma^p \end{pmatrix} \quad \text{and} \quad \nabla_c \begin{pmatrix} \nu \\ \mu_a \end{pmatrix} = \begin{pmatrix} \nabla_c \nu - \mu_c \\ \nabla_c \mu_a + P_{ca} \nu \end{pmatrix}, \quad \text{i.e.,}$$
$$\nabla_c Y_a^A = -X^A \mathsf{P}_{ca}, \ \nabla_c X^A = Y_c^A \ \text{ and } \ \nabla_c Y_A = X_A^a \mathsf{P}_{ca}, \ \nabla_c X_A^a = -Y_A \delta_c^a,$$

and its Ω curvature has the form

$$\Omega_{ab}{}^{E}{}_{F} = Y^{E}_{e} X^{f}_{F} W_{ab}{}^{e}{}_{f} - X^{E} X^{f}_{F} A_{abf} \in \mathcal{E}_{[ab]} \otimes \mathcal{A}.$$

That is, $\mathcal{A} = $ trace-free$(\mathcal{E}^{E}{}_{F})$ is the projective adjoint tractor bundle where "trace-free" denotes the trace-free part. Hence the curvature action on $\mathcal{E}_C$ is $(\nabla_a\nabla_b - \nabla_b\nabla_a)F_C = (\Omega\sharp F)_{abC} = -\Omega_{ab}{}^{D}{}_{C}F_D$. We shall often write $\Omega_{ab}\sharp F_C$ instead of $(\Omega\sharp F)_{abC}$ to simplify the notation.

Using the notation developed above, the inclusions ι and $\bar{\iota}$ defined in 2.2 have the form $Y^{E}_{a^0}Y_F : \mathcal{E}_{\mathbf{a}} \xrightarrow{\bar{\iota}} \mathcal{E}_{a^0\mathbf{a}}{}^{E}{}_{F}$ and $X^{E}X^{a^1}_{F} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\iota} \mathcal{E}_{\dot{\mathbf{a}}}{}^{E}{}_{F}$. Thus

$$\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto Y^{E}_{a^0}Y_F f_{\mathbf{a}} \xrightarrow{\sharp} Y^{E}_{a^0}Y_F \sharp f_{\mathbf{a}} \in \mathcal{E}_{a^0\dot{\mathbf{a}}} \otimes \mathcal{T} \quad \text{and}$$

$$\partial^* : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto X^{E}X^{a^1}_{F} f_{\mathbf{a}} \xrightarrow{\sharp} X^{E}X^{a^1}_{F} \sharp f_{\mathbf{a}} \in \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{T},$$

and we can easily compute $\Box_k$ on $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$ using the action $\sharp$ as demonstrated by the following example.

Example 3.1 We shall compute the case $\mathcal{T} = \mathcal{E}^C$ in detail. One has $\mathcal{E}_{\mathbf{a}}{}^{C} = \mathcal{E}_{\mathbf{a}}{}^{c}[-1] \oplus \mathcal{E}_{\mathbf{a}}[-1]$, where $\mathcal{E}_{\mathbf{a}}$ is irreducible and $\mathcal{E}_{\mathbf{a}}{}^{c}$ has two irreducible components, which are the trace and trace-free parts. We shall compute $\Box_k$ separately for all three irreducible components.

We start with a (not necessarily irreducible) section $\sigma_{\mathbf{a}}{}^{c} \in \mathcal{E}_{\mathbf{a}}{}^{c}[-1]$. Then ∂ is zero on $f_{\mathbf{a}}{}^{C} := Y^{C}_{c}\sigma_{\mathbf{a}}{}^{c}$ and $X^{E}X^{a^1}_{F}\sharp Y^{C}_{c}\sigma_{\mathbf{a}}{}^{c} = X^{C}\sigma_{p\dot{\mathbf{a}}}{}^{p} = (\partial^* f)_{\dot{\mathbf{a}}}{}^{C}$. Thus $\partial^* f = 0$ for trace-free section $\sigma_{\mathbf{a}}{}^{c}$. Assume $\sigma_{\mathbf{a}}{}^{c} = \delta^{c}_{a^1}\tilde{\sigma}_{\dot{\mathbf{a}}}$. Then $f_{\mathbf{a}}{}^{C} = Y^{C}_{a^1}\tilde{\sigma}_{\dot{\mathbf{a}}}$, $(\partial^* f)_{\dot{\mathbf{a}}}{}^{C} = \frac{n-k+1}{k}X^{C}\tilde{\sigma}_{\dot{\mathbf{a}}}$ thus $(\Box_k f)_{\mathbf{a}}{}^{C} = (\partial\partial^* f)_{\mathbf{a}}{}^{C} = Y^{C}_{a^1}\tilde{\sigma}_{\dot{\mathbf{a}}}$. Finally if $\bar{f}_{\mathbf{a}}{}^{C} = X^{C}\rho_{\mathbf{a}}$ then $(\partial^*\bar{f})_{\dot{\mathbf{a}}}{}^{C} = 0$, $(\partial\bar{f})_{\mathbf{a}}{}^{C} = Y^{C}_{a^0}\rho_{\mathbf{a}}$ and $(\Box_k\bar{f})_{\mathbf{a}}{}^{C} = (\partial^*\partial\bar{f})_{\mathbf{a}}{}^{C} = \frac{n-k}{k+1}X^{C}\rho_{\mathbf{a}}$.

In summary, $\Box_k$ acts by zero on the trace-free part of $\mathcal{E}_{\mathbf{a}}{}^{c}[-1] = \mathcal{E}_{\mathbf{a}}{}^{C}/\mathcal{E}_{\mathbf{a}}[-1]$, by $\frac{n-k+1}{k}$ on the trace part, i.e., on $\mathcal{E}_{\dot{\mathbf{a}}}[-1] \subseteq \mathcal{E}_{\mathbf{a}}{}^{C}/\mathcal{E}_{\mathbf{a}}[-1]$ and by $\frac{n-k}{k+1}$ on $\mathcal{E}_{\mathbf{a}}[-1] \subseteq \mathcal{E}_{\mathbf{a}}{}^{C}$. Note that the inclusion $\mathcal{E}_{\mathbf{a}}[-1] \hookrightarrow \mathcal{E}_{\mathbf{a}}{}^{C}$ is realized by $X^{C} : \mathcal{E}_{\mathbf{a}}[-1] \to \mathcal{E}_{\mathbf{a}}{}^{C}$.

3.2 Skew symmetric tractors and tractor forms

The notation for the standard tractor bundle $\mathcal{E}^C$ developed above can be easily generalized to the products $\bigwedge^{\ell}\mathcal{E}^C = \mathcal{E}^{\mathbf{C}} = \mathcal{E}^{\mathbf{c}}(-\ell) \oplus \mathcal{E}^{\dot{\mathbf{c}}}(-\ell)$, where $\mathbf{C} = \mathbf{C}^{\ell}$. Note that $\bigwedge^{\ell}\mathcal{E}^C \cong \bigwedge^{n-\ell+1}\mathcal{E}_D$, hence these products are isomorphic to tractor forms. We put

$$\mathbb{Y}^{\mathbf{C}}_{\mathbf{c}} = Y^{[C^1}_{c^1}\dots Y^{C^\ell]}_{c^\ell} \in \mathcal{E}^{\mathbf{C}}_{\mathbf{c}}(\ell), \quad \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} = X^{[C^1}Y^{C^2}_{c^2}\dots Y^{C^\ell]}_{c^\ell} \in \mathcal{E}^{\mathbf{C}}_{\dot{\mathbf{c}}}(\ell),$$

and write the sections of $\mathcal{E}^{\mathbf{C}}$ as

$$\begin{pmatrix}\sigma^{\mathbf{c}}\\ \rho^{\dot{\mathbf{c}}}\end{pmatrix} = \mathbb{Y}^{\mathbf{C}}_{\mathbf{c}}\sigma^{\mathbf{c}} + \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}\rho^{\dot{\mathbf{c}}} \in \mathcal{E}^{\mathbf{C}}, \quad \sigma^{\mathbf{c}} \in \mathcal{E}^{\mathbf{c}}(-\ell), \quad \rho^{\dot{\mathbf{c}}} \in \mathcal{E}^{\dot{\mathbf{c}}}(-\ell),$$

where $\mathbf{c} = \mathbf{c}^{\ell}$. The change of the projective rescaling parametrized by Υ_a is

$$\widehat{\begin{pmatrix}\sigma^{\mathbf{c}}\\ \rho^{\dot{\mathbf{c}}}\end{pmatrix}} = \begin{pmatrix}\sigma^{\mathbf{c}}\\ \rho^{\dot{\mathbf{c}}} - \ell\,\Upsilon_{c^1}\sigma^{\mathbf{c}}\end{pmatrix}, \quad \text{i.e.,} \quad \hat{\mathbb{Y}}^{\mathbf{C}}_{\mathbf{c}} = \mathbb{Y}^{\mathbf{C}}_{\mathbf{c}} + \ell\,\Upsilon_{c^1}\mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}, \ \hat{\mathbb{X}}^{\mathbf{C}}_{\dot{\mathbf{c}}} = \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}$$

and the normal tractor covariant derivative has the form

$$\nabla_b\begin{pmatrix}\sigma^{\mathbf{c}}\\ \rho^{\dot{\mathbf{c}}}\end{pmatrix} = \begin{pmatrix}\nabla_b\sigma^{\mathbf{c}} + \rho^{\dot{\mathbf{c}}}\delta_b{}^{c^1}\\ \nabla_b\rho^{\dot{\mathbf{c}}} - \ell\,P_{bc^1}\sigma^{\mathbf{c}}\end{pmatrix}, \quad \text{i.e.,} \quad \nabla_b\mathbb{Y}^{\mathbf{C}}_{\mathbf{c}} = -\ell\,P_{bc^1}\mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}, \ \nabla_b\mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}} = \mathbb{Y}^{\mathbf{C}}_{[b\dot{\mathbf{c}}]}$$

Example 3.2 We shall compute the sequence for the tractor bundle $\mathcal{E}^{\mathbf{C}}$, $\mathbf{C} = \mathbf{C}^{\ell}$, i.e., $\mathcal{E}^{\mathbf{C}} \xrightarrow{\tilde{d}} \dots \xrightarrow{\tilde{d}} \mathcal{E}_{\mathbf{a}^n}{}^{\mathbf{C}}$. Since the filtration of $\mathcal{E}^{\mathbf{C}}$ has level 2, it follows immediately from the construction of $\tilde{d}$ that $(\tilde{d}F)_{a^0\mathbf{a}} = (d^{\nabla}F)_{a^0\mathbf{a}}{}^{\mathbf{C}} + (\Box_{k+1})^{-1}(\partial^* d^{\nabla} d^{\nabla} F)_{a^0\mathbf{a}}{}^{\mathbf{C}}$ for every $F_{\mathbf{a}}{}^{\mathbf{C}} \in \mathcal{E}_{\mathbf{a}}{}^{\mathbf{C}}$ (In particular, the difference between d^{∇} and $\tilde{d}$ is algebraic in this case.)

Let us compute $\tilde{d}$ in detail. Assume $F_{\mathbf{a}}{}^{\mathbf{C}} = \mathbb{Y}^{\mathbf{C}}_{\mathbf{c}}\sigma_{\mathbf{a}}{}^{\mathbf{c}} + \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}\rho_{\mathbf{a}}{}^{\dot{\mathbf{c}}}$. Then

$$
\begin{aligned}
(d^{\nabla}d^{\nabla}F)_{a^{-1}a^0\mathbf{a}}{}^{\mathbf{C}} &= \frac{1}{2}\Omega_{a^{-1}a^0}\sharp F_{\mathbf{a}}{}^{\mathbf{C}} = \frac{1}{2}\ell\Omega_{a^{-1}a^0}{}^{C^1}{}_{P}F_{\mathbf{a}}{}^{P\dot{\mathbf{C}}} \\
&= \frac{1}{2}\ell\mathbb{Y}^{\mathbf{C}}_{\mathbf{c}}W_{a^{-1}a^0}{}^{c}{}_{p}\sigma_{\mathbf{a}}{}^{p\dot{\mathbf{c}}} + \mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}\bar{\rho}_{a^{-1}a^0\mathbf{a}}{}^{\dot{\mathbf{c}}}
\end{aligned}
$$

for some section $\bar{\rho}$ which we shall not need explicitly. Therefore

$$
\begin{aligned}
(\partial^* d^{\nabla}d^{\nabla}F)_{a^0\mathbf{a}}{}^{\mathbf{C}} &= \frac{\ell^2}{2}X^{C^1}X^{r}_{Q}\Omega_{[ra^0}{}^{[Q}{}_{|P|}F_{\mathbf{a}]}{}^{|P|\dot{\mathbf{C}}]} = \frac{\ell^2}{2}\mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}W_{[ra^0}{}^{[r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{|p|\dot{\mathbf{c}}]} \\
&= \frac{\ell}{2(k+2)}\mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}\big[-(\ell-1)W_{pr}{}^{c^2}{}_{a^0}\sigma_{\mathbf{a}}{}^{pr\ddot{\mathbf{c}}} + kW_{a^0a^1}{}^{r}{}_{p}\sigma_{r\dot{\mathbf{a}}}{}^{p\dot{\mathbf{c}}}\big].
\end{aligned}
$$

It remains to apply $(\Box_{k+1})^{-1}$. Note that the map $\partial^* d^{\nabla} d^{\nabla} : \mathcal{E}_{\mathbf{a}}{}^{\mathbf{C}} \to \mathcal{E}_{a^0\mathbf{a}}{}^{\mathbf{C}}$ has values in the (completely reducible) subbundle $\mathcal{E}_{a^0\mathbf{a}}{}^{\dot{\mathbf{c}}}(-\ell) \subseteq \mathcal{E}_{a^0\mathbf{a}}{}^{\mathbf{C}}$, cf. the previous display. The irreducible components of this subbundle are the bundles $\mathrm{tf}[\mathcal{E}_{\mathbf{b}^{k+2-i}}{}^{\mathbf{d}^{\ell-i}}](-\ell)$, $1 \le i \le \min\{\ell, k+2\}$, where the notation tf[..] denotes the trace-free part of the enclosed bundle. The Laplace-Kostant operator $\Box_{k+1}$ on $\mathrm{tf}[\mathcal{E}_{\mathbf{b}^s}{}^{\mathbf{d}^t}](-\ell)$ acts by $A^t_s(\ell) := \frac{1}{s+1}[n-s-t+1+(l-t)(n-s)]$. Note that the computation is rather simple if we consider $\mathrm{tf}[\mathcal{E}_{\mathbf{b}^s}{}^{\mathbf{d}^t}](-\ell)$ as the irreducible invariant subbundle of $\mathcal{E}^{\mathbf{D}^t(E_1\dots E_{l-t})}$ and then follow 3.1. Also note $A^t_s(\ell)$ is always nonzero. This of course follows by general means but can be verified directly since $\mathrm{tf}[\mathcal{E}_{\mathbf{b}^s}{}^{\mathbf{d}^t}] \neq \{0\}$ if and only if $s + t \le n$.

