CONDITIONAL OSCILLATION OF HALF-LINEAR EQUATIONS

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Habilitation Thesis

Preface

The nature around us can be described in many ways. One of the most accurate ways is using equations. The type of the investigated equations depends on the analyzed phenomenon itself. There are two fundamental types, namely, differential equations (for continuous time) and difference equations (for discrete time). Of course, during the last decades, the unification of both types, the so called time scale calculus, have been developed. Chapter 2 is devoted to differential equations, then Chapter 3 deals with difference equations, and finally the dynamic equations on time scales are treated in Chapter 4.

Oscillation (or non-oscillation) is one of the fundamental topics from the qualitative theory of differential and difference equations. The main idea of this notion is to count zeros of solutions at infinity. If there exists the greatest zero of a solution, we say that this solution is non-oscillatory. On the other hand, if the zeros of a solution tend to infinity, the solution is said to be oscillatory.

The main goal of this thesis is to study the conditional oscillation of equations and to find the so-called critical oscillation constants. Once we prove that an equation is conditionally oscillatory and find the critical constant (which usually depends on the coefficients of the studied equation), the oscillation properties of such an equation are fully resolved with no more than one exception (the critical case). This fact means that conditionally oscillatory equations are ideal testing equations. In particular, many equations (which are not conditionally oscillatory) can be compared with these testing equations using comparison theorems and, consequently, it is possible to specify their oscillation properties.

This work is based on papers [35, 40, 41, 42, 43, 44, 45, 46] published or submitted for publication during years 2014 and 2015. Hence, I am obliged to my coauthors, I thank them for fruitful collaboration and look forward to solving open problems which appeared during our conjoint work.

Also, I would like to sincerely thank Prof. Ondřej Došlý, who lead my first steps to mathematical analysis, for his advice and willingness.

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Chapter

1 Introduction

1.1 The essentials of the used techniques

The topic of this thesis belongs to the oscillation theory of half-linear equations. The main part (Chapter 2) deals with differential equations. Therefore, we recall the basics of half-linear differential equations at this place and difference equations and dynamic equations on time scales will be introduced in Chapters 3 and 4, respectively. Note that we will use the standard notation $\mathbb{R}^+ = (0, \infty)$ and $\mathbb{R}_a = [a, \infty)$ for arbitrary given $a \in \mathbb{R}$.

Our main interest is to study equations of the form

$$[r(t)\Phi(x')]' + z(t)\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1} \operatorname{sgn} x,\ p > 1, \tag{1.1.1}$$

with continuous coefficients $r > 0$ and z. An equation of this form appeared for the first time in [8] and as the basic pioneering papers in the field of half-linear differential equations we mention [24, 60]. During the last decades, these equations have been widely studied in the literature. A detailed description and a comprehensive literature overview concerning the topic can be found in [21] (see also [2, Chapter 3]).

The name *half-linear equations* was introduced in [7]. This term is motivated by the fact that the solution space of these equations is homogeneous (likewise in the linear case), but it is not additive. There are several differences between linear equations and half-linear equations. Especially, some tools widely used in the theory of linear equations are not available for half-linear equations (e.g., see [25] for the Wronskian identity and [23] for the Fredholm alternative). In fact, these differences are caused, more or less, by the lack of the additivity. On the other hand, many results from the theory of linear equations are extendable to their half-linear counterparts. Nevertheless, according to our best knowledge, the results presented in this thesis are new even for the linear case (i.e., for $p = 2$).

Since the main tools in this thesis are based on the Riccati technique and the Prüfer angle (more precisely on their generalizations and combinations), we recall these notions at this place. To begin with the Prüfer transformation, we have to recall the half-linear trigonometric functions as well. For more comprehensive description, we refer, e.g., to [21, Section 1.1.2]. The half-linear sine function, denoted by $\sin_p$, is defined as the odd $2\pi_p$-periodic extension of the solution of the initial problem

$$[\Phi(x')]' + (p-1)\Phi(x) = 0, \quad x(0) = 0, \quad x'(0) = 1, \tag{1.1.2}$$

where

$$\pi_p := \frac{2}{p} B\left(\frac{1}{p}, \frac{1}{q}\right) = \frac{2\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{q}\right)}{p\Gamma\left(\frac{1}{p}+\frac{1}{q}\right)} = \frac{2\pi}{p \sin\frac{\pi}{p}}. \tag{1.1.3}$$

In the definition of π_p, we use the Euler beta and gamma functions

$$B(x,y) = \int_0^1 \tau^{x-1}(1-\tau)^{y-1}\,\mathrm{d}\tau, \quad x,y>0, \qquad \Gamma(x) = \int_0^\infty \tau^{x-1}\,\mathrm{e}^{-\tau}\,\mathrm{d}\tau, \quad x>0,$$

and the formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad x>0,$$

together with the identity (the conjugacy of the numbers p and q)

$$\frac{1}{p}+\frac{1}{q} = 1, \quad \text{i.e.,} \quad p+q=pq. \tag{1.1.4}$$

The derivative of the half-linear sine function is called the half-linear cosine function and it is denoted by $\cos_p$. Note that the half-linear sine and cosine functions satisfy the half-linear Pythagorean identity

$$|\sin_p t|^p + |\cos_p t|^p = 1, \quad t\in\mathbb{R}. \tag{1.1.5}$$

Especially, the half-linear trigonometric functions are bounded. Therefore, there exists $L>0$ such that

$$|\cos_p y|^p < L, \quad |\Phi(\cos_p y)\sin_p y| < L, \quad |\sin_p y|^p < L, \quad y\in\mathbb{R}. \tag{1.1.6}$$

In fact, (1.1.6) is valid for any $L>1$.

Using the notion of the half-linear trigonometric functions, we can introduce the basic half-linear Prüfer transformation

$$x(t) = \rho(t)\sin_p\varphi(t), \tag{1.1.7}$$

$$r^{q-1}(t)x'(t) = \rho(t)\cos_p\varphi(t), \tag{1.1.8}$$

whose modifications will be used later, and we apply it to Eq. (1.1.1) as follows. We differentiate (1.1.7) and combine it with (1.1.8). It leads to

$$r^{q-1}(t)\rho(t)\cos_p\varphi(t) = \rho'(t)\sin_p\varphi(t) + \rho(t)\varphi'(t)\cos_p\varphi(t). \tag{1.1.9}$$

Then we apply the function Φ to (1.1.8), we differentiate the result, and combine it with Eq. (1.1.1). This results to

$$\begin{aligned} &z(t)\rho^{p-1}(t)\Phi(\sin_p\varphi(t)) \\ &\quad = (p-1)\left[\varphi'(t)\rho^{p-1}(t)\Phi(\sin_p\varphi(t)) - \rho'(t)\rho^{p-2}(t)\Phi(\cos_p\varphi(t))\right]. \end{aligned} \tag{1.1.10}$$

Finally, we combine the $[\Phi(\cos_p \varphi(t))\rho^{-1}(t)]$ multiple of (1.1.9) with $[\sin_p \varphi(t)\rho^{1-p}(t)]$ multiple of (1.1.10). It leads directly to the system of first order differential equations

$$\begin{aligned} \varphi'(t) &= \frac{z(t)|\sin_p \varphi(t)|^p}{p-1} + r^{1-q}(t)|\cos_p \varphi(t)|^p, \\ \rho'(t) &= \rho(t)\Phi(\sin_p \varphi(t)) \cos_p \varphi(t) \left[r^{1-q}(t) - \frac{z(t)}{p-1} \right]. \end{aligned} \tag{1.1.11}$$

Remark 1.1.1. The function φ used above is called the Prüfer angle and the first equation from (1.1.11) is referred as the equation of the Prüfer angle (this equation will be very important later). The connection of the Prüfer angle to the oscillation theory is obvious directly from (1.1.7), i.e., if the Prüfer angle is bounded then there exists the greatest zero point of the solution x. This is equivalent to the definition of oscillation of Eq. (1.1.1) given (more correctly) below.

Now, we turn our attention to the Riccati technique. We derive the Riccati equation associated to Eq. (1.1.1) and we show their mutual connection. To obtain the Riccati equation, we use the transformation

$$w(t) = r(t)\frac{\Phi(x'(t))}{\Phi(x(t))}, \tag{1.1.12}$$

where x is a solution of Eq. (1.1.1) which is non-zero on the interval under consideration. We simply compute the derivative of w and use Eq. (1.1.1) and (1.1.4) as follows

$$\begin{aligned} w'(t) &= \frac{[r(t)\Phi(x')]'\,\Phi(x) - (p-1)r(t)\Phi(x')|x|^{p-2}x'}{\Phi^2(x)} \\ &= -z(t) - (p-1)r(t)|x'|^p|x|^{-p} = -z(t) - (p-1)r^{1-q}(t)|w(t)|^q. \end{aligned}$$

We obtained the half-linear Riccati differential equation

$$w'(t) + z(t) + (p-1)r^{1-q}(t)|w(t)|^q = 0. \tag{1.1.13}$$

The connection of Eq. (1.1.13) and Eq. (1.1.1) is embodied in the below given half-linear Reid roundabout theorem 1.1.1. Nevertheless, to formulate it properly, we have to briefly mention the notion of disconjugacy and the energy functional of Eq. (1.1.1).