Proposition 3.3 *The operator $\tilde{d} : \mathcal{E}_{\mathbf{a}}{}^{\mathbf{C}} \to \mathcal{E}_{a^0\mathbf{a}}{}^{\mathbf{C}}$ for the projective geometry has the form*

$$
(\tilde{d}F)_{a^0\mathbf{a}}{}^{\mathbf{C}} = (d^{\nabla}F)_{a^0\mathbf{a}}{}^{\mathbf{C}} - \frac{\ell^2}{2}\sum_{i=1}^{\min\{\ell,k+2\}}\frac{1}{A^{\ell-i}_{k+2-i}(\ell)}\,\mathrm{Proj}^{\ell-i}_{k+2-i}\,\mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}W_{[ra^0}{}^{[r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{|p|\dot{\mathbf{c}}]}
$$

where $\sigma_{\mathbf{a}}{}^{\mathbf{c}} = \mathbb{X}^{\mathbf{c}}_{\mathbf{C}}F_{\mathbf{a}}{}^{\mathbf{C}}$, $\mathbb{X}^{\mathbf{c}}_{\mathbf{C}} = X^{c^1}_{C^1}\dots X^{c^\ell}_{C^\ell}$ and $\mathrm{Proj}^t_s : \mathcal{E}_{\mathbf{a}^{s+i}}{}^{\mathbf{c}^{t+i}}(\ell) \to tf[\mathcal{E}_{\mathbf{a}^s}{}^{\mathbf{c}^t}](\ell)$, $i \ge 0$ is the projection. □

The operator $\tilde{d}$ simplifies for the special cases $\ell = 1$ and $k = 0$. First assume $\ell = 1$. Then $(\partial^* d^{\nabla} d^{\nabla})_{a^0\mathbf{a}}{}^{C} = \frac{k}{2(k+2)}X^C W_{a^0a^1}{}^{r}{}_{p}\sigma_{r\dot{\mathbf{a}}}{}^{p}$ has values in the irreducible subbundle $\mathcal{E}_{a^0\mathbf{a}}(-\ell)$ of $\mathcal{E}_{a^0\mathbf{a}}{}^{C}$. We computed $\Box_{k+1}$ acts by $\frac{n-(k+1)}{k+2}$ on this subbundle. Inverting this scalar, we obtain the result

$$
(\tilde{d}F)_{a^0\mathbf{a}}{}^{C} = (d^{\nabla}F)_{a^0\mathbf{a}}{}^{C} + \frac{k}{2(n-k-1)}X^C W_{a^0a^1}{}^{r}{}_{p}\sigma_{r\dot{\mathbf{a}}}{}^{p}.
$$

Now assume $k = 0$. Then $(\partial^* d^{\nabla} d^{\nabla} F)_{a}{}^{\mathbf{C}} = -\frac{\ell(\ell-1)}{4}\mathbb{X}^{\mathbf{C}}_{\dot{\mathbf{c}}}W_{pr}{}^{c^2}{}_{a}\sigma^{pr\ddot{\mathbf{c}}}$ has values in the trace-free (thus irreducible) part of the subbundle $\mathcal{E}_{a}{}^{\dot{\mathbf{c}}}(-\ell)$. Since $\Box_{k+1}$ acts on the trace-free part of $\mathcal{E}_{a}{}^{\dot{\mathbf{c}}}(-\ell) \subseteq \mathcal{E}_{a}{}^{\mathbf{C}}$ by $\frac{n-\ell}{2}$, the resulting formula is

$$
(\tilde{d}F)_{a}{}^{\mathbf{C}} = (d^{\nabla}F)_{a}{}^{\mathbf{C}} + \frac{\ell(\ell-1)}{2(n-\ell)}\mathbb{X}^{\mathbf{C}}W_{pr}{}^{c^2}{}_{a}\sigma^{pr\ddot{\mathbf{c}}}.
$$

We claim that $\tilde{d}$ actually coincides with the prolongation covariant derivative $\tilde{\nabla}$. To verify this, first observe $((\tilde{\nabla} - \nabla)F)_a{}^{\mathbf{C}} \in \operatorname{Im} \partial^*$ by the construction of $\tilde{d} = \tilde{\nabla}$. Thus, it remains to verify $(d^{\tilde{\nabla}}\tilde{\nabla}F)_{a^{-1}a^0}{}^{\mathbf{C}} \in \operatorname{Ker} \partial^*$. But since $(d^{\nabla}\tilde{\nabla}F)_{a^{-1}a^0}{}^{\mathbf{C}} \in \operatorname{Ker} \partial^*$ (again by the construction of $\tilde{d} = \tilde{\nabla}$) and $d^{\tilde{\nabla}} - d^{\nabla} : \mathcal{E}_{a^0} \to \ker \partial^* \subseteq \mathcal{E}_{a^{-1}a^0}{}^{\mathbf{C}}$, cf. the last term in the previous display, the claim follows. Using the matrix notation, $\tilde{\nabla} = \tilde{d}$ has the form

$$\tilde{\nabla}_a \begin{pmatrix} \sigma^{\mathbf{c}} \\ \rho^{\dot{\mathbf{c}}} \end{pmatrix} = \nabla_a \begin{pmatrix} \sigma^{\mathbf{c}} \\ \rho^{\dot{\mathbf{c}}} \end{pmatrix} + \frac{\ell(\ell-1)}{2(n-\ell)} \begin{pmatrix} 0 \\ W_{pr}{}^{c^2}{}_a \sigma^{pr\ddot{\mathbf{c}}} \end{pmatrix}.$$

Finally, note that, using the tractor volume form, $\mathcal{E}^{\mathbf{C}} \cong \mathcal{E}_{\mathbf{D}}$ for $\mathbf{C} = \mathbf{C}^{\ell}$ and $\mathbf{D} = \mathbf{D}^{n-\ell+1}$. The case $\ell = n-1$ (i.e., $\mathbf{D} = \mathbf{D}^2$) was solved in [9], where the prolongation of the corresponding BGG-operator $\mathcal{E}_a(2) \to \mathcal{E}_{(ab)}$ (explicitly $f_a \mapsto \nabla_{(a} f_{b)}$) is constructed. They construct the prolongation as the tractor covariant derivative $D_a : \mathcal{E}_{\mathbf{D}^2} \to \mathcal{E}_{a\mathbf{D}^2}$, cf. [17]. Since $D_a - \nabla_a : \mathcal{E}_{\mathbf{D}^2} \to \operatorname{im} \partial^*$ (this follows from the formula for D_a on p. 9, [9] after a short computation) and the curvature of $(D_a D_b - D_b D_a) : \mathcal{E}_{\mathbf{D}^2} \to \operatorname{Ker} \partial^*$ (this is obvious form the formula for $D_a D_b - D_b D_a$ on $\mathcal{E}_{\mathbf{D}^2}$ on the same page) we conclude $D_a = \tilde{\nabla}_a$, cf. 1.1.

Example 3.4 Here we discuss the bundle $\mathcal{E}^{(AB)} = \mathcal{E}^{(ab)}(-2) \oplus \mathcal{E}^a(-2) \oplus \mathcal{E}(-2)$. Consider a section $F_{\mathbf{a}}{}^{BC} \in \mathcal{E}_{\mathbf{a}}{}^{(BC)}$, expanded in the basis of injectors as $F_{\mathbf{a}}{}^{BC} = Y^{(B}_b Y^{C)}_c \sigma_{\mathbf{a}}{}^{bc} + X^{(B} Y^{C)}_c \rho_{\mathbf{a}}{}^c + X^B X^C \nu_{\mathbf{a}}$. Then

$$\begin{aligned}(d^{\nabla} d^{\nabla} F)_{a^{-1}a^0\mathbf{a}}{}^{BC} &= \frac{1}{2} \Omega_{a^{-1}a^0} \sharp F_{\mathbf{a}}{}^{BC} = \Omega_{a^{-1}a^0}{}^{(B}{}_P F_{\mathbf{a}}{}^{C)P} \\ &= Y^{(B}_b Y^{C)}_c W_{a^{-1}a^0}{}^{(b}{}_p \sigma_{\mathbf{a}}{}^{c)p} + X^{(B} Y^{C)}_c \left[\frac{1}{2} W_{a^{-1}a^0}{}^c{}_p \rho_{\mathbf{a}}{}^p - A_{a^{-1}a^0 p} \sigma_{\mathbf{a}}{}^{cp} \right] + X^B X^C \bar{\nu}_{\mathbf{a}}\end{aligned}$$

for some section $\bar{\nu}$. Applying ∂^* we obtain

$$\begin{aligned}(\partial^* d^{\nabla} d^{\nabla} F)_{a^0\mathbf{a}}{}^{BC} &= 2X^{(B} Y^{C)}_c W_{[ra^0}{}^{(r}{}_{|p|} \sigma_{\mathbf{a}]}{}^{c)p} \\ &\quad + X^B X^C \left[\frac{1}{2} W_{[ra^0}{}^r{}_{|p|} \rho_{\mathbf{a}]}{}^p - A_{[ra^0|p|} \sigma_{\mathbf{a}]}{}^{pr} \right].\end{aligned}$$

The filtration degree of $\mathcal{E}^{(AB)}$ is 3 and therefore the construction of $\tilde{d}$ will require (at most) 2 steps. In the first step we put $d' := d^{\nabla} + (\Box^{XY}_{k+1})^{-1} \partial^* d^{\nabla} d^{\nabla} : \mathcal{E}_{\mathbf{a}}{}^{BC} \to \mathcal{E}_{a^0\mathbf{a}}{}^{BC,}$ where $\Box^{XY}_{k+1}$ denotes $\Box_{k+1}$ restricted to the subquotient $\mathcal{E}_{\mathbf{a}}{}^c(-2)$ of $\mathcal{E}_{\mathbf{a}}{}^{(BC)}$, which corresponds to the injector $X^{(B} Y^{C)}_c : \mathcal{E}_{\mathbf{a}}{}^c(-2) \hookrightarrow \mathcal{E}_{\mathbf{a}}{}^{(BC)}$. Note that this subquotient has two irreducible components, but we only need the trace-free part since $W_{[ra^0}{}^{(r}{}_{|p|} \sigma_{\mathbf{a}]}{}^{c)p}$ is trace-free. A short computation reveals $\partial^* \partial = \Box_1$ acts on the corresponding subquotient of $\mathcal{E}_a{}^{(BC)}$ by $\frac{n-k}{k+2}$. Hence

$$\begin{aligned}(d' F)_{a^0\mathbf{a}}{}^{BC} =& \nabla_{a^0} F_{\mathbf{a}}{}^{BC} - \frac{k+2}{n-k} \Big[2X^{(B} Y^{C)}_c W_{[ra^0}{}^{(r}{}_{|p|} \sigma_{\mathbf{a}]}{}^{c)p} \\ &+ X^B X^C \left(\frac{1}{2} W_{[ra^0}{}^r{}_{|p|} \rho_{\mathbf{a}]}{}^p - A_{[ra^0|p|} \sigma_{\mathbf{a}]}{}^{pr} \right) \Big]. \end{aligned} \tag{3}$$

Further computation reveals

$$(d^{\nabla}d'F)_{a^{-1}a^0\mathbf{a}}{}^{BC} =(d^{\nabla}d^{\nabla}F)_{a^{-1}a^0\mathbf{a}}{}^{BC} - \frac{k+2}{n-k}\Bigg[2Y^{(B}_{a^{-1}}Y^{C)}_{c}W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{c)p}$$

$$+ 2X^{(B}Y^{C)}_{c}\left(+\frac{1}{2}\delta^{c}_{a^{-1}}W_{[ra^0}{}^{r}{}_{|p|}\rho_{\mathbf{a}]}{}^{p} - \delta^{c}_{a^{-1}}A_{[ra^0|p|}\sigma_{\mathbf{a}]}{}^{pr}$$

$$+ \nabla_{a^{-1}}W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{c)p}\right)\Bigg] + X^BX^C\gamma_{a^{-1}a^0\mathbf{a}}$$

for some section $\gamma_{a^{-1}a^0\mathbf{a}} \in \mathcal{E}_{a^{-1}a^0\mathbf{a}}(-2)$ and

$$(\partial^* d^{\nabla}d'F)_{a^0\mathbf{a}}{}^{BC} = -\frac{1}{n-k}X^BX^C\Bigg[2\nabla_s W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{s)p}$$

$$+ (n-k-2)\left(\frac{1}{2}W_{[ra^0}{}^{r}{}_{|p|}\rho_{\mathbf{a}]}{}^{p} - A_{[ra^0|p|}\sigma_{\mathbf{a}]}{}^{pr}\right)\Bigg].$$

The previous displays shows that $(\partial^* d^{\nabla}d'F)_{a^0\mathbf{a}}{}^{BC}$ is the section of the subbundle $\mathcal{E}_{a^0\mathbf{a}}(-2) \subseteq \mathcal{E}_{a^0\mathbf{a}}{}^{BC}$. Since $\square_{k+1}$ acts on this subbundle by $\frac{2(n-k-1)}{k+2}$, we obtain the result $\tilde{d} := d' - \frac{k+2}{2(n-k-1)}\partial^* d^{\nabla}d'$.

Proposition 3.5 *The operator $\tilde{d} : \mathcal{E}_{\mathbf{a}}{}^{(BC)} \to \mathcal{E}_{a^0\mathbf{a}}{}^{(BC)}$ for the projective geometry has the form a2*

$$(\tilde{d}F)_{a^0\mathbf{a}}{}^{BC} = \nabla_{a^0}F_{\mathbf{a}}{}^{BC} - \frac{k+2}{n-k}\Bigg[2X^{(B}Y^{C)}_{c}W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{c)p}$$

$$- \frac{1}{2(n-k-1)}X^BX^C\Bigg[2\nabla_s W_{[ra^0}{}^{(r}{}_{|p|}\sigma_{\mathbf{a}]}{}^{s)p}$$

$$-(n-k)\left(\frac{1}{2}W_{[ra^0}{}^{r}{}_{|p|}\rho_{\mathbf{a}]}{}^{p} - A_{[ra^0|p|}\sigma_{\mathbf{a}]}{}^{pr}\right)\Bigg]\Bigg],$$

where $\sigma_{\mathbf{a}}{}^{bc} = X^b_BX^c_CF_{\mathbf{a}}{}^{BC}$ and $\rho_{\mathbf{a}}{}^{b} = 2X^b_BY_CF_{\mathbf{a}}{}^{BC}$. □

We shall discuss the case $k = 0$ in more details. Then the formula in Proposition 3.5 simplifies to

$$(\tilde{d}F)_a{}^{BC} =\nabla_aF^{BC} - \frac{2}{n}X^{(B}Y^{C)}_{c}W_{ra}{}^{c}{}_{p}\sigma^{rp}$$

$$+ \frac{1}{n}X^BX^C\left(2A_{rap}\sigma^{pr} + \frac{1}{n-1}W_{ra}{}^{s}{}_{p}\nabla_s\sigma^{rp}\right).$$

This means that $\tilde{d}$ is not a covariant derivative on $\mathcal{E}^{(BC)}$, as the term $W_{ra}{}^{s}{}_{p}\nabla_s\sigma^{rp}$ is not algebraic in F^{BC}, i.e., $\tilde{d} \neq \tilde{\nabla}$ in this case. To compute $\tilde{\nabla}$ explicitly assume $k = 0$ and put $\nabla' := d'$ (this is a covariant derivative on $\mathcal{E}^{(BC)}$). That is, $\nabla'_aF^{BC} = \nabla_aF^{BC} - \frac{2}{n}(\Psi F)_a{}^{BC}$, where the homomorphism $\Psi_a : \mathcal{E}^{(BC)} \to \mathcal{E}_a{}^{(BC)}$ is given by the formula (3), i.e., $(\Psi F)_a{}^{BC} = X^{(B}Y^{C)}_{c}W_{ra}{}^{c}{}_{p}\sigma^{rp} - X^BX^CA_{rap}\sigma^{pr}$. Extending Ψ_{a^0} to an endomorphism $\mathcal{E}_{a^1}{}^{(BC)} \to \mathcal{E}_{a^0a^1}{}^{(BC)}$, an easy computation shows

$$(\Psi\nabla'F)_{a^0a^1}{}^{BC} = X^{(B}Y^{C)}_{c}\left[W_{ra^0}{}^{c}{}_{p}\nabla_{a^1}\sigma^{rp} - \frac{3}{2}W_{a^0a^1}{}^{c}{}_{p}\rho^{p}\right] + X^BX^C\bar{\nu}$$

for some $\bar{\nu} \in \mathcal{E}(-2)$. Therefore $(\partial^* \Psi \nabla' F)_a^{BC} = -\frac{1}{2} X^B X^C W_{ra}{}^c{}_p \nabla_c \sigma^{rp}$, and we finally obtain $(\partial^* d^{\nabla'} \nabla' F)_a{}^{BC} = (\partial^* d^{\nabla} \nabla' F)_a{}^{BC} - \frac{2}{n}(\partial^* \Psi \nabla' F)_a^{BC} = 0$. Since the left hand side is the curvature of ∇' (applied to F^{BC}), this curvature is a map $\mathcal{E}^{(BC)} \to \operatorname{Ker} \partial^*$. Thus we verified $\widetilde{\nabla} = \nabla'$, cf. Theorem 1.1. Rewriting $\widetilde{\nabla}$ in matrix notation, we obtain

$$\widetilde{\nabla}_a \begin{pmatrix} \sigma^{bc} \\ \rho^c \\ \nu \end{pmatrix} = \nabla_a \begin{pmatrix} \sigma^{bc} \\ \rho^c \\ \nu \end{pmatrix} - \frac{2}{n} \begin{pmatrix} 0 \\ W_{ra}{}^c{}_p \sigma^{pr} \\ -A_{rap} \sigma^{pr} \end{pmatrix}.$$

Note that $\widetilde{\nabla}_a$ provides the prolongation of the corresponding (first order) BGG operator from $\mathcal{E}^{(bc)}(-2)$ to the totally trace-free part of $\mathcal{E}_a{}^{(bc)}(-2)$. The same problem was solved in [10] in terms of the connection defined by (3.6) or the left hand side of (5.2) there. Let us denote this connection on $\mathcal{E}^{(BC)}$ by D_a. Note that the formula for D_a differs from $\widetilde{\nabla}_a$ in the middle term of the last matrix in the previous display: this term is $-\frac{2}{n} W_{ra}{}^c{}_p \sigma^{pr}$ for $\widetilde{\nabla}_a$ whereas $\frac{1}{n} W_{ra}{}^c{}_p \sigma^{pr}$ in the case of D_a, cf. [10, (3.6)]. The reason is purely notational, specifically in the choice of the projectors. If one replaces $X^{(B} Y_c^{C)}$ by $-\frac{1}{2} X^{(B} Y_c^{C)}$—which means, e.g., $F_{\mathbf{a}}{}^{BC} = Y_b^{(B} Y_c^{C)} \sigma^{bc} + (-\frac{1}{2} X^{(B} Y_c^{C)}) \rho^c + X^B X^C \nu$—both terms will coincide. Note also that formulas for ∇_a and the normal covariant derivative defined in the display preceding to [10, Theorem 5.1] coincide after the change of projectors. This confirms that the results here coincide with those in [10].

4 Conformal geometry

4.1 Conformal geometry and tractor calculus

We summarize some notation and background. Further details may be found in [12]. Let M be a smooth manifold of dimension $n \geq 3$. Recall that a *conformal structure* of signature (p, q) on M is a smooth ray subbundle $\mathcal{Q} \subset S^2 T^* M$ whose fiber over x consists of conformally related signature-(p, q) metrics at the point x. Sections of $\mathcal{Q}$ are metrics g on M. So we may equivalently view the conformal structure as the equivalence class $[g]$ of conformally related metrics. The principal bundle $\pi : \mathcal{Q} \to M$ has structure group $\mathbb{R}_+$, and so each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \operatorname{End}(\mathbb{R})$ induces a natural line bundle on $(M, [g])$ that we term the conformal density bundle $E[w]$. We shall write $\mathcal{E}[w]$ for the space of sections of this bundle. We write $\mathcal{E}^a$ for the space of sections of the tangent bundle TM and $\mathcal{E}_a$ for the space of sections of T^*M. The indices here are abstract in the sense of [19] and we follow the usual conventions from that source. So, for example, $\mathcal{E}_{ab}$ is the space of sections of $\otimes^2 T^* M$. Here and throughout, sections, tensors, and functions are always smooth. When no confusion is likely to arise, we will use the same notation for a bundle and its section space.

We write $\boldsymbol{g}$ for the *conformal metric*, that is the tautological section of $S^2 T^* M \otimes E[2]$ determined by the conformal structure. This is used to identify TM with $T^*M[2]$. For many calculations we employ abstract indices in an obvious way. Given a choice of metric g from $[g]$, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab} \nabla_a \nabla_b = \nabla^b \nabla_b$. Here we are raising indices and contracting using the (inverse) conformal metric. Indices will be raised and lowered in this way without further comment. Note that $E[w]$ is trivialized by a choice of metric g from the conformal class, and we also write ∇ for the connection corresponding to this trivialization. The coupled covariant derivative ∇_a preserves the conformal metric.

The curvature $R_{ab}{}^{c}{}_{d}$ of the Levi-Civita connection (the Riemannian curvature) is given by $[\nabla_a, \nabla_b]v^c = R_{ab}{}^{c}{}_{d}v^d$ ($[\cdot,\cdot]$ indicates the commutator bracket). It can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab}, according to

$$R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}\mathsf{P}_{b]d} + 2\boldsymbol{g}_{d[b}\mathsf{P}_{a]c}, \tag{4}$$

where $[\cdots]$ indicates anti-symmetrization over the enclosed indices. The Schouten tensor is a trace modification of the Ricci tensor $\mathrm{Ric}_{ab} = R_{ca}{}^{c}{}_{b}$, and vice versa: $\mathrm{Ric}_{ab} = (n-2)\mathsf{P}_{ab} + \mathsf{J}\boldsymbol{g}_{ab}$, where we write J for the trace $\mathsf{P}_a{}^a$ of P. The *Cotton tensor* is defined by $A_{abc} := 2\nabla_{[a}\mathsf{P}_{b]c}$. Via the Bianchi identity this is related to the divergence of the Weyl tensor as follows:

$$(n-3)A_{abc} = \nabla^d C_{dcab}. \tag{5}$$

Finally we put

$$B_{ab} = \nabla^p A_{pab} + \mathsf{P}^{pq}C_{paqb} \in \mathcal{E}_{(ab)_0}[-2]. \tag{6}$$

In dimension $n = 4$ this is the conformally invariant *Bach* tensor.

Under a *conformal transformation* we replace a choice of metric g by the metric $\hat{g} = e^{2\Upsilon}g$, where Υ is a smooth function. We recall that, in particular, the Weyl curvature is conformally invariant $\widehat{C}_{abcd} = C_{abcd}$. With $\Upsilon_a := \nabla_a\Upsilon$, the Schouten tensor transforms according to

$$\widehat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \tfrac{1}{2}\Upsilon^c\Upsilon_c\boldsymbol{g}_{ab}. \tag{7}$$

Explicit formulae for the corresponding transformation of the Levi-Civita connection and its curvatures are given in, e.g., [1,12]. From these one can easily compute the transformation for a general valence (i.e., rank) s section $f_{bc\cdots d} \in \mathcal{E}_{bc\cdots d}[w]$ using the Leibniz rule:

$$\begin{aligned}\hat{\nabla}_{\bar{a}} f_{bc\cdots d} =& \nabla_{\bar{a}} f_{bc\cdots d} + (w-s)\Upsilon_{\bar{a}} f_{bc\cdots d} - \Upsilon_b f_{\bar{a}c\cdots d} \cdots - \Upsilon_d f_{bc\cdots\bar{a}} \\ &+ \Upsilon^p f_{pc\cdots d}\boldsymbol{g}_{b\bar{a}} \cdots + \Upsilon^p f_{bc\cdots p}\boldsymbol{g}_{d\bar{a}}.\end{aligned} \tag{8}$$

Next we define the standard tractor bundle over $(M,[g])$. It is a vector bundle of rank $n+2$ that is defined, for each $g \in [g]$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\widehat{g} = e^{2\Upsilon}g$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\widehat{\alpha}, \widehat{\mu}_a, \widehat{\tau}) \in [\mathcal{E}^A]_{\widehat{g}}$ by the transformation

$$\begin{pmatrix}\widehat{\alpha}\\ \widehat{\mu}_a\\ \widehat{\tau}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ \Upsilon_a & \delta_a{}^b & 0\\ -\frac{1}{2}\Upsilon_c\Upsilon^c & -\Upsilon^b & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \mu_b\\ \tau\end{pmatrix}. \tag{9}$$

It is straightforward to verify that these identifications are consistent with a change to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{E}^A$ over the conformal manifold. On a conformal structure of signature (p,q) the bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(p+1, q+1)$ and an invariant connection, which we shall also denote by ∇_a, that preserves h_{AB}. Up to an isomorphism this is the unique *normal conformal tractor connection* and therefore induces the normal connection on $\bigotimes\mathcal{E}^A$ that will be denoted ∇_a and termed the (normal) tractor connection. In a conformal scale g the metric h_{AB} and ∇_a on $\mathcal{E}^A$ are given by

$$h_{AB} = \begin{pmatrix}0 & 0 & 1\\ 0 & \boldsymbol{g}_{ab} & 0\\ 1 & 0 & 0\end{pmatrix} \text{ and } \nabla_a\begin{pmatrix}\alpha\\ \mu_b\\ \tau\end{pmatrix} = \begin{pmatrix}\nabla_a\alpha - \mu_a\\ \nabla_a\mu_b + \boldsymbol{g}_{ab}\tau + \mathsf{P}_{ab}\alpha\\ \nabla_a\tau - \mathsf{P}_{ab}\mu^b\end{pmatrix}. \tag{10}$$

It is readily verified that both of these are conformally well-defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A, \mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we see that $Y_A X^A = 1$, $\;Z_{Ab}Z^A{}_c = \boldsymbol{g}_{bc}$, and all other quadratic combinations that contract the tractor index vanish. In (9) note that $\widehat{\alpha} = \alpha$, hence X^A is conformally invariant. Reformulating (10), we obtain

$$\nabla_a Y_B = Z^b_B P_{ab}, \quad \nabla_a Z^b_B = -Y_B \delta^b_a - X_B P_a{}^b \quad \text{and} \quad \nabla_a X_B = Z^b_B \boldsymbol{g}_{ab}.$$

Given a choice of $g \in [g]$, the *tractor-D operator*

$$D_A : \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1]$$

is defined by

$$D_A V := (n+2w-2)wY_A V + (n+2w-2)Z_{Aa}\nabla^a V - X_A(\Delta V + w\mathsf{J}V). \tag{11}$$

This is conformally invariant, as can be checked directly using the formula above.

The curvature Ω of the tractor connection is defined on $\mathcal{E}^C$ by $[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E$. Using (10) and the formulae for the Riemannian curvature yields

$$\Omega_{abEF} = Z^e_E Z^f_F C_{abef} - 2X_{[E}Z^f_{F]}A_{abf} \in \mathcal{E}_{[ab][EF]} = \mathcal{E}_{[ab]} \otimes \mathcal{A}, \tag{12}$$

where $\mathcal{A} = \mathcal{E}_{[EF]}$ is the conformal adjoint tractor bundle. We shall write $\Omega_{ab}\sharp F_C$ or $(\Omega\sharp F)_{abC}$ for the curvature action $(\nabla_a\nabla_b - \nabla_b\nabla_a)F_C = -\Omega_{ab}{}^D{}_C F_D$.