Definition 1.1.1. Eq. (1.1.1) is said to be *disconjugate* on the closed interval $[a, b]$ if the solution x given by the initial condition $x(a) = 0, r(a)\Phi(x'(a)) = 1$ has no zero in $(a, b]$ (by a zero of a solution x we mean such a t_0 that $x(t_0) = 0$). In the opposite case Eq. (1.1.1) is said to be *conjugate* on $[a, b]$.

We recall that the Sobolev space $W_0^{1,p}(a, b)$ contains absolutely continuous functions f such that

$$f' \in L^p(a, b) = \left\{ y : (a, b) \to \mathbb{R}; \left(\int_a^b |y(t)|^p + |y'(t)|^p \,\mathrm{d}t \right)^{\frac{1}{p}} < \infty \right\}, \qquad f(a) = 0 = f(b).$$

The *energy functional* of Eq. (1.1.1) is

$$\mathcal{F}(y;a,b) = \int_a^b r(t)|y'|^p - z(t)|y|^p \,\mathrm{d}t, \quad y \in W_0^{1,p}(a,b).$$

Now, we can formulate and prove the half-linear Reid roundabout theorem.

Theorem 1.1.1 (Half-linear Reid roundabout theorem)**.** *The following statements are equivalent.*

(i) Eq. (1.1.1) *is disconjugate on the interval* $[a,b]$*.*

(ii) There exists a solution of Eq. (1.1.1) *having no zero in* $[a,b]$*.*

(iii) There exists a solution w *of the generalized Riccati equation* (1.1.13) *which is defined on the whole interval* $[a,b]$*.*

(iv) The energy functional $\mathcal{F}(y;a,b)$ *is positive for every* $0 \not\equiv y \in W_0^{1,p}(a,b)$*.*

Proof. To prove that all the statements in the theorem are equivalent, we prove the validity of the implications

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i).$$

The first implications $(i) \Rightarrow (ii)$ is a consequence of the continuous dependence of solutions of Eq. (1.1.1) on initial conditions. More precisely, we suppose that Eq. (1.1.1) is disconjugate and we consider a solution $\hat{x}$ of the initial value problem given by Eq. (1.1.1) and the conditions $\hat{x}(a) = \varepsilon, r(a)\Phi(\hat{x}'(a)) = 1$ with sufficiently small $\varepsilon > 0$. Then the solution $\hat{x}$ is positive on $[a,b]$

The second implication $(ii) \Rightarrow (iii)$ comes directly from the Riccati substitution

$$w(t) = r(t)\frac{\Phi(x'(t))}{\Phi(x(t))},$$

i.e., whenever there exists a solution of Eq. (1.1.1) with no zero, we obtain a solution w of Eq. (1.1.13).

The third implication $(iii) \Rightarrow (iv)$ can be proved by a direct computation. We suppose that there exists a solution w of Eq. (1.1.13) which is defined for all $t \in [a,b]$. Then we have for $x \in W_0^{1,p}(a,b)$

$$\begin{aligned}\mathcal{F}(x;a,b) &= \int_a^b r(t)|x'|^p - z(t)|x|^p \,\mathrm{d}t \\ &= \int_a^b (w(t)|x|^p)' + pr^{1-q}(t)\left[\frac{1}{p}|r^{q-1}(t)x'|^p - r^{q-1}(t)x'\Phi(x)w(t) + \frac{1}{q}|\Phi(x)w(t)|^q\right]\mathrm{d}t \\ &= p\int_a^b r^{1-q}(t)\left[\frac{1}{p}|r^{q-1}(t)x'|^p - r^{q-1}(t)x'\Phi(x)w(t) + \frac{1}{q}|\Phi(x)w(t)|^q\right]\mathrm{d}t \geq 0.\end{aligned}$$

In the above computation, we used the definition of w and the Young inequality

$$\frac{A^p}{p} - AB + \frac{B^q}{q} \geq 0$$

with $A = r^{q-1}(t)x'$ and $B = \Phi(x)w(t)$ where the equality holds for $\Phi(A) = B$. Of course, the identity $\mathcal{F}(x; a, b) = 0$ holds if and only if $\Phi(r^{q-1}(t)x') = \Phi(x)w(t)$ which can be rewritten to $x' = x\Phi^{-1}\left(\frac{w(t)}{r(t)}\right)$. Immediately, together with the fact that $x(a) = 0$, we obtain

$$x(t) = x(a) \exp\left[\int_a^t \Phi^{-1}\left(\frac{w(s)}{r(s)}\right) \mathrm{d}s\right] \equiv 0.$$

Hence, the energy functional $\mathcal{F}(y; a, b)$ is positive for every $0 \not\equiv y \in W_0^{1,p}(a, b)$ and it is equal to zero only if $y \equiv 0$.

To prove the final implication $(iv) \Rightarrow (i)$, we use a contradiction. We suppose that Eq. (1.1.1) is conjugate and, at the same time, there exists $0 \not\equiv y \in W_0^{1,p}(a, b)$ such that the energy functional $\mathcal{F}(y; a, b) > 0$. The fact that Eq. (1.1.1) is conjugate means that the solution x given by the initial conditions $x(a) = 0, r(a)\Phi(x'(a)) = 1$ has at least one zero in $(a, b]$. We denote this zero (or one from these zeros) by t_0 and we introduce the function

$$y(t) = \begin{cases} x(t) & \text{for } t \in [a, t_0], \\ 0 & \text{for } t \in [t_0, b] \end{cases}.$$

Evidently, $y \in W_0^{1,p}(a, b)$ and using integration by parts we obtain

$$\begin{aligned} \mathcal{F}(y; a, b) &= \mathcal{F}(y; a, t_0) = \mathcal{F}(x; a, t_0) \\ &= \left[r(t)x(t)\Phi(x'(t))\right]_a^{t_0} - \int_a^{t_0} x(t)\left\{[r(t)\Phi(x'(t))]' + z(t)\Phi(x(t))\right\} \mathrm{d}t = 0. \end{aligned} \tag{1.1.14}$$

Which is a contradiction and the proof is complete. □

To introduce properly the definition of (non-)oscillation of Eq. (1.1.1), we formulate and prove the half-linear Sturm comparison theorem. Since its usefulness in the upcoming chapter, we state the half-linear Sturm comparison theorem as well.

Theorem 1.1.2 (Sturm separation theorem). *Let $t_1 < t_2$ be two consecutive zeros of a nontrivial solution x of Eq.* (1.1.1). *Then any other solution of this equation which is not proportional to x has exactly one zero in the interval* (t_1, t_2).

Proof. Since t_1 and t_2 are consecutive zeros of a solution x, we can, without loss of generality, suppose that $x(t) > 0$ for $t \in (t_1, t_2)$. Hence, we can introduce a solution $w(t) = r(t)\frac{\Phi(x')}{\Phi(x)}$ of Eq. (1.1.13) defined on the interval (t_1, t_2). Then we have

$$\lim_{t \to t_1^+} w(t) = \infty, \qquad \lim_{t \to t_2^-} w(t) = -\infty.$$

Now, we suppose by contradiction that there exists a solution $\hat{x}$ of Eq. (1.1.1) with no zero between t_1 and t_2 which is linearly independent of the solution x. Hence, $\hat{x}(t_1) \neq 0 \neq \hat{x}(t_2)$. Since the solution $\hat{x}$ is non-zero for all $t \in [t_1, t_2]$, there is a solution $\hat{w}(t) = r(t)\frac{\Phi(\hat{x}')}{\Phi(\hat{x})}$ of Eq. (1.1.13) which is finite at t_1 and t_2. Especially, $\hat{w}(t_1) < \infty$ and $\hat{w}(t_2) > -\infty$. Therefore, $w(t_0) = \hat{w}(t_0)$ for some $t_0 \in (t_1, t_2)$ which contradicts the uniqueness of solutions of Eq. (1.1.13). We note that the unique solvability of Eq. (1.1.13) comes from the fact that it can be rewritten as

$$w' = -z(t) - (p-1)r^{1-q}(t)|w|^q$$

which is a first order equation whose right-hand side has the Lipschitz property with respect to w. □

Definition 1.1.2. We say that Eq. (1.1.1) is non-oscillatory (more precisely, non-oscillatory at infinity), if there exists $t_0 \in \mathbb{R}$ such that Eq. (1.1.1) is disconjugate on any interval of the form $[t_0, T]$, $T > t_0$. Otherwise, Eq. (1.1.1) is said to be oscillatory.

Theorem 1.1.3 (Sturm comparison theorem)**.** *We consider Eq.* (1.1.1) *together with the equation*

$$[R(t)\Phi(x')]' + Z(t)\Phi(x) = 0, \tag{1.1.15}$$

where $R > 0, Z$ are continuous functions. Let us denote $t_1 < t_2$ two consecutive zeros of a nontrivial solution x of Eq. (1.1.1) *and let the coefficients satisfy*

$$R(t) \leq r(t), \qquad Z(t) \geq z(t), \qquad t \in [t_1, t_2]. \tag{1.1.16}$$

Then any solution of Eq. (1.1.15) *has a zero in (t_1, t_2) or it is a multiple of the solution x. The last possibility is excluded if at least one of the inequalities in* (1.1.16) *is strict on a set of positive measure.*

Proof. Let us consider a nontrivial solution x of Eq. (1.1.1) with two consecutive zeros $t_1 < t_2$. Then, using integration by parts (analogous to (1.1.14)), we obtain $\mathcal{F}(x; t_1, t_2) = 0$. Combined with (1.1.16) it implies

$$\mathcal{F}_{RZ}(x; t_1, t_2) = \int_{t_1}^{t_2} R(t)|x'|^p - Z(t)|x|^p \, \mathrm{d}t \leq 0.$$

Using Theorem 1.1.1 (equivalency $(i) \Leftrightarrow (iv)$), we know that the solution $\hat{x}$ of Eq. (1.1.15) given by the initial conditions $\hat{x}(t_1) = 0, r(t_1)\Phi(\hat{x}'(t_1)) = 1$ has a zero in $(a, b]$. Hence, by Sturm separation theorem 1.1.2, any solution of Eq. (1.1.15) which is linearly independent on $\hat{x}$ has a zero in (t_1, t_2).