Using the notation developed above, the inclusions ι and $\bar{\iota}$ defined in 2.2 have the form $-2Y_{[E}Z_{F]a^0} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\bar{\iota}} \mathcal{E}_{a^0\mathbf{a}[EF]}$ and $-2X_{[E}Z^{a^1}_{F]} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\iota} \mathcal{E}_{\dot{\mathbf{a}}[EF]}$. (The scalar -2 is used for the sake of compatibility of ∂ and ∇, cf. [6].) Thus

$$\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto -2Y_{[E}Z_{F]a^0} f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{T} \quad \text{and}$$
$$\partial^* : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto -2X_{[E}Z^{a^1}_{F]} f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{T}$$

and we can easily compute $\Box_k$ on $\mathcal{E}_{\mathbf{a}} \otimes \mathcal{T}$ using the tensorial action $\sharp$.

Example 4.1 We shall compute $\tilde{d}$ on forms twisted by $\mathcal{E}_C$. Let $\mathbf{a} = \mathbf{a}^k$ and consider $F_{\mathbf{a}C} = Y_C\sigma_{\mathbf{a}} + Z^c_C\mu_{c\mathbf{a}} + X_C\nu_{\mathbf{a}} \in \mathcal{E}_{\mathbf{a}C}$. Then

$$\begin{aligned}(d^\nabla d^\nabla F)_{a^{-1}a^0\mathbf{a}C} &= \frac{1}{2}\Omega_{a^{-1}a^0}\sharp F_{\mathbf{a}C} = \frac{1}{2}\Omega_{a^{-1}a^0C}{}^P F_{\mathbf{a}P} \\ &= \frac{1}{2}Z^c_C\big[C_{a^{-1}a^0c}{}^p\mu_{\mathbf{a}p} + A_{a^{-1}a^0c}\sigma_{\mathbf{a}}\big] - X_C A_{a^{-1}a^0}{}^p\mu_{\mathbf{a}p},\end{aligned}$$

hence $(\partial^* d^\nabla d^\nabla F)_{a^0\mathbf{a}C} = -\frac{k}{2(k+2)}X_C\big[C_{a^0a^1}{}^{rp}\mu_{r\dot{\mathbf{a}}p} + A_{a^0a^1}{}^r\sigma_{r\dot{\mathbf{a}}}\big]$. This is a section of the subbundle $\mathcal{E}_{a^0\mathbf{a}}[-1] \subseteq \mathcal{E}_{a^0\mathbf{a}C}$ and one easily computes that $\Box_k$ acts on this (irreducible) subbundle by $-\frac{n-k-1}{k+2}$. Therefore $(\tilde{d}F)_{a^0\mathbf{a}C} = \nabla_{a^0}F_{\mathbf{a}C} - \frac{k}{2(n-k-1)}X_C\big[C_{a^0a^1}{}^{rp}\mu_{r\dot{\mathbf{a}}p} + A_{a^0a^1}{}^r\sigma_{r\dot{\mathbf{a}}}\big]$ for $0 \le k \le n-1$, and $\tilde{d} = d^\nabla$ for $k \ge n-1$. Finally, note that the prolongation covariant derivative coincides with the normal one for $k = 0$, i.e., $\tilde{\nabla} = \nabla$ on $\mathcal{E}_C$.

Example 4.2 The computation of the prolongation covariant derivative is getting rather technical for more complicated bundles. We shall demonstrate it for the prolongation covariant derivative $\widetilde{\nabla}$ on $\mathcal{E}_{(BC)_0}$. (Note that $\mathcal{E}_{(BC)_0}$ and $\mathcal{E}^{(BC)_0}$ are isomorphic using the tractor metric.) The computation consists of three steps: we start with ∇ and then define the covariant derivatives $\overline{\nabla}$, $\overline{\overline{\nabla}}$ and $\widetilde{\nabla}$. Taking a section $F_{BC} = Y_{(B}Y_{C)}\sigma + Y_{(B}Z^c_{C)}\rho_c + Z^b_{(B}Z^c_{C)}\omega_{bc} + X_{(B}Y_{C)}\nu + X_{(B}Z^c_{C)}\mu_c + X_{(B}X_{C)}\kappa$ we get

$$
\begin{aligned}
&(d^\nabla d^\nabla F)_{a^0a^1BC}\\
&= \frac{1}{2}\Omega_{a^0a^1}\sharp F_{BC} = \frac{1}{2}\Omega'_{a^0a^1BC}{}^{PQ}F_{PQ}\\
&= Y_{(B}Z^c_{C)}\left[\frac{1}{2}C_{a^0a^1c}{}^p\rho_p + A_{a^0a^1c}\sigma\right] + Z^b_{(B}Z^c_{C)}\left[C_{a^0a^1(b}{}^p\omega_{c)p} + \frac{1}{2}A_{a^0a^1(b}\rho_{c)}\right]\\
&\quad -\frac{1}{2}X_{(B}Y_{C)}A_{a^0a^1}{}^p\rho_p + X_{(B}Z^c_{C)}\left[\frac{1}{2}C_{a^0a^1c}{}^p\mu_p - A_{a^0a^1}{}^p\omega_{cp} + \frac{1}{2}A_{a^0a^1c}\nu\right]\\
&\quad -\frac{1}{2}X_BX_CA_{a^0a^1}{}^p\mu_p,
\end{aligned}
$$

where $\Omega'_{a^0a^1BC}{}^{PQ} := 2\Omega_{a^0a^1(B}{}^{(P}h_{C)}{}^{Q)}$. Applying ∂^* to the previous display we obtain $(\partial^* d^\nabla d^\nabla F)_{a^1BC} = -2\mathbb{X}_{(B}{}^{Pr}\Omega_{|ra^1|C)}{}^QF_{PQ}$, because $\Omega_{a^0a^1EF}$ is ∂^*-closed (i.e. $\mathbb{X}_{A^0}{}^{Pp}\Omega_{pa^1PA^1} = 0$). We put $\Psi_{a^1BC}{}^{PQ} := -2\mathbb{X}_{(B}{}^{Pr}\Omega_{|ra^1|C)}{}^Q$. Equivalently, $\Psi_{a^1BC}{}^{PQ}$ can be obtained by applying ∂^* to the $\mathcal{E}_{BC}$-factor of $\Omega'_{a^0a^1(BC)}{}^{PQ}$. This is exactly the operator ∂^*_V from [17] since the notation therein means $V = \mathcal{E}_{(BC)_0}$, $V^* = \mathcal{E}^{(PQ)_0}$, and therefore $\Omega'_{a^0a^1BC}{}^{PQ} \in \mathcal{E}_{a^0a^1} \otimes \mathrm{End}(V)$ is the curvature tensor of ∇_a on $V = \mathcal{E}_{(BC)_0}$. We shall denote the operator ∂^*_V by $\partial^*_{BC} : \mathcal{E}_{a^0a^1BC}{}^{PQ} \to \mathcal{E}_{a^1BC}{}^{PQ}$ here. Thus we have $\Psi_{a^1BC}{}^{PQ} = \frac{1}{2}(\partial^*_{BC}\Omega')_{a^1BC}{}^{PQ}$, explicitly

$$
\begin{aligned}
\Psi_{a^1BC}{}^{PQ} =& - Z^b_{(B}Z^c_{C)}\left[X^{(P}Z^{Q)q}C_{a^1(bc)q} + X^PX^QA_{a^1(bc)}\right]\\
&+ X_{(B}Z^c_{C)}\left[Z^{p(P}Z^{Q)q}C_{a^1pcq} + 2X^{(P}Z^{Q)q}A_{a^1(cq)}\right]\\
&+ X_{(B}X_{C)}Z^{p(P}Z^{Q)q}A_{pa^1q}.
\end{aligned} \tag{13}
$$

Since $\frac{1}{2}C_{a^1(bc)}{}^p\rho_p + A_{a^1(bc)}\sigma$ is a section of the Cartan component of the subquotient $\mathcal{E}_{[a^1b]} \otimes \mathcal{E}_c$ of $\mathcal{E}_{a^1(BC)_0}$ and $\Box_1$ acts on this subquotient by $-\frac{3}{2}$, we put $\overline{\nabla}_aF_{BC} = \nabla_aF_{BC} + \frac{2}{3}\Psi_{aBC}{}^{PQ}F_{PQ}$ as the first "approximation" of $\widetilde{\nabla}$. We need to know $\nabla_{a^0}\Psi_{a^1BC}{}^{PQ}$ to compute the curvature $\overline{\Omega}_{a^0a^1BC}{}^{PQ}$ of $\overline{\nabla}$. First, it easily follows from $\Psi_{a^1BC}{}^{PQ} := -2\mathbb{X}_{(B}{}^{Pr}\Omega_{|ra^1|C)}{}^Q$ that

$$
\begin{aligned}
(d^\nabla\Psi)_{a^0a^1BC}{}^{PQ} &= \nabla_{a^0}\Psi_{a^1BC}{}^{PQ} = -2\nabla_{a^0}\mathbb{X}_{(B}{}^{Pr}\Omega_{|ra^1|C)}{}^Q\\
&= -2\mathbb{Z}_{(B}^{e^0Pe^1}\boldsymbol{g}_{|a^0e^0}\Omega_{e^1a^1|C)}{}^Q + 2\mathbb{W}_{(B}{}^P\Omega_{|a^0a^1|C)}{}^Q - \mathbb{X}_{(B}{}^{Pr}\nabla_{|r}\Omega_{a^0a^1|C)}{}^Q,
\end{aligned}
$$

since $\nabla_{a^{-1}}\Omega_{a^0a^1CQ} = 0$. Expanding the expressions in the previous display we obtain, after some computations that use the differential Bianchi identity, in particular the relation [14, (29)],

$$
\begin{aligned}
&(d^{\nabla}\Psi)_{a^0a^1BC}{}^{PQ}\\
&= -\frac{3}{2}Y_{(B}Z^c_{C)}\big[X^{(P}Z^{Q)q}C_{a^0a^1cq} + X^PX^QA_{a^0a^1c}\big] + \frac{3}{2}X_{(B}Y_{C)}X^{(P}Z^{Q)q}A_{a^0a^1q}\\
&\quad + Z^b_{(B}Z^c_{C)}\Big[-2Z^{p(P}Z^{Q)q}\boldsymbol{g}_{a^0[b}C_{p]a^1cq} + \frac{1}{2}X^PX^Q\big(\nabla_bA_{a^0a^1c} + P_b{}^rC_{a^0a^1rc}\big)\\
&\quad + X^{(P}Z^{Q)q}\big(-2\boldsymbol{g}_{a^0[b}A_{q]a^1c} + \frac{1}{2}\nabla_bC_{a^0a^1cq} - \boldsymbol{g}_{b[c}A_{|a^0a^1|q]}\big)\Big]\\
&\quad + X_{(B}Z^c_{C)}\Big[\frac{3}{2}Y^{(P}Z^{Q)q}C_{a^0a^1cq} - X^{(P}Z^{Q)q}\big(\nabla_{(c}A_{|a^0a^1|q)} + P_{(c}{}^sC_{|a^0a^1s|q)}\big)\\
&\quad + Z^{p(P}Z^{Q)q}\big(2\boldsymbol{g}_{a^0[c}A_{p]a^1q} - \frac{1}{2}\nabla_pC_{a^0a^1cq} + \boldsymbol{g}_{p[c}A_{|a^0a^1|q]}\big)\Big]\\
&\quad + X_BX_C\Big[-\frac{3}{2}Y^{(P}Z^{Q)q}A_{a^0a^1q} + \frac{1}{2}Z^{p(P}Z^{Q)q}\big(\nabla_pA_{a^0a^1q} + P_p{}^sC_{a^0a^1sq}\big)\Big].
\end{aligned}
$$

Now we need to apply ∂^*_{BC} to the previous display. This yields

$$
\begin{aligned}
(\partial^*_{BC}\, d^{\nabla}\Psi)_{a^1BC}{}^{PQ} = &\,\mu\frac{3}{2}Z^b_{(B}Z^c_{C)}\big[X^{(P}Z^{Q)q}C_{a^1(bc)q} + X^PX^QA_{a^1(bc)}\big]\\
&+ X_{(B}Z^c_{C)}\Big[\frac{1}{2}(n-1)Z^{p(P}Z^{Q)q}C_{a^1(pq)c} - \frac{1}{2}X^PX^QB_{a^1c}\\
&+ X^{(P}Z^{Q)q}\big((n-4)A_{q(a^1c)} - 3A_{a^1(qc)}\big)\Big]\\
&+ X_BX_C\Big[\frac{1}{2}(n-1)Z^{p(P}Z^{Q)q}A_{a^1(pq)} + \frac{1}{2}X^{(P}Z^{Q)q}B_{a^1q}\Big].
\end{aligned}
$$