Next, we suppose that the only zero of $\hat{x}$ in $(t_1, t_2]$ is t_2. Hence, there exists a solution $\hat{w}(t) = R(t)\frac{\Phi(\hat{x}'(t))}{\Phi(\hat{x}(t))}, t \in (t_1, t_2)$, of the Riccati equation associated to Eq. (1.1.15) and we can calculate the energy functional $\mathcal{F}_{RZ}(x; t_1, t_2)$ as follows

$$\mathcal{F}_{RZ}(x; t_1, t_2) = \int_{t_1}^{t_2} R(t)|x'|^p - Z(t)|x|^p \,\mathrm{d}t$$
$$= \int_{t_1}^{t_2} (\hat{w}(t)|x|^p)' + pR^{1-q}(t)\left[\frac{|R^{q-1}(t)x'|^p}{p} - R^{q-1}(t)x'\Phi(x)\hat{w}(t) + \frac{|\Phi(x)\hat{w}(t)|^q}{q}\right] \mathrm{d}t \geq 0.$$

Indeed, we used Young inequality and the facts

$$\lim_{t\to t_1^+} \hat{w}(t)|x|^p = \lim_{t\to t_1^+} x(t)R(t)\Phi(\hat{x}')\frac{\Phi(x(t))}{\Phi(\hat{x}(t))} = 0, \qquad \lim_{t\to t_2^-} \hat{w}(t)|x|^p = 0,$$

which are consequences of the existence of the limits

$$\lim_{t\to t_1^+} \frac{x(t)}{\hat{x}(t)} = \lim_{t\to t_1^+} \frac{x'(t)}{\hat{x}'(t)} = \frac{x'(t_1)}{\hat{x}'(t_1)}, \qquad \lim_{t\to t_2^-} \frac{x(t)}{\hat{x}(t)} = \frac{x'(t_2)}{\hat{x}'(t_2)}.$$

Altogether, we obtained that $0 \leq \mathcal{F}_{RZ}(x; t_1, t_2) \leq 0$, i.e., $\mathcal{F}_{RZ}(x; t_1, t_2) = 0$. Hence, $\hat{w}(t) = R(t)\frac{\Phi(x'(t))}{\Phi(x(t))}$ (see also the proof of the implication $(iii) \Rightarrow (iv)$ of Theorem 1.1.1), i.e., the solutions x and $\hat{x}$ are proportional which cannot happen if at least one of the inequalities in (1.1.16) is strict on a set of positive measure. □

Remark 1.1.2. If inequalities in (1.1.16) hold, Eq. (1.1.1) is said to be the minorant equation of Eq. (1.1.15) and Eq. (1.1.15) is said to be the majorant equation of Eq. (1.1.1). Then we can summarize Theorem 1.1.3 as follows. If minorant equation is oscillatory then the original equation is oscillatory as well. If majorant equation is non-oscillatory then the original equation is non-oscillatory as well.

The main objective of this thesis is to study the so-called *conditional oscillation* of Eq. (1.1.1). It means that there exists the so-called *critical oscillation constant*, which is a value dependent on coefficients r and c with the following property. Any equation of the form (1.1.1) whose coefficients:

1) indicate a value greater than the critical one is oscillatory;

2) indicate a value less than the critical one is non-oscillatory.

We can reformulate the notion of conditional oscillation as follows. We say that the equation

$$[r(t)\Phi(x')]' + \gamma z(t)\Phi(x) = 0, \qquad \gamma \in \mathbb{R}, \tag{1.1.17}$$

is conditionally oscillatory if there exists a constant $\Gamma \in \mathbb{R}$ such that Eq. (1.1.17) is non-oscillatory for $\gamma < \Gamma$ and oscillatory for $\gamma > \Gamma$. This constant Γ is called the critical oscillation constant and forms a sharp borderline between oscillation and non-oscillation of Eq. (1.1.17). This fact is also one of the reasons why identifying of conditionally oscillatory equations and their critical constants is an important part in the field of oscillation theory. Using many comparison theorems (e.g., the Sturm comparison theorem 1.1.3), we can test equations which are not conditionally oscillatory towards appropriate conditionally oscillatory equation. This makes conditionally oscillatory equation ideal testing equations. Of course, a natural question is a behavior of conditionally oscillatory equations in the critical case $\gamma = \Gamma$. It turns out that some equations (e.g., with constant or periodic coefficients, see [19]) are non-oscillatory in the border case. Nevertheless, the oscillation properties of more complicated equations may be generally unsolvable (see Remark 2.1.3). In this thesis, we will solve critical cases in theorems 2.2.1, 2.3.1, and 2.5.2. Another importance of half-linear equations lies in their connection with the partial differential equations with p-Laplacian. Eq. (1.1.1) can be considered the equation with one dimensional p-Laplacian and results dealing with Eq. (1.1.1) are helpful tools in the study of more general partial differential equations (for more details and an example of such use, see § 2.1.2).

1.2 Short history overview

In this section, we collect the milestones in the theory of the conditional oscillation with respect to the topic of this thesis. It appears that appropriate half-linear equations for the study of the conditional oscillation are the Euler type equations, i.e., the equations written in the form

$$[r(t)\Phi(x')]' + \frac{\gamma s(t)}{t^p}\Phi(x) = 0 \tag{1.2.1}$$

with continuous coefficient s.

The conditional oscillation (as well as many other areas in the oscillation theory) of half-linear equations originates from the oscillation theory of linear differential equations. The first result about the conditional oscillation of the considered differential equations was obtained by A. Kneser in [52], where the oscillation constant $\Gamma = 1/4$ was found for the equation

$$x'' + \frac{\gamma}{t^2}x = 0. \tag{1.2.2}$$

More than one hundred years later, in [33, 70], the above result concerning Eq. (1.2.2) was extended for the linear equations

$$[r(t)x']' + \frac{\gamma s(t)}{t^2}x = 0 \tag{1.2.3}$$

with positive α-periodic coefficients r, s, where the critical constant is

$$\Gamma = \frac{\alpha^2}{4}\left(\int_0^\alpha \frac{\mathrm{d}\tau}{r(\tau)}\right)^{-1}\left(\int_0^\alpha s(\tau)\,\mathrm{d}\tau\right)^{-1}. \tag{1.2.4}$$

We should also mention, at least as references, papers [53, 54, 55, 56, 71] containing more general results (see also [31, 32]). Note that the critical case $\gamma = \Gamma$ of Eq. (1.2.3) was solved as non-oscillatory (see [71]).

In the field of half-linear equations, the basic critical constant

$$\Gamma = \left(\frac{p-1}{p}\right)^p \tag{1.2.5}$$

for the equation

$$[\Phi(x')]' + \frac{\gamma}{t^p}\Phi(x) = 0$$

comes from [26] (see also [27]). Then, in [34, 38], the conditional oscillation was proved for more general equations of the form

$$[r(t)\Phi(x')]' + \frac{\gamma s(t)}{t^p}\Phi(x) = 0. \tag{1.2.6}$$

Especially, the critical constant of Eq. (1.2.6) with positive α-periodic functions r, s was identified as (cf. (1.2.4), (1.2.5))

$$\Gamma_1 = \left(\frac{\alpha(p-1)}{p}\right)^p \left(\int_0^\alpha r^{\frac{1}{1-p}}(\tau)\,\mathrm{d}\tau\right)^{1-p} \left(\int_0^\alpha s(\tau)\,\mathrm{d}\tau\right)^{-1} \tag{1.2.7}$$

in [34].

Let us turn our attention to the perturbed Euler type equations. The linear case of such equations with periodic coefficients is studied in [54, 71]. The half-linear case is treated in [19], where the equations

$$[r(t)\Phi(x')]' + \left[\gamma s(t) + \frac{\mu d(t)}{\log^2 t}\right]\frac{\Phi(x)}{t^p} = 0 \tag{1.2.8}$$

are analyzed for positive α-periodic coefficients r, s, and d. There is proved that, in the critical case $\gamma = \Gamma_1$ (see (1.2.7)), Eq. (1.2.8) is oscillatory for

$$\mu > \Gamma_2 := \frac{\alpha^p}{2}\left(\frac{p-1}{p}\right)^{p-1} \left(\int_0^\alpha r^{\frac{1}{1-p}}(\tau)\,\mathrm{d}\tau\right)^{1-p} \left(\int_0^\alpha d(\tau)\,\mathrm{d}\tau\right)^{-1}$$

and non-oscillatory for $\mu < \Gamma_2$. For further generalizations, we refer to [16, 17, 22, 39] (see also [28]).

Another direction of researches, which is related to the one presented here, is based on the oscillation of Euler type equations generalizing Eq. (1.2.2) in a different way. We point out at least papers [4, 5, 73, 74, 79], where the equations of the following form (and generalizations of this form)

$$x'' + f(t)g(x) = 0$$

are considered and oscillation theorems are proved.

1.3 Organization of the thesis

In this section, we briefly mention only the main result of each upcoming section. As we already mentioned above, the main part of this thesis is the analysis of half-linear differential equations which is the topic of Chapter 2. In Section 2.1, we study Euler-type equation (1.2.1) with coefficients r, s having mean values. From technical reasons, we rewrite Eq. (1.2.1) into the form

$$\left[r^{-\frac{p}{q}}(t)\Phi(x')\right]' + \frac{s(t)}{t^p}\Phi(x) = 0 \tag{1.3.1}$$

Since the coefficient r is considered positive, it does not mean any loss of generality. We prove that this equation is conditionally oscillatory and we identify the borderline of oscillation and non-oscillation. Of course, this result cover all previously known results concerning Euler-type equations (1.2.3) and (1.2.6) whose coefficients are constant, periodic, (asymptotically) almost periodic etc. This section is based on paper [35].

The critical case remains unsolved in Section 2.1 and for such general equations it is not possible to solve it in general (for details see Remark 2.1.3). Therefore, Sections 2.2 and 2.3 are devoted to this problem. Section 2.2 (based on paper [40]) deals with the study of the critical case of Eq. (1.3.1) with periodic coefficients r, s whose periods do not have to coincide (e.g., one may be rational and the other irrational). In Section 2.3, we solve the critical case for the equation

$$\left[\left(r(t) + \frac{\sum_{i=1}^m R_i(t)}{\log^2 t}\right)^{-\frac{p}{q}} \Phi\left(x'\right)\right]' + \left(s(t) + \frac{\sum_{i=1}^n S_i(t)}{\log^2 t}\right)\frac{\Phi(x)}{t^p} = 0 \tag{1.3.2}$$

with α-periodic coefficients r, s and general periodic coefficients $R_1, \ldots, R_m$, $S_1, \ldots, S_n$ defined on some interval $[a, \infty)$, $a \in \mathbb{R}^+$ (see [41]). Moreover, we are able to combine results of Section 2.3 to prove that Eq. (1.3.2) is non-oscillatory if and only if

$$\lim_{t\to\infty}\left[\frac{t}{\alpha^p}\left(\int_a^{a+\alpha} r(\tau)\,\mathrm{d}\tau\right)^{\frac{p}{q}}\left(\int_a^{a+\alpha} s(\tau)\,\mathrm{d}\tau\right) - q^{-p}t \right.$$
$$\left. + \frac{1}{t^p}\left(\int_a^{a+t}\sum_{i=1}^n S_i(\tau)\,\mathrm{d}\tau\right)\left(\int_a^{a+t} r(\tau)\,\mathrm{d}\tau\right)^{\frac{p}{q}} + \frac{p\int_a^{a+t}\sum_{i=1}^m R_i(\tau)\,\mathrm{d}\tau}{q^{p+1}\int_a^{a+t} r(\tau)\,\mathrm{d}\tau}\right] \le \frac{q^{1-p}}{2},$$

see Theorem 2.3.4.

Then we turn our attention to another type of conditionally oscillatory equations

$$\left[r(t)t^{p-1}\Phi(x')\right]' + \frac{s(t)}{t\log^p t}\Phi(x) = 0, \tag{1.3.3}$$

where $r > 0$ and s are continuous. Eq. (1.3.3) is discussed in Section 2.4 (see paper [42]). We prove that this equation remains conditionally oscillatory and we identify the sharp borderline. Remarkable is the fact that the obtained results cover any equation of the form (1.3.3) with coefficients $r > 0, s$ satisfying

$$\lim_{t\to\infty} \frac{\int_t^{t+\alpha} r^{1-q}(\tau)\,\mathrm{d}\tau}{\sqrt{t\log t}} = 0, \qquad \lim_{t\to\infty} \frac{\int_t^{t+\alpha} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t\log t}} = 0$$

for some positive α. Unfortunately, as well as above, the critical case is complicated and remains unsolved in general. Hence, we dedicate Section 2.5 to the critical case of Eq. (1.3.3) with periodic coefficients r, s whose period may differ, i.e., without any common period (see [44]).

The last section in Chapter 2 deals with the behavior of the perturbed Eq. (1.3.3). The objective of Section 2.6 is to identify the form of perturbations which preserve the conditional oscillation of the original equation and to solve its behavior (see [45]). More precisely, we perturb Eq. (1.3.3) in the critical case and we analyze (non-)oscillation of the resulting equation

$$\left[\left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]' \\ + \frac{1}{t\log^p t}\left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2}\right)\Phi(x) = 0,$$

where $r_1 > 0$ and s_1 are α-periodic continuous functions and r_2, s_2 are continuous functions satisfying

$$r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} > 0, \qquad t > \mathrm{e},$$

$$\lim_{t\to\infty} \frac{1}{\sqrt{t\log t}} \int\limits_t^{t+\alpha} |r_2(u)|\,\mathrm{d}u = 0, \qquad \lim_{t\to\infty} \frac{1}{\sqrt{t\log t}} \int\limits_t^{t+\alpha} |s_2(u)|\,\mathrm{d}u = 0.$$

Chapters 3 and 4 are devoted to a presentation of the situation in the discrete case and the case of dynamic equations on time scales. Since the discrete and time scale methods are more complicated than in the continuous case, the results are in general not as advanced as in the continuous case. Nevertheless, we succeeded to obtain a discrete version of the result from Section 2.1. This result is contained in Chapter 3 (which is based on paper [43]). We find the oscillation constant of the half-linear difference equation

$$\Delta\left[r_k\Phi(\Delta x_k)\right] + \frac{s_k}{(k+1)^{(p)}}\,\Phi(x_{k+1}) = 0,$$

where $k^{(p)}$ stands for the generalized power function, the sequence $\{r_k\}$ is a positive bounded sequence such that there exists a (positive) mean value of $\{r_k^{1-q}\}$ and the sequence $\{s_k\}$ has a mean value.

Finally, in Chapter 4 (see [46]) are treated dynamic half-linear equations on time scales

$$\left[r(t)\Phi(y^{\Delta})\right]^{\Delta} + \frac{s(t)}{t^{(p-1)}\sigma(t)}\Phi(y^{\sigma}) = 0, \tag{1.3.4}$$

where $\sigma, f^{\Delta}, t^{(p)}$ stand for the forward jump operator, Δ-derivative, and generalized power function, respectively. The considered coefficients r, s are rd-continuous, positive, and periodic. The obtained result shows that Eq. (1.3.4) is conditionally oscillatory and reveals its critical constant. In spite of the fact that this result is much weaker than the results in continuous and discrete case, as far as we know, it is the most general result about conditional oscillation on time scales and it could be a step to this rich field.

Chapter 2 Differential equations

2.1 Equations with coefficients having mean values

In this section, we will study Eq. (1.1.1) in the form

$$\left[r^{-\frac{p}{q}}(t)\Phi(x')\right]' + \frac{s(t)}{t^p}\Phi(x) = 0, \tag{2.1.1}$$

where $r : \mathbb{R}_a \to \mathbb{R}$ is a continuous function having mean value $M(r) = 1$ and satisfying

$$0 < r^- := \inf_{t\in\mathbb{R}_a} r(t) \le r^+ := \sup_{t\in\mathbb{R}_a} r(t) < \infty \tag{2.1.2}$$

and $s : \mathbb{R}_a \to \mathbb{R}$ is a continuous function having mean value $M(s) > 0$.

The Riccati equation associated to Eq. (2.1.1) has the form (see (1.1.13))

$$w'(t) + \frac{s(t)}{t^p} + (p-1)r(t)|w(t)|^q = 0. \tag{2.1.3}$$

Finally, using the substitution $w(t) = -\zeta(t)t^{1-p}$, we obtain the adapted Riccati equation

$$\zeta'(t) = \frac{1}{t}\left[(p-1)\zeta(t) + s(t) + (p-1)r(t)|\zeta(t)|^q\right], \tag{2.1.4}$$

which will play a crucial role in the proof of the announced result (see the below given Theorem 2.1.3).

To prove the main results, we will apply the Riccati technique for Eq. (1.1.1). The fundamental connection between the non-oscillation of Eq. (1.1.1) and the solvability of Eq. (1.1.13) is described by the following theorem.

Theorem 2.1.1. *Eq.* (1.1.1) *is non-oscillatory if and only if there exists a function w which solves Eq.* (1.1.13) *on some interval* $[T, \infty)$.

Proof. The theorem is a consequence of the Reid roundabout theorem 1.1.1. □

We will also use the Sturm comparison theorem in the form given below.

Theorem 2.1.2. *Let $z, Z : \mathbb{R}_a \to \mathbb{R}$ be continuous functions satisfying $Z(t) \ge z(t)$ for all sufficiently large t. Let us consider Eq.* (1.1.1) *and the equation*

$$[r(t)\Phi(x')]' + Z(t)\Phi(x) = 0. \tag{2.1.5}$$

(i) *If Eq.* (1.1.1) *is oscillatory, then Eq.* (2.1.5) *is oscillatory as well.*

(ii) *If Eq.* (2.1.5) *is non-oscillatory, then Eq.* (1.1.1) *is non-oscillatory as well.*

Proof. The theorem is a (weaker) reformulation of Theorem 1.1.3 (see Remark 1.1.2). □

Now, we recall the concept of mean values which is necessary to find an explicit oscillation constant for general half-linear equations.