We need to compute $\overline{\Psi}_{a_1BC}{}^{PQ} = \frac{1}{2}(\partial^*_{BC}\overline{\Omega})_{a^1BC}{}^{PQ}$ satisfying $\overline{\Psi}_{a^1BC}{}^{PQ}F_{PQ} = (\partial^*d^{\overline{\nabla}}\overline{\nabla}F)_{a^1BC}$. Since $\overline{\nabla}_aF_{BC} = \nabla_a + \frac{2}{3}\Psi_{aBC}{}^{PQ}$ we have

$$
\frac{1}{2}\overline{\Omega}_{a^0a^1BC}{}^{PQ} = \frac{1}{2}\Omega'_{a^0a^1BC}{}^{PQ} + \frac{2}{3}(d^{\nabla}\Psi)_{a^0a^1BC}{}^{PQ} + \frac{4}{9}(\Psi\wedge\Psi)_{a^0a^1BC}{}^{PQ},
$$

where $(\Psi\wedge\Psi)_{a^0a^1BC}{}^{PQ} = \Psi_{a^0BC}{}^{RS}\Psi_{a^1RS}{}^{PQ}$. Since $\frac{1}{2}(\partial^*_{BC}\Omega')_{a^1BC}{}^{PQ} = \Psi_{a^1BC}{}^{PQ}$ by definition of Ψ, applying ∂^*_{BC} to the previous display yields

$$
\begin{aligned}
\overline{\Psi}_{a_1BC}{}^{PQ} &= \frac{1}{2}(\partial^*_{BC}\overline{\Omega})_{a^1BC}{}^{PQ}\\
&= \Psi_{a^1BC}{}^{PQ} + \frac{2}{3}(\partial^*_{BC}d^{\nabla}\Psi)_{a^1BC}{}^{PQ} + \frac{4}{9}(\partial^*(\Psi\wedge\Psi))_{a^1BC}{}^{PQ}\\
&= \frac{1}{3}X_{(B}Z^c_{C)}\big[(n-4)Z^{p(P}Z^{Q)q}C_{a^1(pq)c} + 2(n-4)X^{(P}Z^{Q)q}A_{q(a^1c)} - X^PX^QB_{a^1c}\big]\\
&\quad + \frac{1}{3}X_BX_C\big[(n-4)Z^{p(P}Z^{Q)q}A_{a^1(pq)} + X^{(P}Z^{Q)q}B_{a^1q}\big] + \frac{4}{9}(\partial^*_{BC}(\Psi\wedge\Psi))_{a^1BC}{}^{PQ},
\end{aligned}
\tag{14}
$$

where

$$
(\partial^*_{BC}(\Psi\wedge\Psi))_{a^1BC}{}^{PQ} = \frac{1}{2}X_BX_C\big[X^{(P}Z^{Q)q}C_{a^1}{}^{(rs)p}C_{qrsp} + X^PX^QC_{a^1}{}^{(rs)q}A_{qrs}\big].
\tag{15}
$$

Remark 4.3 The section $(\partial^* d^\nabla \Psi)_{aBC}{}^{PQ}$ is closely related to the conformally invariant curvature quantity

$$
\begin{aligned}
W_{\mathbf{EF}} =& (n-4)\mathbb{Z}_{\mathbf{E}}^{\mathbf{e}}\mathbb{Z}_{\mathbf{F}}^{\mathbf{f}}C_{\mathbf{ab}} - 2(n-4)\mathbb{Z}_{\mathbf{E}}^{\mathbf{e}}\mathbb{X}_{\mathbf{F}}^{f}A_{\mathbf{e}f} \\
& - 2(n-4)\mathbb{X}_{\mathbf{E}}^{e}\mathbb{Z}_{\mathbf{F}}^{\mathbf{f}}A_{\mathbf{f}e} + 4\mathbb{X}_{\mathbf{E}}^{e}\mathbb{X}_{\mathbf{F}}^{f}B_{ef},
\end{aligned}
$$

cf. [11], where all the form indices **E**, **F**, **e**, **f** have the valence 2. In fact, one easily computes $(\partial^* d^\nabla \Psi)_{aBC}{}^{PQ} = -\frac{1}{3}Z_a^R X_{(B}W_{C)}{}^{(P}{}_R{}^{Q)}$. Since $(\partial^* d^\nabla \Psi)_{aBC}{}^{PQ}$ coincides with $\overline{\Psi}_{aBC}{}^{PQ}$ up to the terms involving $C_{a^1}{}^{(rs)p}C_{qrsp}$ and $C_{a^1}{}^{(rs)q}A_{prs}$, cf. (14), conformal invariance of $W_{\mathbf{EF}}$ verifies the invariance of the previous computations.

Looking at the form of $\overline{\Psi}_{a_1 BC}{}^{PQ}F_{PQ}$, we see that we need the action of $\Box_1$ on the subquotient $\mathcal{E}_{(a^1c)_0}$ of $\mathcal{E}_{a_1BC}$ (corresponding to the injector $X_{(B}Z_{C)}^c$). A short computation reveals this is $-\frac{n}{2}$ hence the next "approximation" of $\widetilde{\nabla}$ will be the covariant derivative

$$
\overline{\overline{\nabla}}_a := \overline{\nabla}_a + \frac{2}{n}\overline{\Psi}_{aBC}{}^{PQ} = \nabla_a + \frac{2}{3}\Psi_{aBC}{}^{PQ} + \frac{2}{n}\overline{\Psi}_{aBC}{}^{PQ} : \mathcal{E}_{(PQ)} \to \mathcal{E}_{a(BC)}.
$$

Now we need the curvature $\overline{\overline{\Omega}}_{a^0a^1BC}{}^{PQ}$ of $\overline{\overline{\nabla}}_a$ and then to apply ∂^*_{BC} on $\frac{1}{2}\overline{\overline{\Omega}}_{a^0a^1BC}{}^{PQ}$. It follows from the definition of $\overline{\overline{\nabla}}_a$ that

$$
\frac{1}{2}\overline{\overline{\Omega}}_{a^0a^1BC}{}^{PQ} = \frac{1}{2}\overline{\Omega}_{a^0a^1BC}{}^{PQ} + \frac{2}{n}\nabla_{a^0}\overline{\Psi}_{a^1BC}{}^{PQ} + \frac{4}{3n}\overline{\Psi}_{a^0BC}{}^{RS}\Psi_{a^1RS}{}^{PQ}, \tag{16}
$$

since $\overline{\Psi}_{a^0BC}{}^{RS}\overline{\Psi}_{a^1RS}{}^{PQ} = \Psi_{a^0BC}{}^{RS}\overline{\Psi}_{a^1RS}{}^{PQ} = 0$.

The next step is to compute $\overline{\overline{\Psi}}_{a^1BC}{}^{PQ} := \frac{1}{2}(\partial^*_{BC}\overline{\overline{\Omega}})_{a^1BC}{}^{PQ}$. We apply ∂^*_{BC} to the three terms on the right hand side of (16). Firstly recall $\frac{1}{2}(\partial^*_{BC}\overline{\Omega})_{a^1BC}{}^{PQ} = \overline{\Psi}_{a^1BC}{}^{PQ}$ by definition. Secondly, one gets

$$
\begin{aligned}
(d^\nabla\overline{\Psi})_{a^0a^1BC}{}^{PQ} =& \frac{1}{3}Z_{(B}^bZ_{C)}^c\Big[(n-4)Z^{p(P}Z^{Q)q}\boldsymbol{g}_{ba^0}C_{a^1(pq)c} \\
& + 2(n-4)X^{(P}Z^{Q)q}\boldsymbol{g}_{ba^0}A_{q(a^1c)} - X^PX^Q\boldsymbol{g}_{ba^0}B_{a^1c}\Big] \\
& + \frac{1}{3}X_{(B}Z_{C)}^c\Big[\frac{3}{2}(n-4)Y^{(P}Z^{Q)q}C_{a^0a^1qc} - \frac{3}{2}(n-4)X^{(P}Y^{Q)}A_{a^0a^1c} \\
& + (n-4)Z^{p(P}Z^{Q)q}\big(\nabla_{a^0}C_{a^1(pq)c} + 2\boldsymbol{g}_{a^0(p}A_{q)(a^1c)} + 2\boldsymbol{g}_{a^0c}A_{a^1(pq)}\big) \\
& + 2X^{(P}Z^{Q)q}\big((n-4)\nabla_{a^0}A_{q(a^1c)} - (n-4)P_{a^0}{}^pC_{a^1(pq)c} + 2\boldsymbol{g}_{a^0[c}B_{q]a^1} \\
& + \frac{2}{3}\boldsymbol{g}_{ca^0}C_{a^1}{}^{(rs)p}C_{qrsp}\big) + X^PX^Q\big(-\nabla_{a^0}B_{a^1c} - 2(n-4)P_{a^0}{}^qA_{q(a^1c)} \\
& + \frac{4}{3}\boldsymbol{g}_{ca^0}C_{a^1}{}^{(rs)p}A_{p(rs)}\big)\Big] + X_BX_C\varphi_{a^1}{}^{PQ}
\end{aligned}
$$

for some $\varphi_{a^1}{}^{PQ} \in \mathcal{E}_{a^1}{}^{PQ}$ after some computation. Using the last display, it is not difficult to verify that

$$
(\partial^*_{BC}d^\nabla\overline{\Psi})_{a^1BC}{}^{PQ} = -\frac{n}{2}\overline{\Psi}_{a^1BC}{}^{PQ} - \frac{2}{9}(n-2)(\partial^*_{BC}(\Psi\wedge\Psi))_{a^1BC}{}^{PQ}.
$$

Thirdly, one easily derives $\overline{\Psi}_{a^0BC}{}^{RS}\Psi_{a^1RS}{}^{PQ} = -\frac{n-4}{3}\Psi_{a^0BC}{}^{RS}\Psi_{a^1RS}{}^{PQ}$. Hence we finally obtain

$$\overline{\overline{\Psi}}_{a^1BC}{}^{PQ} = \frac{1}{2}(\partial^*_{BC}\overline{\overline{\Omega}})_{a^1BC}{}^{PQ} = -\frac{8}{9n}(n-3)(\partial^*_{BC}(\Psi\wedge\Psi))_{a^1BC}{}^{PQ}, \tag{17}$$

where $-\frac{8}{9n}(n-3) = -\frac{4}{9n}(n-2) - \frac{4}{9n}(n-4)$.

In the last step we need the action of $\Box_1$ on the subbundle $\mathcal{E}_{a^1}[-2] \subseteq \mathcal{E}_{a^1(BC)_0}$ corresponding to the injector X_BX_C. This is the scalar $-(n-1)$, so by adding $\frac{1}{n-1}\overline{\overline{\Psi}}_{a^1BC}{}^{PQ}$ to $\overline{\overline{\nabla}}_a$ we obtain the resulting prolongation covariant derivative

$$\widetilde{\nabla}_a := \nabla_a + \frac{2}{3}\Psi_{aBC}{}^{PQ} + \frac{2}{n}\overline{\Psi}_{aBC}{}^{PQ} + \frac{1}{n-1}\overline{\overline{\Psi}}_{aBC}{}^{PQ} : \mathcal{E}_{(PQ)} \to \mathcal{E}_{a(BC)}.$$

Proposition 4.4 *The prolongation connection $\widetilde{\nabla} : \mathcal{E}_{(BC)} \to \mathcal{E}_{a(BC)}$ in the conformal geometry has the form $\widetilde{\nabla}_a F_{BC} = \nabla_a F_{BC} + \frac{2}{3}\widetilde{\Psi}_{aBC}{}^{PQ}F_{PQ}$, where*

$$\begin{aligned}\widetilde{\Psi}_{aBC}{}^{PQ} =& -Z_{(B}^{b}Z_{C)}^{c}\Big[X^{(P}Z^{Q)q}C_{a(bc)q} + X^PX^QA_{a(bc)}\Big]\\ &+ X_{(B}Z_{C)}^{c}\Big[-\frac{4}{n}Z^{p(P}Z^{Q)q}C_{a(pq)c} + 2X^{(P}Z^{Q)q}\big(A_{a(cq)} + \frac{n-4}{n}A_{q(ac)}\big)\\ &-\frac{1}{n}X^PX^QB_{ac}\Big]\\ &+ X_BX_C\Big[-\frac{4}{n}Z^{p(P}Z^{Q)q}A_{a(pq)} + \frac{1}{n}X^{(P}Z^{Q)q}\big(B_{aq} + \frac{4}{3(n-1)}C_a{}^{(rs)p}C_{qrsp}\big)\\ &+\frac{4}{3n(n-1)}X^PX^QC_a{}^{(rs)p}A_{prs}\Big].\end{aligned}$$

□

Example 4.5 The prolongation covariant derivative $\widetilde{\nabla}$ on tractor form bundles $\mathcal{E}_{A^0\mathbf{A}}$, $\mathbf{A} = \mathbf{A}^k$ was computed in [16]. Consider a section $F_{A^0\mathbf{A}} = \mathbb{Y}_{A^0\mathbf{A}}^{\mathbf{a}}\sigma_{\mathbf{a}} + \frac{1}{k+1}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\mathbf{a}}\mu_{a^0\mathbf{a}} + \mathbb{W}_{A^0\mathbf{A}}^{\dot{\mathbf{a}}}\nu_{\dot{\mathbf{a}}} + \mathbb{X}_{A^0\mathbf{A}}^{\mathbf{a}}\rho_{\mathbf{a}} \in \mathcal{E}_{A^0\mathbf{A}}$. Then

$$\begin{aligned}\widetilde{\nabla}_c F_{A^0\mathbf{A}} =& \nabla_c F_{A^0\mathbf{A}} + \frac{1}{2}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\mathbf{a}}\Big[C_c{}^p{}_{a^0a^1}\sigma_{p\dot{\mathbf{a}}} + \frac{k-1}{n}\boldsymbol{g}_{ca^0}C_{a^1a^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}}\Big]\\ &-\frac{k(k-1)}{2n(n-k)}\mathbb{W}_{A^0\mathbf{A}}^{\dot{\mathbf{a}}}\Big[(n-2)C_{ca^2}{}^{pq}\sigma_{pq\ddot{\mathbf{a}}} - (k-2)C_{a^2a^3}{}^{pq}\sigma_{cpq\dddot{\mathbf{a}}}\Big]\\ &+\mathbb{X}_{A^0\mathbf{A}}^{\mathbf{a}}\Big[-A_c{}^p{}_{a^1}\sigma_{p\dot{\mathbf{a}}} - \frac{(k-1)(k-2)}{2nk}\boldsymbol{g}_{ca^1}C_{a^2a^3}{}^{pq}\nu_{pq\dddot{\mathbf{a}}}\\ &+\frac{k-1}{2(n-k)}\Big(\frac{n-2k}{2n}(\nabla_cC_{a^1a^2}{}^{pq})\sigma_{pq\ddot{\mathbf{a}}} + \boldsymbol{g}_{ca^1}A^{pq}{}_{a^2}\sigma_{pq\ddot{\mathbf{a}}}\\ &-2A_{ca^1}{}^p\sigma_{p\dot{\mathbf{a}}} - A_{a^1a^2}{}^p\sigma_{cp\ddot{\mathbf{a}}} + C_{ca^1}{}^{pq}\mu_{pq\dot{\mathbf{a}}}\\ &+\frac{n(n-k+1)-2k}{nk}C_c{}^p{}_{a^1a^2}\nu_{p\ddot{\mathbf{a}}} - \frac{k}{n}C_{a^1a^2}{}^{pq}\mu_{cpq\ddot{\mathbf{a}}}\Big)\Big],\end{aligned}$$

cf. [16, Remark 4.2].