Definition 2.1.1. Let a continuous function $f : \mathbb{R}_a \to \mathbb{R}$ be such that the limit

$$M(f) := \lim_{t\to\infty} \frac{1}{t} \int_b^{b+t} f(s)\,\mathrm{d}s \tag{2.1.6}$$

is finite and exists uniformly with respect to $b \in \mathbb{R}_a$. The number $M(f)$ is called the mean value of f.

§ 2.1.1 Results and examples

To prove the announced result, we need the following lemmata.

Lemma 2.1.1. *If there exists a solution of Eq.* (2.1.4) *on some interval* $[T, \infty)$, *then Eq.* (2.1.1) *is non-oscillatory.*

Proof. A solution ζ of Eq. (2.1.4) on an interval $[T, \infty)$ gives the solution $w(t) = -\zeta(t)t^{1-p}$ of Eq. (2.1.3) on the same interval. Thus, the lemma follows from Theorem 2.1.1. □

Lemma 2.1.2. *Let Eq.* (2.1.1) *be non-oscillatory and let there exists* $M > 0$ *such that*

$$\left| \int_b^c \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \right| < M, \quad a \le b < c \le \infty. \tag{2.1.7}$$

For any solution w *of Eq.* (2.1.3) *on* $[T, \infty)$, *it holds*

$$\int_T^\infty r(\tau)\,|w(\tau)|^q\,\mathrm{d}\tau < \infty. \tag{2.1.8}$$

Proof. The lemma follows, e.g., from [21, Theorem 2.2.3], where it suffices to use (2.1.2). □

Lemma 2.1.3. *If Eq.* (2.1.1) *is non-oscillatory, then there exists a solution* ζ *of Eq.* (2.1.4) *on some interval* $[T, \infty)$ *with the property that* $\zeta(t) < A$ *for all* $t \ge T$ *and for some* $A > 0$.

Proof. Considering Theorem 2.1.1, the non-oscillation of Eq. (2.1.1) implies that there exists a solution w of Eq. (2.1.3) on some interval $[T,\infty)$ which gives the solution $\zeta(t) = -w(t)t^{p-1}$ of Eq. (2.1.4) on the interval. We show that this solution ζ is bounded above.

At first, we prove the convergence of the integral

$$\int_T^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \in \mathbb{R} \tag{2.1.9}$$

and the inequality

$$\sup_{t\geq T}\left| t^{p-1}\int_t^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau\right| < L \quad \text{for some } L>0. \tag{2.1.10}$$

Evidently, it suffices to prove (2.1.9) and

$$\limsup_{t\to\infty}\left| t^{p-1}\int_t^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau\right| < \infty. \tag{2.1.11}$$

Let $b>0$ be such that

$$\int_t^{t+b} s(\tau)\,\mathrm{d}\tau > 0, \qquad \left|\int_t^{t+b} s(\tau)\,\mathrm{d}\tau - bM(s)\right| < b, \quad t\geq T, \tag{2.1.12}$$

where we use directly Definition 2.1.1 (the existence of $M(s)>0$). The symbol $[f(\cdot)]^+$ and $[f(\cdot)]^-$ will denote the positive and negative part of function f, respectively. We choose $t_0\geq T$. We can express

$$\int_{t_0+(k-1)b}^{t_0+kb} \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau = \int_{t_0+(k-1)b}^{t_0+kb} \frac{[s(\tau)]^+}{\tau^p}\,\mathrm{d}\tau - \int_{t_0+(k-1)b}^{t_0+kb} \frac{[s(\tau)]^-}{\tau^p}\,\mathrm{d}\tau, \quad k\in\mathbb{N}.$$

For an arbitrarily given positive integer k, we have

$$\begin{aligned} I_k := \int_{t_0+(k-1)b}^{t_0+kb} \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \leq \frac{1}{[t_0+(k-1)b]^p} &\int_{t_0+(k-1)b}^{t_0+kb} [s(\tau)]^+\,\mathrm{d}\tau \\ &- \frac{1}{[t_0+kb]^p}\int_{t_0+(k-1)b}^{t_0+kb} [s(\tau)]^-\,\mathrm{d}\tau \end{aligned} \tag{2.1.13}$$

if $I_k > 0$, and

$$\begin{aligned} -\int\limits_{t_0+(k-1)b}^{t_0+kb} \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \le -\frac{1}{[t_0+kb]^p} \int\limits_{t_0+(k-1)b}^{t_0+kb} [s(\tau)]^+\,\mathrm{d}\tau \\ + \frac{1}{[t_0+(k-1)b]^p} \int\limits_{t_0+(k-1)b}^{t_0+kb} [s(\tau)]^-\,\mathrm{d}\tau \end{aligned} \tag{2.1.14}$$

if $I_k < 0$. Using

$$\lim_{k\to\infty} \frac{[t_0+(k-1)b]^p}{[t_0+kb]^p} = 1$$

and using (2.1.12), (2.1.13), (2.1.14), we obtain the existence of $n_0 \in \mathbb{N}$ such that it holds

$$\left| \int\limits_{t_0+(k-1)b}^{t_0+kb} \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \right| < \frac{2b[M(s)+1]}{[t_0+(k-1)b]^p}, \quad k \ge n_0,\ k \in \mathbb{N}.$$

Since $t_0 \ge T$ is arbitrary, it also holds

$$\left| \int\limits_{t}^{t+b} \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \right| < \frac{2b[M(s)+1]}{t^p} \tag{2.1.15}$$

for all sufficiently large t.

Hence, the integral $\int_T^\infty s(\tau)\tau^{-p}\,\mathrm{d}\tau$ is convergent. Especially,

$$\sup_{t\ge T} \left| \int\limits_{T}^{t} \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \right| < K \quad \text{for some } K > 0. \tag{2.1.16}$$

Moreover, we have (see (2.1.15))

$$\begin{aligned} \limsup_{t\to\infty} \left| t^{p-1} \int\limits_{t}^{\infty} \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \right| &= \limsup_{t\to\infty} \left| t^{p-1} \sum_{k=1}^{\infty} \left(\int_{t+(k-1)b}^{t+kb} \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \right) \right| \\ &\le \limsup_{t\to\infty} \left| t^{p-1} \sum_{k=1}^{\infty} \frac{2b[M(s)+1]}{[t+(k-1)b]^p} \right| < \infty. \end{aligned}$$

Thus, (2.1.11) is valid, i.e., there exists $L > 0$ for which (2.1.10) is valid.

Integrating Eq. (2.1.3), we obtain

$$w(t) = w(T) - \int\limits_{T}^{t} \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau - (p-1)\int\limits_{T}^{t} r(\tau)\,|w(\tau)|^q\,\mathrm{d}\tau, \quad t \ge T. \tag{2.1.17}$$

We know that

$$\int_T^\infty r(\tau)\,|w(\tau)|^q\,\mathrm{d}\tau < \infty, \quad \text{i.e. (see (2.1.2))}, \quad \int_T^\infty |w(\tau)|^q\,\mathrm{d}\tau < \infty. \tag{2.1.18}$$

Indeed, considering (2.1.7) together with (2.1.16), one can get (2.1.8) from Lemma 2.1.2. From (2.1.9) and (2.1.17) it follows that there exists the limit $\lim_{t\to\infty} w(t) \in \mathbb{R}$. In addition, the convergence of the integral in (2.1.18) gives

$$\lim_{t\to\infty} |w(t)|^q = 0, \quad \text{i.e.,} \quad \lim_{t\to\infty} w(t) = 0. \tag{2.1.19}$$

Again, we consider arbitrarily given $t_0 \geq T$. We can rewrite Eq. (2.1.17) into (or see directly Eq. (2.1.3))

$$w(t) = w(t_0) - \int_{t_0}^t \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau - (p-1)\int_{t_0}^t r(\tau)\,|w(\tau)|^q\,\mathrm{d}\tau, \quad t \geq T. \tag{2.1.20}$$

Putting $t_0 \to \infty$, from (2.1.9), (2.1.18), (2.1.19), and (2.1.20), we obtain

$$w(t) = \int_t^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau + (p-1)\int_t^\infty r(\tau)\,|w(\tau)|^q\,\mathrm{d}\tau, \quad t \geq T. \tag{2.1.21}$$

Finally, let us denote $w(t) = f_1(t) + f_2(t)$, where

$$f_1(t) := \int_t^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau, \quad f_2(t) := (p-1)\int_t^\infty r(\tau)\,|w(\tau)|^q\,\mathrm{d}\tau, \quad t \geq T.$$

We know that (see (2.1.10) and (2.1.21))

$$\sup_{t\geq T} t^{p-1}\,|\,w(t) - f_2(t)\,| < L. \tag{2.1.22}$$

We denote $S := \{t \geq T : w(t) < 0\}$. If w is positive, then the statement of the lemma is true for all $A > 0$. Therefore, we can assume that $S \neq \emptyset$. Since f_2 is non-increasing and $\lim_{t\to\infty} f_2(t) = 0$, function f_2 is nonnegative. From (2.1.22) it follows

$$\sup_{t\in S} t^{p-1}\left([w(t)]^- + f_2(t)\right) = \sup_{t\in S} t^{p-1}\,|\,w(t) - f_2(t)\,| < L.$$

Hence, we have

$$\sup_{t\in S} t^{p-1}\,|w(t)| = \sup_{t\in S} t^{p-1}\left([w(t)]^- + 0\right) \leq \sup_{t\in S} t^{p-1}\left([w(t)]^- + f_2(t)\right) < L$$

and, consequently, we obtain that $\zeta(t) = -t^{p-1}w(t) < L$, $t \geq T$. It means that the statement of the lemma is valid for $A = L$. $\square$

Remark 2.1.1. Let Eq. (2.1.1) be non-oscillatory. If the considered function s is positive for all $t \geq a$, then the statement of Lemma 2.1.3 is true for a negative solution ζ of Eq. (2.1.4). See, e.g., [21, Lemma 2.2.5].

Theorem 2.1.3. *Eq.* (2.1.1) *is oscillatory for* $M(s) > q^{-p}$*, and non-oscillatory for* $M(s) < q^{-p}$.

Proof. The proof is organized as follows. In the first part, we derive upper bounds for two integrals involving function s. Then we prove the oscillatory part and, finally, the non-oscillatory part.

At first, we use the existence of $M(s)$ and the continuity of function s. There exists $\beta \geq 1$ with the property that

$$\left| \int_b^{b+\beta} s(\tau)\, \mathrm{d}\tau \right| < \beta \left[M(s) + 1 \right], \quad b \in [a, \infty), \tag{2.1.23}$$

$$\left| \frac{1}{\beta} \int_b^{b+\beta} s(\tau)\, \mathrm{d}\tau - \frac{1}{\beta + \xi} \int_b^{b+\beta+\xi} s(\tau)\, \mathrm{d}\tau \right| < 1, \quad b \in [a, \infty),\ \xi \in (0, 1], \tag{2.1.24}$$

and, consequently, there exists $R > 0$ with the property that

$$\left| \int_c^{c+\xi} s(\tau)\, \mathrm{d}\tau \right| < R, \quad c \in [a, a + \beta],\ \xi \in (0, 1]. \tag{2.1.25}$$

We can rewrite (2.1.24) into the form

$$\left| \frac{\xi}{\beta + \xi} \int_b^{b+\beta} s(\tau)\, \mathrm{d}\tau - \frac{\beta}{\beta + \xi} \int_{b+\beta}^{b+\beta+\xi} s(\tau)\, \mathrm{d}\tau \right| < \beta, \quad b \in [a, \infty),\ \xi \in (0, 1].$$

Using (2.1.23), we obtain

$$\left| \frac{\beta}{\beta + \xi} \int_{b+\beta}^{b+\beta+\xi} s(\tau)\, \mathrm{d}\tau \right| < \beta + \beta[M(s) + 1], \quad b \in [a, \infty),\ \xi \in (0, 1],$$

i.e.,

$$\left| \int_{b+\beta}^{b+\beta+\xi} s(\tau)\, \mathrm{d}\tau \right| < 2\beta \left[M(s) + 2 \right], \quad b \in [a, \infty),\ \xi \in (0, 1]. \tag{2.1.26}$$

Combining (2.1.25) and (2.1.26), we have

$$\left| \int_{b}^{b+\xi} s(\tau)\ \mathrm{d}\tau \right| < S, \quad b \in [a,\infty),\ \xi \in (0,1], \tag{2.1.27}$$

where $S := \max\{R, 2\beta[M(s)+2]\}$. Since the function $y(t) = 1/t$ is decreasing and positive on $\mathbb{R}_a$, it holds

$$\int_{t_1}^{t_2} \frac{s(\tau)}{\tau}\ \mathrm{d}\tau = \frac{1}{t_1}\int_{t_1}^{t_3} s(\tau)\ \mathrm{d}\tau + \frac{1}{t_2}\int_{t_3}^{t_2} s(\tau)\ \mathrm{d}\tau$$

for all $t_2 \geq t_1 \geq a$ and for some $t_3 \in [t_1, t_2]$. Analogously, for any $t_2 \geq t_1 \geq a$, there exists $t_4 \in [t_1, t_2]$ such that

$$\int_{t_1}^{t_2} \frac{s(\tau)}{\tau}\ \mathrm{d}\tau = \frac{1}{t_1}\int_{t_1}^{t_4} s(\tau)\ \mathrm{d}\tau.$$

Hence, from (2.1.27) it follows

$$\left| \int_{b}^{b+\xi} \frac{s(\tau)}{\tau}\ \mathrm{d}\tau \right| < \frac{S}{b}, \quad b \in [a,\infty),\ \xi \in (0,1]. \tag{2.1.28}$$

Now, we prove the oscillatory part. Let $M(s) > q^{-p}$. By contradiction, in this part of the proof, we will suppose that Eq. (2.1.1) is non-oscillatory. Lemma 2.1.3 says that there exists a solution ζ of Eq. (2.1.4) on some interval $[T,\infty)$ and that $\zeta(t) < A$ for all $t \geq T$ and for a certain number $A > 0$. Evidently, we can assume that $T > 1$.

We show that there exists $K < -1$ satisfying

$$\zeta(t) > K, \quad t \geq T. \tag{2.1.29}$$

On contrary, let us assume that $\liminf_{t\to\infty} \zeta(t) = -\infty$. Let $\zeta(t) \leq -P$ for all t from some interval $[t_1, t_2]$, where $t_2 \in (t_1 + j - 1, t_1 + j]$, $j \in \mathbb{N}$, and let $P > 0$ be such that (see (2.1.2))

$$h(x) := (p-1)r^{-}|x|^{q} - S - (p-1)|x| > 0, \quad |x| \geq P. \tag{2.1.30}$$

Indeed, $\lim_{x\to\pm\infty} h(x) = \infty$. We can assume that h is increasing for $x \geq P$. Using (2.1.4), (2.1.28), (2.1.30), it holds

$$\begin{aligned}
\zeta(t_2) - \zeta(t_1) &= \int\limits_{t_1}^{t_2} \zeta'(\tau)\,\mathrm{d}\tau = \int\limits_{t_1}^{t_2} \frac{(p-1)\zeta(\tau) + s(\tau) + (p-1)r(\tau)|\zeta(\tau)|^q}{\tau}\,\mathrm{d}\tau \\
&\geq \int\limits_{t_1}^{t_2} \frac{-(p-1)P + (p-1)r^- P^q}{\tau}\,\mathrm{d}\tau - \left| \int\limits_{t_1}^{t_2} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau \right| \\
&\geq \int\limits_{t_1}^{t_1+1} \frac{-(p-1)P + (p-1)r^- P^q}{\tau}\,\mathrm{d}\tau - \left| \int\limits_{t_1}^{t_1+1} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau \right| + \cdots \\
&\qquad \cdots + \int\limits_{t_1+j-2}^{t_1+j-1} \frac{-(p-1)P + (p-1)r^- P^q}{\tau}\,\mathrm{d}\tau - \left| \int\limits_{t_1+j-2}^{t_1+j-1} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau \right| \\
&\qquad + \int\limits_{t_1+j-1}^{t_2} \frac{-(p-1)P + (p-1)r^- P^q}{\tau}\,\mathrm{d}\tau - \left| \int\limits_{t_1+j-1}^{t_2} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau \right| \\
&> \frac{S}{t_1+1} - \frac{S}{t_1} + \cdots + \frac{S}{t_1+j-1} - \frac{S}{t_1+j-2} + 0 - \frac{S}{t_1+j-1} = -\frac{S}{t_1}.
\end{aligned}$$

Thus, $\zeta(t) \geq -P - ST^{-1}$ for all $t \geq T$ which proves (2.1.29). Indeed, it suffices to consider $\zeta(t_1) = -P$. In addition, we can assume that $-K > A$, i.e.,

$$K < \zeta(t) < -K, \quad t \geq T. \tag{2.1.31}$$

Thus (see directly (2.1.4) and (2.1.28)), we have

$$\begin{aligned}
|\,\zeta(t_2) - \zeta(t_1)\,| = \left| \int\limits_{t_1}^{t_2} \zeta'(\tau)\,\mathrm{d}\tau \right| &\leq \int\limits_{t_1}^{t_2} \frac{p-1}{\tau}\left[|\zeta(\tau)| + r(\tau)|\zeta(\tau)|^q\right]\mathrm{d}\tau + \left| \int\limits_{t_1}^{t_2} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau \right| \\
&\leq \frac{1}{t_1}\left[(p-1)|K|\,(t_2-t_1) + S + (p-1)r^+|K|^q\,(t_2-t_1)\right]
\end{aligned}$$

for all $t_2 > t_1 \geq T$, where $t_2 \leq t_1 + 1$. The previous inequality implies

$$|\,\zeta(t_2) - \zeta(t_1)\,| \leq \frac{(p-1)|K| + S + (p-1)r^+|K|^q}{t_1}, \quad t_1 \geq T,\, t_2 \in (t_1, t_1+1]. \tag{2.1.32}$$

Considering Definition 2.1.1 and $M(s) > q^{-p}$, there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\frac{1}{n}\int\limits_{b}^{b+n} s(\tau)\,\mathrm{d}\tau - q^{-p} > \varepsilon, \quad b \in [a, \infty), \tag{2.1.33}$$

and, at the same time, such that

$$1 - \frac{\varepsilon}{4(p-1)|K|^q} < \frac{1}{n} \int\limits_b^{b+n} r(\tau)\, \mathrm{d}\tau < 2, \quad b \in [a, \infty). \tag{2.1.34}$$

For such an integer n, we define

$$\vartheta(t) := \frac{1}{n} \int\limits_t^{t+n} \zeta(\tau)\, \mathrm{d}\tau, \quad t \geq T. \tag{2.1.35}$$

Since

$$\frac{1}{n} \int\limits_t^{t+n} \zeta(\tau)\, \mathrm{d}\tau < \frac{1}{n} \int\limits_t^{t+n} A\, \mathrm{d}\tau = A, \quad t \geq T,$$

we have

$$\vartheta(t) < A, \quad t \geq T. \tag{2.1.36}$$

Hence, to prove the first implication in the statement of the theorem, it suffices to show that (2.1.36) is not true.