The prolongation covariant derivative $\widetilde{\nabla}$ simplifies for $k=2$ in dimension $n=4$. Then we have (at least locally) the conformal volume form

$$\epsilon_{\mathbf{c}} \in \mathcal{E}_{\mathbf{c}}[4] \quad \text{such that} \quad \epsilon^{\mathbf{c}}\epsilon_{\mathbf{c}} = 4!, \text{ i.e. } \epsilon^{\mathbf{e}}\epsilon_{\mathbf{c}} = 4!\delta_{c^1}^{e^1}\,\delta_{c^2}^{e^2}\,\delta_{c^3}^{e^3}\,\delta_{c^4}^{e^4}, \tag{18}$$

where $\mathbf{c} = \mathbf{c}^4$, $\mathbf{e} = \mathbf{e}^4$. Recall $\nabla\epsilon = 0$ for any connection ∇ from the conformal class. Then the Hodge-star operator $* : \mathcal{E}_{\mathbf{a}^k} \to \mathcal{E}_{\mathbf{a}^{4-k}}$, $k = 0, \ldots, 4$ has the form $(*f)_{\mathbf{a}^k} =$

$\epsilon_{\mathbf{a}^k}{}^{\mathbf{r}^{4-k}} f_{\mathbf{r}^{4-k}}$. The eigenvalues of $*$ for $k = 2$ are ± 2. The induced tractor volume form $E_{\mathbf{C}^6} = -30\,\mathbb{W}_{\mathbf{C}^6}^{\mathbf{c}^4}\epsilon_{\mathbf{c}^4} \in \mathcal{E}_{\mathbf{C}^6}$ yields analogously the tractor Hodge-star operator $* : \mathcal{E}_{\mathbf{B}^\ell} \to \mathcal{E}_{\mathbf{B}^{6-\ell}}$. The eigenvalues of E for $\ell = 3$ are ± 6.

Henceforth we assume $k = 2$ and $n = 4$ and $*F = 6F$. If not stated otherwise, all form indices will have valence 2, e.g., $\mathbf{A} = \mathbf{A}^2$ or $\mathbf{a} = \mathbf{a}^2$. Our normalization of the volume forms E and ϵ means that

$$* \sigma = 2\sigma, \quad *\mu = -3\nu, \quad *\nu = 2\mu, \quad *\rho = -2\rho, \tag{19}$$

i.e., $\sigma_{\mathbf{a}}$ is self-adjoint. Using this and (18), one easily verifies

$$\boldsymbol{g}_{ca^0} C_{\mathbf{a}}{}^{\mathbf{r}} \sigma_{\mathbf{r}} = -2C_c{}^p{}_{\mathbf{a}} \sigma_{pa^0}, \quad C_{\mathbf{a}}{}^{\mathbf{r}} \mu_{c\mathbf{r}} = -2C_{ca^1}{}^{\mathbf{r}} \mu_{a^2\mathbf{r}}. \tag{20}$$

Thus the prolongation covariant derivative $\widetilde{\nabla}$ has the form

$$\begin{aligned}\widetilde{\nabla}_c F_{A^0\mathbf{A}} =& \nabla_c F_{A^0\mathbf{A}} + \frac{1}{4}\mathbb{Z}_{A^0\mathbf{A}}^{a^0\mathbf{a}} C_c{}^p{}_{\mathbf{a}} \sigma_{pa^0} - \frac{1}{4}\mathbb{W}_{A^0\mathbf{A}}{}^{a} C_{ca}{}^{\mathbf{r}}\sigma_{\mathbf{r}} \\ &+ \frac{1}{4}\mathbb{X}_{A^0\mathbf{A}}^{\mathbf{a}}\big[-4A_c{}^p{}_{a^1}\sigma_{pa^2} + \boldsymbol{g}_{ca^1}A^{\mathbf{r}}{}_{a^2}\sigma_{\mathbf{r}} - 2A_{ca^1}{}^p\sigma_{pa^2} \\ &- A_{\mathbf{a}}{}^p\sigma_{cp} + 2C_{ca^1}{}^{\mathbf{r}}\mu_{a^2\mathbf{r}} + C_c{}^p{}_{\mathbf{a}}\nu_p\big].\end{aligned}$$

The connection $\widetilde{\nabla}$ simplifies considerably for half-flat structures, i.e., when

$$\epsilon_{\mathbf{a}}{}^{\mathbf{r}} C_{\mathbf{rb}} + \epsilon_{\mathbf{b}}{}^{\mathbf{r}} C_{\mathbf{ar}} = 4\lambda C_{\mathbf{ab}}, \quad \lambda \in \{+1, -1\}. \tag{21}$$

The self-adjoint structure $\lambda = 1$ equivalently means $C_{\mathbf{a}}{}^{\mathbf{r}} f_{\mathbf{r}} = 0$ for every anti-self-adjoint two form $f_{\mathbf{a}}$ and the anti-self-adjoint structure $\lambda = -1$ analogously means $C_{\mathbf{a}}{}^{\mathbf{r}} f_{\mathbf{r}} = 0$ for every self-adjoint $f_{\mathbf{a}}$. It follows from (21), (19), and (18) that

$$C_c{}^p{}_{\mathbf{a}}\nu_p = \lambda C_{\mathbf{a}}{}^{\mathbf{r}}\mu_{c\mathbf{r}}. \tag{22}$$

We shall discuss the anti-self-dual case $\lambda = -1$ in detail. A short computation reveals

$$C_{\mathbf{a}}{}^{\mathbf{r}}\sigma_{\mathbf{r}} = 0, \quad A^{\mathbf{r}}{}_a\sigma_{\mathbf{r}} = 0 \quad \text{and} \quad A_{\mathbf{a}}{}^p\sigma_{cp} = 2A_{a^1c}{}^p\sigma_{a^2p},$$

where the second and the third equally follow by applying ∇^{a^1} and ∇_{a^0}, respectively, to the first one and using $\nabla_{a^0}C_{\mathbf{ar}} = 2\boldsymbol{g}_{a^0r^1}A_{\mathbf{a}r^2}$. (Note that the last equality says $A_{[\mathbf{a}}{}^p\sigma_{c]p} = 0$.) From the last display and (22) for $\lambda = -1$ we finally obtain the following:

Proposition 4.6 *Consider an anti-self-dual conformal structure in the dimension 4. Then the prolongation connection* $\widetilde{\nabla} : \mathcal{E}^+_{[A^0\mathbf{A}]} \to \mathcal{E}^+_{c[A^0\mathbf{A}]}$, $\mathbf{A} = \mathbf{A}^2$ *on the bundle of self-dual tractor 3-forms* $\mathcal{E}^+_{[A^0\mathbf{A}]} \subseteq \mathcal{E}_{[A^0\mathbf{A}]}$ *has the form*

$$\widetilde{\nabla}_c F_{A^0\mathbf{A}} = \nabla_c F_{A^0\mathbf{A}} + \mathbb{X}_{A^0\mathbf{A}}^{\mathbf{a}}\left[-2A_{c(pa^1)}\sigma^p{}_{a^2} + \frac{1}{2}C_c{}^p{}_{\mathbf{a}}\nu_p\right].$$

for $F_{A^0\mathbf{A}} \in \mathcal{E}^+_{[A^0\mathbf{A}]}$ *where* $\sigma_{\mathbf{a}} = 3\mathbb{X}^{A^0\mathbf{A}}_{\mathbf{a}}F_{A^0\mathbf{A}}$ *and* $\nu_a = -6\mathbb{W}^{A^0\mathbf{A}}_{a}F_{A^0\mathbf{A}}$.

Note that a modification of ∇ on $\mathcal{E}^+_{A^0\mathbf{A}}$ was also obtained in [8, (2.27)], where spinorial notation is used.

5 Commutation of higher projective BGG squares for the tractor forms

Let us denote the normal tractor connection by ∇. We shall use the notation from Section 2.1 for forms. In particular, all sequentially labeled indexes are implicitly skewed over.

We denote the standard projective tractor bundle by $\mathcal{T}$. The composition series of $(\mathcal{T}^*)^k := \bigwedge^k \mathcal{T}^*$ is $\mathcal{E}_{\dot{\mathbf{a}}}(k) \oplus \mathcal{E}_{\mathbf{a}}(k)$ and the normal tractor covariant derivative ∇ and its curvature Ω take the form

$$\nabla_c \begin{pmatrix} \sigma_{\dot{\mathbf{a}}} \\ \mu_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} \nabla_c \sigma_{\dot{\mathbf{a}}} + \mu_{c\dot{\mathbf{a}}} \\ \nabla_c \mu_{\mathbf{a}} - k P_{ca^1} \sigma_{\dot{\mathbf{a}}} \end{pmatrix}, \quad \Omega_{c^0c^1} \sharp \begin{pmatrix} \sigma_{\dot{\mathbf{a}}} \\ \mu_{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} (k-1) W_{c^0c^1}{}^p{}_{a^2} \sigma_{p\ddot{\mathbf{a}}} \\ k W_{c^0c^1}{}^p{}_{a^1} \mu_{p\dot{\mathbf{a}}} + * \end{pmatrix}.$$

Here $\mathbf{a} = \mathbf{a}^k$. We shall consider sections $f_{\mathbf{c}} \in \mathcal{E}_{\mathbf{c}} \otimes \Gamma((\mathcal{T}^*)^k)$, i.e., tractor indices are suppressed in the notation for f. Here $\mathbf{c} = \mathbf{c}^\ell$.

Following Theorem 1.1, we shall start with the sequence $E_i = d^\nabla$. Our aim is to find a suitable modification $D : \mathcal{E}_{\mathbf{c}} \otimes (\mathcal{T}^*)^k \to \mathcal{E}_{c^0\mathbf{c}} \otimes (\mathcal{T}^*)^k$ of d^∇ such that $(D \circ Df)_{c^{-1}c^0\mathbf{c}}) \in \operatorname{Ker} \partial^* \subseteq \mathcal{E}_{c^{-1}c^0\mathbf{c}}$ for every $f_{\mathbf{c}} \in \mathcal{E}_{\mathbf{c}} \otimes (\mathcal{T}^*)^k$. Specifically, we shall use an algebraic operator $\Psi_{c^0} \in \mathcal{E}_c \otimes \operatorname{End}((\mathcal{T}^*)^k)$ and put $(\Psi(f))_{c^0\mathbf{c}} := (\Psi \wedge f)_{c^0\mathbf{c}} \in \operatorname{Im} \partial^* \subseteq \mathcal{E}_{c^0\mathbf{c}} \otimes \operatorname{End}((\mathcal{T}^*)^k)$, and we put $(Df)_{c^0\mathbf{c}} := (d^\nabla f)_{c^0\mathbf{c}} + (\Box_k)^{-1} \Psi_{c^0} f_{\mathbf{c}} \in \mathcal{E}_{c^0} \otimes (\mathcal{T}^*)^k$. We shall usually write the endomorphism Ψ as $\Psi_{c^0} f_{\mathbf{c}}$, since the notation already requires the skew symmetrization. Here $\Box_k$ is a scalar multiple determined by the Kostant's Laplacian.

Now we shall describe the difference between d^∇ and D on $\mathcal{E}_{\mathbf{c}} \otimes \operatorname{End}((\mathcal{T}^*)^k)$. Consider the section

$$f_{\mathbf{c}} = \begin{pmatrix} \sigma_{\dot{\mathbf{a}}} \\ \mu_{\mathbf{a}} \end{pmatrix} \in \mathcal{E}_{\mathbf{c}} \otimes \operatorname{End}((\mathcal{T}^*)^k)$$

and

$$(d^\nabla d^\nabla f)_{c^{-1}c^0\mathbf{c}} = \Omega_{c^{-1}c^0} \sharp f_{\mathbf{c}} = \begin{pmatrix} W_{c^{-1}c^0}{}^p{}_{a^2} \sigma_{\mathbf{c}p\ddot{\mathbf{a}}} \\ * \end{pmatrix} \in \mathcal{E}_{c^{-1}c^0\mathbf{c}} \otimes \operatorname{End}((\mathcal{T}^*)^k).$$

Now we need ∂^* of the previous display and this will define the endomorphism Ψ. Here we need it only up to a scalar multiple and a short computation shows

$$(\Psi(f))_{\mathbf{c}^0\mathbf{c}} = (\partial^* \Omega \sharp f)_{c^0\mathbf{c}} = \begin{pmatrix} 0 \\ W_{a^1a^2}{}^p{}_{c^0} \sigma_{\mathbf{c}p\ddot{\mathbf{a}}} - \ell W_{c^{-1}c^0}{}^p{}_{a^1} \sigma_{a^2\dot{\mathbf{c}}p\ddot{\mathbf{a}}} \end{pmatrix}.$$

We use the modification $D := d^\nabla + \alpha \Box_k^{-1} \Psi$ and it remains to determine the operator $\Box_k$, i.e., how we need to rescale particular ($\mathfrak{g}_0$-)irreducible components of $\Psi(f)$.

6 More complicated examples: projective geometry

The standard projective tractor bundle $\mathcal{E}^A$ has the composition series $\mathcal{E}^A = \mathcal{E}^a(-1) \oplus \mathcal{E}(-1)$. We shall write a section of this bundle as

$$f^A = \begin{pmatrix} \sigma^a \\ \rho \end{pmatrix} = Y^A_a \sigma^a + X^A \rho$$

where $X_A : \mathcal{E}(-1) \to \mathcal{E}^A$ is invariant. The covariant derivative is then

$$\nabla_c F^A = \begin{pmatrix} \nabla_c \sigma^a \\ \nabla_c \rho \end{pmatrix}, \quad \text{i.e.,} \quad \nabla_c Y^A =, \nabla_c X^A = .$$

7 Almost Grassmannian geometry

A complex almost Grassmannian (or AG-) structure on a smooth manifold M is given by two auxiliary vector bundles $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ and the identification

$$\mathcal{E}^a = \mathcal{E}_{A'} \otimes \mathcal{E}^A = \mathcal{E}^A_{A'}, \quad \bigwedge^q \mathcal{E}^A \cong \bigwedge^p \mathcal{E}_{A'}, \tag{23}$$

where p is the rank of $\mathcal{E}_{A'}$ and q is the rank of $\mathcal{E}^A$. In fact, all results we obtain hold for all real forms of a given complex geometry, [13]. Motivated by the case $p = q = 2$, when the structure is the spin conformal structure, we shall term $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ spinor bundles.

Following [13] and Eq. 23, we adopt the convention

$$\mathcal{E}[-1] \cong \mathcal{E}_{\mathbf{A}^q} \cong \mathcal{E}^{\mathbf{B}'^p}, \quad \mathcal{E}[1] \cong \mathcal{E}^{\mathbf{A}^q} \cong \mathcal{E}_{\mathbf{B}'^p}$$

for line bundles. This isomorphism is given explicitly by the tautological section $\boldsymbol{\epsilon}_{\mathbf{A}^q} \in \mathcal{E}_{\mathbf{A}^q}[1]$ as $\mathcal{E}[-1] \ni f \mapsto f\boldsymbol{\epsilon}_{\mathbf{A}^q} \in \mathcal{E}_{\mathbf{A}^q}$. A choice of a *scale* $\xi \in \mathcal{E}[1]$ is equivalent to the choice of spinor volume forms $\boldsymbol{\epsilon}^{\xi}_{\mathbf{A}^q} := \xi^{-1}\boldsymbol{\epsilon}_{\mathbf{A}^q} \in \mathcal{E}_{\mathbf{A}^q}$, and analogously for $\mathcal{E}^{\mathbf{A}'^p}$.

Our conventions for the torsion $T_{ab}{}^c$ and the curvature $R_{ab}{}^d{}_c$ of a covariant derivative ∇_a on TM are given by the equation

$$2\nabla_{[a}\nabla_{b]}v^c = T_{ab}{}^d\nabla_d v^c + R_{ab}{}^c{}_d v^d.$$

Summarizing [13, Theorem 2.1], for a scale $\xi \in \mathcal{E}[1]$ on an AG-structure, there are unique covariant derivatives on $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ such that the torsion $F^{A'}_{A}{}^{B'}_{B}{}^{C}_{C'}$ of the induced covariant derivative on TM is totally trace-free, the induced covariant derivative preserves (23), and in addition ξ is parallel. We denote this class of covariant derivatives, parametrized by sections of $\mathcal{E}[1]$, by $[\nabla]$. Changing the scale $\xi \to \hat{\xi} = e^{\Upsilon}\xi \in \mathcal{E}[1]$ with Υ a smooth function, the covariant derivative ∇ changes to $\hat{\nabla}$ in a way that

$$\begin{aligned}
\hat{\nabla}^{A'}_{A} u^C &= \nabla^{A'}_{A} u^C + \delta^C_A \Upsilon^{A'}_B u^B, \quad \text{for } u^C \in \mathcal{E}^A,\\
\hat{\nabla}^{A'}_{A} u_{C'} &= \nabla^{A'}_{A} u_{C'} + \delta^{A'}_{C'} \Upsilon^{B'}_A u_{B'}, \quad \text{for } u_{C'} \in \mathcal{E}_{C'},\\
\hat{\nabla}^{A'}_{A} v_B &= \nabla^{A'}_{A} v_B - \Upsilon^{A'}_B v_A, \quad \text{for } v_B \in \mathcal{E}_B,\\
\hat{\nabla}^{A'}_{A} v^{B'} &= \nabla^{A'}_{A} v^{B'} - \Upsilon^{B'}_A v^{A'}, \quad \text{for } v^{B'} \in \mathcal{E}^{B'} \text{ and also}\\
\hat{\nabla}_a f &= \nabla_a f + w\Upsilon_a f, \quad \text{for } f \in \mathcal{E}[w],
\end{aligned} \tag{24}$$

where $\Upsilon_a = \nabla_a \Upsilon$. From now on we will use a hat sign to denote quantities corresponding to the changed scale $\hat{\xi} = e^{\Upsilon}\xi$ without further notice.

Given $\nabla \in [\nabla]$, we denote all covariant derivatives on tensor products of $\mathcal{E}^A$ and $\mathcal{E}_{A'}$ also by ∇. The curvature on spinor bundles is given by

$$(2\nabla_{[a}\nabla_{b]} - T_{ab}{}^d\nabla_d)v^C = R_{ab}{}^C_D v^D, \quad (2\nabla_{[a}\nabla_{b]} - T_{ab}{}^d\nabla_d)v_{D'} = -R_{ab}{}^{C'}_{D'} v_{C'}.$$

The curvature of ∇ is $R_{abc}^{\ \ \ d} = R_{ab}{}^D_C\delta^{C'}_{D'} - R_{ab}{}^{C'}_{D'}\delta^D_C$, where $R_{ab}{}^{D'}_{C'}$ and $R_{ab}{}^C_D$ are trace-free on the spinor indices displayed. The relations

$$\begin{aligned}
R_{ab}{}^C_D &= U_{ab}{}^C_D - \delta^C_B \mathsf{P}^{A'}_{A}{}^{B'}_{D} + \delta^C_A \mathsf{P}^{B'}_{B}{}^{A'}_{D},\\
R_{ab}{}^{C'}_{D'} &= U_{ab}{}^{C'}_{D'} + \delta^{B'}_{D'} \mathsf{P}^{A'}_{A}{}^{C'}_{B} - \delta^{A'}_{D'} \mathsf{P}^{B'}_{B}{}^{C'}_{A},
\end{aligned}$$

together with the condition $U^{A'}_{R}{}^{B'}_{B}{}^{R}_{A} - U^{R'}_{A}{}^{B'}_{B}{}^{A'}_{R'} = 0$ (and the algebraic Bianchi identity) determine $U_{ab}{}^C_D$, $U_{ab}{}^{C'}_{D'}$ and the *Rho-tensor* P_{ab}. In more details, the curvature on the (co)tangent bundle is

$$R_{abd}{}^{c} = U_{abd}{}^{c} + \delta_{C'}^{D'}\delta_A^C \mathsf{P}_{B}{}^{B'}{}_{D}{}^{A'} - \delta_{C'}^{D'}\delta_B^C \mathsf{P}_{A}{}^{A'}{}_{D}{}^{B'} + \delta_D^C\delta_{C'}^{A'} \mathsf{P}_{B}{}^{B'}{}_{A}{}^{D'} - \delta_D^C\delta_{C'}^{B'} \mathsf{P}_{A}{}^{A'}{}_{B}{}^{D'},$$

where $U_{abd}{}^{c} = U_{ab}{}_{D}^{C}\delta_{C'}^{D'} - U_{ab}{}_{C'}^{D'}\delta_D^C$. In this form the tensor U is determined by $U_{rba}{}^{r} = U_{R}{}^{A'}{}_{B}{}^{B'}{}_{A}{}^{R} - U_{A}{}^{R'}{}_{B}{}^{B'}{}_{R'}{}^{A'} = 0$. (Note that the previous display shows the decomposition $U = R + \partial\mathsf{P}$, where U is ∂^*-closed, cf. the theory of Weyl structures in [6].) Furthermore,

$$U_{ab}{}_{C}^{C} = -U_{ab}{}_{C'}^{C'} = 2\mathsf{P}_{[ab]} \quad \text{and} \quad -2(p+q)\mathsf{P}_{[ab]} = \nabla_c T_{ab}{}^{c}, \tag{25}$$

where the last identity follows from the algebraic Bianchi identity.

We will mostly be interested in the case $p = 2$ and $q > 2$. In this case the only invariants are the trace-free part of $T_{[A}{}^{(A'}{}_{B]}{}^{B')}{}_{C'}{}^{C}$ and the trace-free part of $U_{(A}{}^{[A'}{}_{B}{}^{B']}{}_{C)}{}^{D}$, [13]. That is, if these two vanish, the geometry is locally isomorphic to the homogeneous model. Finally, note that using the algebraic Bianchi identity, we obtain

$$\begin{aligned}
&U_{(A}{}^{R'}{}_{B)}{}^{[A'}{}_{R'}{}^{B']} = U_{R}{}^{[A'}{}_{(A}{}^{B']}{}_{B)}{}^{R} = U_{[A}{}^{R'}{}_{B]}{}^{(A'}{}_{R'}{}^{B')} = U_{R}{}^{(A'}{}_{[A}{}^{B')}{}_{B]}{}^{R'} = 0,\\
&U_{(A}{}^{R'}{}_{B)}{}^{(A'}{}_{R'}{}^{B')} = U_{R}{}^{(A'}{}_{(A}{}^{B')}{}_{B)}{}^{R} = \frac{1}{q} T_{r(A}{}^{(A'|e|} T_{B)e}{}^{B')r},\\
&U_{[A}{}^{R'}{}_{B]}{}^{[A'}{}_{R'}{}^{B']} = U_{R}{}^{[A'}{}_{[A}{}^{B']}{}_{B]}{}^{R} = -\frac{1}{q+4} T_{r[A}{}^{[A'|e|} T_{B]e}{}^{B']r}.
\end{aligned} \tag{26}$$

7.1 Grassmannian tractor calculus

We follow [13] here. The standard tractor bundle is the (spinor tractor) bundle $\mathcal{E}^\alpha = \mathcal{E}^A \oplus \mathcal{E}^{A'}$ and we denote its dual by $\mathcal{E}_\alpha = \mathcal{E}_{A'} \oplus \mathcal{E}_A$. That is, we use Greek letters for spinor tractor abstract indices. Using the injectors $Y_A^\alpha \in \mathcal{E}_A^\alpha$, $X_{A'}^\alpha \in \mathcal{E}_{A'}^\alpha$, and $Y_\alpha^{A'} \in \mathcal{E}_\alpha^{A'}$, $X_\alpha^A \in \mathcal{E}_\alpha^A$, sections of $\mathcal{E}^\alpha$ and $\mathcal{E}_\alpha$ are written conveniently as

$$\begin{pmatrix}\sigma^A\\ \rho^{A'}\end{pmatrix} = Y_A^\alpha\sigma^A + X_{A'}^\alpha\rho^{A'} \in \mathcal{E}^\alpha, \quad \text{respectively,} \quad \begin{pmatrix}\nu_{A'}\\ \mu_A\end{pmatrix} = Y_\alpha^{A'}\nu_{A'} + X_\alpha^A\mu_A \in \mathcal{E}_\alpha.$$

Splittings of $\mathcal{E}^\alpha$ and $\mathcal{E}_\alpha$ are parametrized by a choice of scale $\xi \in \mathcal{E}[1]$. The change of the splitting has the form

$$\begin{aligned}
&\widehat{\begin{pmatrix}\sigma^A\\ \rho^{A'}\end{pmatrix}} = \begin{pmatrix}\sigma^A\\ \rho^{A'} - \Upsilon_B^{A'}\sigma^B\end{pmatrix}, \quad \text{i.e.,} \quad \hat{Y}_A^\alpha = Y_A^\alpha + X_{B'}^\alpha\Upsilon_A^{B'}, \ \hat{X}_{A'}^\alpha = X_{A'}^\alpha \quad \text{and}\\
&\widehat{\begin{pmatrix}\nu_{A'}\\ \mu_A\end{pmatrix}} = \begin{pmatrix}\nu_{A'}\\ \mu_A + \Upsilon_A^{A'}\nu_{A'}\end{pmatrix}, \quad \text{i.e.,} \quad \hat{Y}_\alpha^{A'} = Y_\alpha^{A'} - X_\alpha^B\Upsilon_B^{A'}, \ \hat{X}_\alpha^A = X_\alpha^A.
\end{aligned}$$

That is, the sections $X_{A'}^\alpha$ and X_α^A are invariant and Y_A^α and $Y_\alpha^{A'}$ depend on the choice of the scale. They are normalized in such a way that $Y_B^\beta X_\alpha^B + Y_\alpha^{B'} X_{B'}^\beta = \delta_\alpha{}^\beta$, i.e., $X_\alpha^B Y_A^\alpha = \delta_A{}^B$, and $X_{A'}^\alpha Y_\alpha^{B'} = \delta_{A'}{}^{B'}$.

The normal covariant tractor derivative is given by

$$\nabla_A^{P'}\begin{pmatrix}\sigma^B\\ \rho^{B'}\end{pmatrix} = \begin{pmatrix}\nabla_A^{P'}\sigma^B + \rho^{P'}\delta_A{}^B\\ \nabla_A^{P'}\rho^{B'} - \mathsf{P}_{A}{}^{P'}{}_{B}{}^{B'}\sigma^B\end{pmatrix} \text{ and } \nabla_A^{P'}\begin{pmatrix}\nu_{B'}\\ \mu_B\end{pmatrix} = \begin{pmatrix}\nabla_A^{P'}\nu_{B'} - \delta_{B'}^{P'}\mu_A\\ \nabla_A^{P'}\mu_B + \mathsf{P}_{A}{}^{P'}{}_{B}{}^{B'}\nu_{B'}\end{pmatrix}.$$

That is,

$$\nabla_A^{P'} Y_B^\alpha = -X_{B'}^\alpha \mathsf{P}_{A\ B}^{P'B'}, \nabla_A^{P'} X_{B'}^\alpha = Y_A^\alpha \delta_{B'}^{P'} \text{ and}$$
$$\nabla_A^{P'} Y_\alpha^{B'} = X_\alpha^B \mathsf{P}_{A\ B}^{P'B'},\ \nabla_A^{P'} X_\alpha^B = -Y_\alpha^{P'} \delta_A^B.$$

Its curvature $\Omega_{ab}{}^\alpha_\beta$ is trace-free on the spinor tractor bundle and has the explicit form

$$\Omega_{ab}{}^\alpha_\beta = -\,Y_C^\alpha Y_\beta^{C'} T_{abC'}{}^C + Y_C^\alpha X_\beta^D U_{abD}{}^C + X_{C'}^\alpha Y_\beta^{D'} U_{abD'}{}^{C'}$$
$$+\,X_{C'}^\alpha X_\beta^C Q_{abC}{}^{C'} \in \mathcal{E}_{[ab]}{}^\alpha_\beta \subseteq \mathcal{E}_{[ab]} \otimes \text{trace-free}(\mathcal{E}_\beta^\alpha),$$

where $Q_{abc} = -2\nabla_{[a}\mathsf{P}_{b]c} + T_{ab}{}^e \mathsf{P}_{ec} \in \mathcal{E}_{[ab]c}$ and trace-free$(\mathcal{E}_\beta^\alpha) = \mathcal{A}$ is the adjoint tractor bundle. That is, $(\nabla_a\nabla_b - \nabla_b\nabla_a - T_{ab}{}^e\nabla_e) f^\alpha = \Omega_{ab}{}^\alpha_\beta f^\beta = (\Omega\sharp f)_{ab}{}^\alpha = \Omega_{ab}\sharp f^\alpha$ in our notation.