From (2.1.32) it follows

$$|\, \zeta(t+\tau) - \zeta(t)\, | \leq \frac{C}{t}, \quad t \geq T, \tau \in [0, n], \tag{2.1.37}$$

where

$$C = n \left[(p-1)|K| + S + (p-1) r^+ |K|^q \right].$$

Especially (see (2.1.35)), (2.1.37) gives

$$|\, \vartheta(t) - \zeta(t)\, | \leq \frac{C}{t}, \quad t \geq T. \tag{2.1.38}$$

Next, we consider the function

$$g_1(t) := (p-1)\zeta(t) + q^{-p} + \frac{p-1}{n} \int\limits_t^{t+n} r(\tau)\, |\zeta(\tau)|^q\, \mathrm{d}\tau, \quad t \geq T.$$

If $\zeta(t) \geq 0$ for some $t \geq T$, then $g_1(t) > 0$. Henceforth (in this paragraph), we consider the case when $\zeta(t) < 0$, $t \geq T$. Let us define

$$f(x) := x + \frac{q^{-p}}{p-1} + (-x)^q, \quad x \leq 0.$$

It can be directly verified that function f has the global minimum

$$f\left(-q^{\frac{1}{1-q}}\right) = -q^{\frac{1}{1-q}} + \frac{q^{-p}}{p-1} + q^{\frac{1+q-1}{1-q}} = q^{\frac{1}{1-q}}\left(-1 + q^{\frac{q-1}{1-q}}\right) + \frac{q^{-\frac{q}{q-1}}}{\frac{q}{q-1}-1}$$
$$= q^{\frac{1}{1-q}}\left(\frac{1}{q}-1\right) + \frac{q^{\frac{1+q-1}{1-q}}}{\frac{q}{q-1}-1} = q^{\frac{1}{1-q}}\left[\frac{1-q}{q} + \frac{q^{-1}}{\frac{q-q+1}{q-1}}\right] = q^{\frac{1}{1-q}}\left[\frac{1-q}{q} + \frac{q-1}{q}\right] = 0.$$

It means that $f(x) \geq 0$, $x \leq 0$. Especially, it gives the inequality

$$(p-1)\zeta(t) + q^{-p} + (p-1)\,|\zeta(t)|^q \geq 0, \quad t \geq T. \tag{2.1.39}$$

Considering (2.1.34) and (2.1.39), we have

$$g_2(t) := (p-1)\zeta(t) + q^{-p} + \frac{p-1}{n}\int_t^{t+n} r(\tau)\,|\zeta(t)|^q \,\mathrm{d}\tau \geq -\frac{\varepsilon}{4}, \quad t \geq T. \tag{2.1.40}$$

Applying (2.1.37) and the inequalities $K < \zeta(t) < A$, $t \geq T$, there exists $\tilde{t} \geq T$ such that

$$|\,|\zeta(t)|^q - |\zeta(t+\tau)|^q\,| < \frac{\varepsilon}{8(p-1)}, \quad t \geq \tilde{t},\ \tau \in [t, t+n].$$

Hence (see also (2.1.34)), we get

$$\begin{aligned} |\,g_1(t) - g_2(t)\,| &\leq \frac{p-1}{n}\int_t^{t+n} r(\tau)\,|\,|\zeta(t)|^q - |\zeta(\tau)|^q\,|\,\mathrm{d}\tau \\ &\leq \frac{p-1}{n}\int_t^{t+n} r(\tau)\frac{\varepsilon}{8(p-1)}\,\mathrm{d}\tau < 2\,(p-1)\,\frac{\varepsilon}{8(p-1)} = \frac{\varepsilon}{4}, \quad t \geq \tilde{t}. \end{aligned} \tag{2.1.41}$$

From (2.1.40) and (2.1.41), we know that

$$g_1(t) \geq -\frac{\varepsilon}{2}, \quad t \geq \tilde{t}. \tag{2.1.42}$$

Of course, (2.1.42) remains true for $\zeta(t) \geq 0$ as well.

Let us consider $\bar{t} \geq \tilde{t}$ for which

$$|\,\zeta(t) - \vartheta(t)\,| < \frac{\varepsilon}{4(p-1)}, \quad t \geq \bar{t}. \tag{2.1.43}$$

Note that the existence of such a number $\bar{t}$ follows from (2.1.38). It is seen that (2.1.42) and (2.1.43) imply

$$(p-1)\vartheta(t) + q^{-p} + \frac{p-1}{n}\int_t^{t+n} r(\tau)\,|\zeta(\tau)|^q\,\mathrm{d}\tau \geq -\frac{3\varepsilon}{4}, \quad t \geq \bar{t}. \tag{2.1.44}$$

Evidently, we can consider the solution ζ in an arbitrarily given neighborhood of $+\infty$. Hence, we can assume that

$$\frac{Sn}{\bar{t}} < \frac{\varepsilon}{24}, \tag{2.1.45}$$
$$\frac{-Kn(p-1)}{\bar{t}} < \frac{\varepsilon}{24}, \tag{2.1.46}$$
$$\frac{r^{+}(-K)^{q}n(p-1)}{\bar{t}} < \frac{\varepsilon}{24}. \tag{2.1.47}$$

From (2.1.27) and (2.1.45), we see

$$\begin{aligned} \left| \frac{1}{n} \int_{t}^{t+n} \frac{s(\tau)}{\tau} \,\mathrm{d}\tau - \frac{1}{n} \int_{t}^{t+n} \frac{s(\tau)}{t} \,\mathrm{d}\tau \right| &= \frac{1}{n} \left| \int_{t}^{t+n} s(\tau) \left(\frac{1}{\tau} - \frac{1}{t} \right) \mathrm{d}\tau \right| \\ &\leq \frac{1}{n} \left| \int_{t}^{t+n} s(\tau) \,\mathrm{d}\tau \right| \frac{n}{t^{2}} < \frac{Sn}{t^{2}} < \frac{\varepsilon}{24t}, \quad t \geq \bar{t}, \end{aligned} \tag{2.1.48}$$

from (2.1.31) and (2.1.46), we have

$$\begin{aligned} \left| \frac{p-1}{n} \int_{t}^{t+n} \frac{\zeta(\tau)}{\tau} \,\mathrm{d}\tau - \frac{p-1}{n} \int_{t}^{t+n} \frac{\zeta(\tau)}{t} \,\mathrm{d}\tau \right| &= \frac{p-1}{n} \left| \int_{t}^{t+n} \zeta(\tau) \left(\frac{1}{\tau} - \frac{1}{t} \right) \mathrm{d}\tau \right| \\ &\leq \frac{p-1}{t^{2}} \int_{t}^{t+n} |\zeta(\tau)| \,\mathrm{d}\tau < \frac{-Kn(p-1)}{t^{2}} < \frac{\varepsilon}{24t}, \quad t \geq \bar{t}, \end{aligned} \tag{2.1.49}$$

and, analogously, from (2.1.2), (2.1.31), and (2.1.47) it follows

$$\begin{aligned} &\left| \frac{p-1}{n} \int_{t}^{t+n} r(\tau) \,\frac{|\zeta(\tau)|^{q}}{\tau} \,\mathrm{d}\tau - \frac{p-1}{n} \int_{t}^{t+n} r(\tau) \frac{|\zeta(\tau)|^{q}}{t} \,\mathrm{d}\tau \right| \\ &\qquad = \frac{p-1}{n} \int_{t}^{t+n} r(\tau) \,|\zeta(\tau)|^{q} \left(\frac{1}{t} - \frac{1}{\tau} \right) \mathrm{d}\tau \\ &\qquad \leq \frac{p-1}{t^{2}} \int_{t}^{t+n} r(\tau) \,|\zeta(\tau)|^{q} \,\mathrm{d}\tau < \frac{r^{+}(-K)^{q}n(p-1)}{t^{2}} < \frac{\varepsilon}{24t}, \quad t \geq \bar{t}. \end{aligned} \tag{2.1.50}$$