The inclusions ι and $\bar\iota$ from 2.2 are of the form $Y_{A^0}^\alpha Y_\beta^{A^{0\prime}} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\bar\iota} \mathcal{E}_{a^0\mathbf{a}}{}^\alpha{}_\beta$ and $X_{A^{1\prime}}^\alpha X_\beta^{A^1} : \mathcal{E}_{\mathbf{a}} \xrightarrow{\iota} \mathcal{E}_{\dot{\mathbf{a}}}{}^\alpha{}_\beta$, where we use the identification $\mathcal{E}_{a^0} = \mathcal{E}_{A^0}^{A^{0\prime}}$ and $\mathcal{E}^{a^1} = \mathcal{E}_{A^{1\prime}}^{A^1}$. Therefore

$$\partial : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto Y_{A^0}^\alpha Y_\beta^{A^{0\prime}} f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{a^0\mathbf{a}} \otimes \mathcal{T} \quad \text{and}$$
$$\partial^* : \mathcal{E}_{\mathbf{a}} \otimes \mathcal{T} \ni f_{\mathbf{a}} \mapsto X_{A^{1\prime}}^\alpha X_\beta^{A^1} f_{\mathbf{a}} \xrightarrow{\sharp} \mathcal{E}_{\dot{\mathbf{a}}} \otimes \mathcal{T}$$

for any subbundle $\mathcal{T}$ of $\bigotimes \mathcal{E}_\alpha \otimes \bigotimes \mathcal{E}^\beta \otimes \mathcal{E}[w]$. This does not cover all tractor bundles but will be sufficient in the examples treated below.

Henceforth we assume $p = 2, q > 2$. Note we have the decomposition $\Omega_{ab}{}^\alpha_\beta = \Omega_{(AB)}^{[A'B']\alpha}{}_\beta + \Omega_{[AB]}^{(A'B')\alpha}{}_\beta$, where the component $\Omega_{[AB]}^{(A'B')\alpha}{}_\beta$ vanishes in the torsion-free case.

7.2 Skew symmetric tractors and tractor forms

We shall also need tractor bundles $\bigwedge^\ell \mathcal{E}^\alpha = \mathcal{E}^{\boldsymbol\alpha}$ with the notation $\boldsymbol\alpha = \boldsymbol\alpha^\ell$ for the multiindex. Since $\bigwedge^\ell \mathcal{E}^\alpha \cong \bigwedge^{q+2-\ell} \mathcal{E}_\beta$ (we assume orientability here), these are just tractor forms. Specifically, the case $\ell = q+1$ is just the bundle $\mathcal{E}_\beta$.

It follows from the structure of $\mathcal{E}^\alpha$ that

$$\mathcal{E}^{\boldsymbol\alpha} = \mathcal{E}^{\mathbf{A}} \oplus \mathcal{E}^{B'\dot{\mathbf{A}}} \oplus \mathcal{E}^{[B'C']\ddot{\mathbf{A}}}, \qquad \boldsymbol\alpha = \boldsymbol\alpha^\ell,\ \mathbf{A} = \mathbf{A}^\ell,\ 2 \le \ell \le q.$$

Of course we have the isomorphism $\mathcal{E}^{[B'C']\ddot{\mathbf{A}}} \cong \mathcal{E}^{\ddot{\mathbf{A}}}[-1]$ using the spinor volume form $\boldsymbol\epsilon_{B'C'} \in \mathcal{E}_{[B'C']}[-1]$, but it turns out to be more convenient for the computation to use the form as in the display.

We put

$$\mathbb{Y}_{\mathbf{A}}^{\boldsymbol\alpha} = Y_{A^1}^{[\alpha^1} \ldots Y_{A^\ell}^{\alpha^\ell]} \in \mathcal{E}_{\mathbf{A}}^{\boldsymbol\alpha}, \quad \mathbb{W}_{B'\dot{\mathbf{A}}}^{\boldsymbol\alpha} = X_{B'}^{[\alpha^1} Y_{A^2}^{\alpha^2} \ldots Y_{A^\ell}^{\alpha^\ell]} \in \mathcal{E}_{B'\dot{\mathbf{A}}}^{\boldsymbol\alpha},$$
$$\mathbb{X}_{B'C'\ddot{\mathbf{A}}}^{\boldsymbol\alpha} = X_{B'}^{[\alpha^1} X_{C'}^{\alpha^2} Y_{A^3}^{\alpha^3} \ldots Y_{A^\ell}^{\alpha^\ell]} \in \mathcal{E}_{[B'C']\ddot{\mathbf{A}}}^{\boldsymbol\alpha},$$

where $\mathbb{X}_{B'C'\ddot{\mathbf{A}}}^{\boldsymbol\alpha}$ is invariant and $\mathbb{Y}_{\mathbf{A}}^{\boldsymbol\alpha}$ and $\mathbb{W}_{B'\dot{\mathbf{A}}}^{\boldsymbol\alpha}$ are scale dependent. Finally, the normal tractor connection on these section is

$$\nabla_c \mathbb{Y}_{\mathbf{A}}^{\boldsymbol\alpha} = -\ell\, \mathbb{W}_{B'[\dot{\mathbf{A}}}^{\boldsymbol\alpha} \mathsf{P}_{|c|A^1]}{}^{B'},$$
$$\nabla_c \mathbb{W}_{B'\dot{\mathbf{A}}}^{\boldsymbol\alpha} = \mathbb{Y}_{C\dot{\mathbf{A}}}^{\boldsymbol\alpha} \delta_{B'}^{C'} - (\ell-1)\, \mathbb{X}_{B'D'[\ddot{\mathbf{A}}}^{\boldsymbol\alpha} \mathsf{P}_{|c|A^2]}{}^{D'}, \quad \text{and}$$
$$\nabla_c \mathbb{X}_{B'D'\ddot{\mathbf{A}}}^{\boldsymbol\alpha} = 2\, \mathbb{W}_{B'C\ddot{\mathbf{A}}}^{\boldsymbol\alpha} \delta_{D'}^{C'}.$$

Example 7.1 We shall demonstrate the prolongation covariant derivative $\widetilde{\nabla}$ for AG-geometries on tractor bundles corresponding to fundamental representations. These are the bundles $\bigwedge^{\ell} \mathcal{E}^{\alpha}$ for $1 \leq \ell \leq q+1$. Since the computation is getting very technical for $1 < \ell < q+1$, we later restrict to torsion-free manifolds.

First we discuss the cases $\mathcal{E}^{\alpha}$ and $\mathcal{E}_{\beta} \cong \bigwedge^{q+1} \mathcal{E}^{\alpha}$. Considering $F^{\alpha} \in \mathcal{E}^{\alpha}$ and $G_{\beta} \in \mathcal{E}_{\beta}$, a short computation gives

$$
\begin{aligned}
(\partial^* d^{\nabla} d^{\nabla} F)_c{}^{\alpha} &= \frac{1}{2} X^{\alpha}_{D'} X^{D}_{\omega} S^{D'C'}_{C\,D} F^{\omega} \quad \text{and} \\
(\partial^* d^{\nabla} d^{\nabla} G)_{c\beta} &= -\frac{1}{2} X^{\omega}_{D'} X^{D}_{\beta} S^{D'C'}_{C\,D} G_{\omega},
\end{aligned}
$$

where

$$
S^{D'C'}_{C\,D} = U^{R'A'B'}_{A\,B\,R'} = U^{A'B'R}_{R\,A\,B} = \frac{1}{q} T_{r(A}{}^{(A'|e|} T^{B')\,r}_{B)\,e} - \frac{1}{q+4} T_{r[A}{}^{[A'|e|} T^{B']\,r}_{B]\,e}.
$$

Hence we need the action of the Kostant-Laplace operator $\Box$ on $\mathcal{E}^{D'C'}_{C} = \mathcal{E}^{(D'C')}_{C} \oplus \mathcal{E}^{[D'C']}_{C}$. The eigenvalues are, respectively, $\frac{1}{2}(q-1)$ and $\frac{1}{2}(q+1)$. Therefore the prolongation connection $\widetilde{\nabla}$ has the form

$$
\begin{aligned}
\widetilde{\nabla}_c F^{\alpha} &= \nabla_c F^{\alpha} - X^{\alpha}_{D'} X^{D}_{\omega} \left[\frac{1}{q-1} S^{(D'C')}_{C\,D} + \frac{1}{q+1} S^{[D'C']}_{C\,D} \right] F^{\omega} \quad \text{for } F^{\alpha} \in \mathcal{E}^{\alpha}, \\
\widetilde{\nabla}_c G_{\beta} &= \nabla_c G_{\beta} + X^{\omega}_{D'} X^{D}_{\beta} \left[\frac{1}{q-1} S^{(D'C')}_{C\,D} + \frac{1}{q+1} S^{[D'C']}_{C\,D} \right] G_{\omega} \quad \text{for } G_{\beta} \in \mathcal{E}_{\beta}.
\end{aligned}
$$

It remains to consider the bundles $\mathcal{E}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{\ell}$ for $2 \leq \ell \leq q$. Consider the section $F^{\boldsymbol{\alpha}} = \mathbb{Y}^{\boldsymbol{\alpha}}_{\mathbf{A}} \sigma^{\mathbf{A}} + \mathbb{W}^{\boldsymbol{\alpha}}_{B'\dot{\mathbf{A}}} \mu^{B'\dot{\mathbf{A}}} + \mathbb{X}^{\boldsymbol{\alpha}}_{B'C'\ddot{\mathbf{A}}} \rho^{B'C'\ddot{\mathbf{A}}}$, where $\sigma^{\mathbf{A}} \in \mathcal{E}^{\mathbf{A}}$, $\mu^{B'\dot{\mathbf{A}}} \in \mathcal{E}^{B'\dot{\mathbf{A}}}$ and $\rho^{B'C'\ddot{\mathbf{A}}} \in \mathcal{E}^{[B'C']\ddot{\mathbf{A}}}$. A straightforward computation shows that

$$
\begin{aligned}
(d^{\nabla} d^{\nabla} F)_{de}^{\boldsymbol{\alpha}} = \frac{1}{2} \Omega_{de} \sharp F^{\boldsymbol{\alpha}} &= \frac{1}{2} \ell\, \Omega_{de}{}^{[\alpha^1}_{\omega} F^{|\omega|\dot{\boldsymbol{\alpha}}]} \\
&= \frac{1}{2} \Big\{ \mathbb{Y}^{\boldsymbol{\alpha}}_{\mathbf{A}} \Big[\ell U_{de}{}^{[A^1}_{Q} \sigma^{|Q|\dot{\mathbf{A}}]} - T_{de}{}^{[A^1}_{Q'} \mu^{|Q'|\dot{\mathbf{A}}]} \Big] \\
&\quad + \mathbb{W}^{\boldsymbol{\alpha}}_{B'\dot{\mathbf{A}}} \Big[(\ell-1) U_{de}{}^{[A^2}_{Q} \mu^{|B'Q|\ddot{\mathbf{A}}]} + \ell Q_{de}{}^{B'}_{Q} \sigma^{Q\dot{\mathbf{A}}} + U_{de}{}^{B'}_{Q'} \mu^{Q'\dot{\mathbf{A}}} \\
&\quad - 2 T_{de}{}^{[A^2}_{Q'} \rho^{|B'Q'|\ddot{\mathbf{A}}]} \Big] + \mathbb{X}^{\boldsymbol{\alpha}}_{B'C'\ddot{\mathbf{A}}} \varphi^{B'C'\ddot{\mathbf{A}}} \Big\}
\end{aligned}
$$

for a section $\varphi^{B'C'\ddot{\mathbf{A}}} \in \mathcal{E}^{B'C'\ddot{\mathbf{A}}}$. We need to compute ∂^* on the terms in the previous display.

It turns out the computation is getting too technical in general, so we compute $\widetilde{\nabla}$ in the torsion-free case only. That is, we assume $T_{ef}{}^{C'}_{C} = 0$ (hence also $S^{D'C'}_{C\,D} = 0$) from now on. Then we obtain

$$
\begin{aligned}
(\partial^* d^{\nabla} d^{\nabla} F)_e{}^{\boldsymbol{\alpha}} = &\frac{1}{2} (\ell - 1) \Big\{ \ell\, \mathbb{W}^{\boldsymbol{\alpha}}_{B'\dot{\mathbf{A}}} U_{eR}{}^{B'[A^2}_{Q} \sigma^{|QR|\ddot{\mathbf{A}}]} \\
&+ \mathbb{X}^{\boldsymbol{\alpha}}_{B'C'\ddot{\mathbf{A}}} \Big[(\ell-2) U_{eR}{}^{C'[A^3}_{Q} \mu^{|B'QR|\dddot{\mathbf{A}}]} - \ell\, Q_{eR}{}^{C'B'}_{Q} \sigma^{QR\dot{\mathbf{A}}} - U_{eR}{}^{C'B'}_{Q'} \mu^{Q'R\ddot{\mathbf{A}}} \Big] \Big\}.
\end{aligned}
$$

Since $U^{A'B'D}_{A\,B\,C} = U^{[A'B']D}_{(A\,B\,C)}$ in the torsion-free case, we conclude that

$$
(\partial^* d^{\nabla} d^{\nabla} F)_e{}^{\boldsymbol{\alpha}} = 0.
$$

This yields the surprising result $\widetilde{\nabla} = \nabla$ on $\mathcal{E}^{\boldsymbol{\alpha}}$. The same is obviously true also for $\ell = 1$ and $\ell = q + 1$. Hence we obtain

Proposition 7.2 *The prolongation connection* $\widetilde{\nabla}_c : \mathcal{E}^{\boldsymbol{\alpha}} \to \mathcal{E}_c{}^{\boldsymbol{\alpha}}, \boldsymbol{\alpha} = \boldsymbol{\alpha}^\ell$ *for* $1 \leq \ell \leq q + 1$ *on torsion-free AG-manifolds is equal to the normal tractor connection, i.e.,* $\widetilde{\nabla} = \nabla$.

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