For all $t \geq \bar{t}$, using (2.1.33), (2.1.44), (2.1.48), (2.1.49), and (2.1.50), we obtain

$$\begin{aligned}
\vartheta'(t) &= \frac{1}{n}\int_t^{t+n} \zeta'(\tau)\ \mathrm{d}\tau = \frac{1}{n}\int_t^{t+n} \frac{(p-1)\zeta(\tau)+s(\tau)+(p-1)r(\tau)\,|\zeta(\tau)|^q}{\tau}\ \mathrm{d}\tau \\
&= \frac{1}{n}\int_t^{t+n} \frac{s(\tau)}{\tau}\ \mathrm{d}\tau - \frac{q^{-p}}{t} + \frac{p-1}{n}\int_t^{t+n}\frac{\zeta(\tau)}{\tau}\ \mathrm{d}\tau + \frac{q^{-p}}{t} + \frac{p-1}{n}\int_t^{t+n} r(\tau)\frac{|\zeta(\tau)|^q}{\tau}\ \mathrm{d}\tau \\
&> \frac{1}{n}\int_t^{t+n} \frac{s(\tau)}{t}\ \mathrm{d}\tau - \frac{q^{-p}}{t} + \frac{p-1}{n}\int_t^{t+n}\frac{\zeta(\tau)}{t}\ \mathrm{d}\tau + \frac{q^{-p}}{t} + \frac{p-1}{n}\int_t^{t+n} r(\tau)\frac{|\zeta(\tau)|^q}{t}\ \mathrm{d}\tau - \frac{\varepsilon}{8t} \\
&= \frac{1}{t}\left[\frac{1}{n}\int_t^{t+n} s(\tau)\ \mathrm{d}\tau - q^{-p} + (p-1)\vartheta(t) + q^{-p} + \frac{p-1}{n}\int_t^{t+n} r(\tau)\,|\zeta(\tau)|^q\ \mathrm{d}\tau - \frac{\varepsilon}{8}\right] > \frac{\varepsilon}{8t}.
\end{aligned}$$

Thus, it holds

$$\vartheta(t) - \vartheta(\bar{t}) = \int_{\bar{t}}^{t} \vartheta'(\tau)\ \mathrm{d}\tau \geq \int_{\bar{t}}^{t} \frac{\varepsilon}{8\tau}\ \mathrm{d}\tau, \quad t \geq \bar{t}.$$

Since

$$\lim_{t\to\infty} \int_{\bar{t}}^{t} \frac{\varepsilon}{8\tau}\ \mathrm{d}\tau = \infty,$$

we obtain that $\lim_{t\to\infty} \vartheta(t) = \infty$. The contradiction with (2.1.36) proves the first implication.

In the non-oscillatory part of the proof, we consider $M(s) < q^{-p}$. Let $n \in \mathbb{N}$ and $\varepsilon > 0$ satisfy

$$\frac{1}{n}\int_b^{b+n} s(\tau)\ \mathrm{d}\tau - q^{-p} < -\varepsilon, \quad b \in [a,\infty). \tag{2.1.51}$$

Let us consider solution ζ of Eq. (2.1.4) given by $\zeta(T_0) = -1$ for some sufficiently large $T_0 \geq a$. Since the right-hand side of Eq. (2.1.4) is continuous, the considered solution ζ can be defined on an interval $[T_0, T_1)$, where $T_0 < T_1 \leq \infty$. In addition, if $T_1 < \infty$, we can assume that

$$\limsup_{t\to T_1^-} |\zeta(t)| = \infty. \tag{2.1.52}$$

If $T_1 = \infty$, then the considered solution of Eq. (2.1.4) satisfies the condition of Lemma 2.1.1. It means that it suffices to find $B, C \in \mathbb{R}$ for which

$$B \leq \zeta(t) \leq C, \quad t \in [T_0, T_1). \tag{2.1.53}$$

As in the oscillatory part of the proof (see (2.1.29)), we can prove that $\zeta(t) > K$ for some $K < -1$ and for all $t \in [T_0, T_1)$. Indeed, we can analogously show that the inequality

$\zeta(t) < -P - ST_0^{-1}$ cannot be valid for any $t \in [T_0, T_1)$, where S is taken from (2.1.27) and P from (2.1.30). We want to prove that $T_1 = \infty$. On contrary, let (2.1.52) be valid for some $T_1 \in \mathbb{R}$. Especially, solution ζ has to be positive on some interval $[T_2, T_3] \subset [T_0, T_1)$ in this case.

We denote

$$\tilde{t} := -\left(\frac{p}{p-1}\right)^{1-p} = -q^{1-p} \tag{2.1.54}$$

and we compute

$$(p-1)\tilde{t} + q^{-p} + (p-1)\left|\tilde{t}\right|^q = (1-p)q^{1-p} + pq^{-p} = q^{-p}\left[(1-p)q + p\right] = 0. \tag{2.1.55}$$

We know that ζ is negative on an interval $[T_0, \tilde{T_1}) \subseteq [T_0, T_1)$. Let $\tilde{T_1}$ have the property that $\zeta(\tilde{T_1}) = 0$. For all $t_1, t_2 \in [T_0, \tilde{T_1}]$, $t_1 \le t_2 \le t_1 + 1$, we have (see (2.1.28))

$$\begin{aligned} |\,\zeta(t_2) - \zeta(t_1)\,| &= \left|\int_{t_1}^{t_2} \zeta'(\tau)\ \mathrm{d}\tau\right| \le (p-1)\int_{t_1}^{t_2} \frac{|\zeta(\tau)| + r^+|\zeta(\tau)|^q}{\tau}\ \mathrm{d}\tau + \left|\int_{t_1}^{t_2} \frac{s(\tau)}{\tau}\ \mathrm{d}\tau\right| \\ &\le (p-1)\frac{|K| + r^+|K|^q}{t_1} + \frac{S}{t_1} = \frac{(p-1)\left[|K| + r^+|K|^q\right] + S}{t_1}. \end{aligned}$$

Thus, for general $t_1, t_2 \in [T_0, \tilde{T_1}]$ satisfying $t_1 \le t_2 \le t_1 + 2n$, we have

$$|\,\zeta(t_2) - \zeta(t_1)\,| \le \frac{2n\left([p-1]\left[|K| + r^+|K|^q\right] + S\right)}{t_1}. \tag{2.1.56}$$

We can assume that T_0 is so large that

$$\zeta(\tau) < 0, \quad \tau \in [t, t+2n] \subset [T_0, T_1),$$

if $\zeta(t) \le \tilde{t}$ (see (2.1.54)). Namely ($\zeta(T_0) = -1 < \tilde{t}$), we can define the function

$$\vartheta(t) := \frac{1}{n}\int_{t}^{t+n} \zeta(\tau)\ \mathrm{d}\tau$$

for all $t \in [T_0, T_0 + n)$ and for all $t \ge T_0 + n$ when $\zeta(t-n) \le \tilde{t}$. Especially, let T_0 be so large that $\vartheta(T_0) < \tilde{t}$.

We repeat that we assume the positivity of ζ which implies the inequality $\vartheta(t) > \tilde{t}$ for t from some interval. The continuity of ϑ gives the existence of $\bar{t} > T_0$ such that

$$\vartheta(\bar{t}) = \tilde{t}, \quad \vartheta'(\bar{t}) \ge 0. \tag{2.1.57}$$

From (2.1.56) it follows that, for any $\delta > 0$, one can choose T_0 such that

$$|\,\zeta(t_2) - \zeta(t_1)\,| < \delta, \quad T_0 \le t_1 \le t_2 \le t_1 + n \le \tilde{T_1}.$$

Thus, we can assume that

$$\left| \vartheta(\bar{t}) - \zeta(\tau) \right| < \delta, \quad \tau \in \left[\bar{t}, \bar{t} + n\right]. \tag{2.1.58}$$

Consequently, let

$$\left| |\vartheta(\bar{t})|^q - |\zeta(\tau)|^q \right| < \frac{\varepsilon}{8(p-1)}, \quad \tau \in \left[\bar{t}, \bar{t} + n\right]. \tag{2.1.59}$$

At the same time, we can assume that $n \in \mathbb{N}$ was chosen in such a way that it is valid

$$\left| \frac{1}{n} \int\limits_{b}^{b+n} r(\tau)\, \mathrm{d}\tau - 1 \right| < \min\left\{ \frac{\varepsilon}{4(p-1)}, 1 \right\}, \quad b \in [a, \infty). \tag{2.1.60}$$

Using (2.1.59) and (2.1.60), we have

$$\left| \left[(p-1)\vartheta(\bar{t}) + q^{-p} + \frac{p-1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau)\, |\zeta(\tau)|^q \, \mathrm{d}\tau \right] - \left[(p-1)\vartheta(\bar{t}) + q^{-p} + (p-1) \left|\vartheta(\bar{t})\right|^q \right] \right|$$

$$= (p-1) \left| \frac{1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau)\, |\zeta(\tau)|^q \, \mathrm{d}\tau - \left|\vartheta(\bar{t})\right|^q + \frac{1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau) \left|\vartheta(\bar{t})\right|^q \mathrm{d}\tau - \frac{1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau) \left|\vartheta(\bar{t})\right|^q \mathrm{d}\tau \right|$$

$$\leq (p-1) \left| \frac{1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau)\, |\zeta(\tau)|^q \, \mathrm{d}\tau - \frac{1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau) \left|\vartheta(\bar{t})\right|^q \mathrm{d}\tau \right|$$

$$+ (p-1) \left| \left|\vartheta(\bar{t})\right|^q - \frac{1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau) \left|\vartheta(\bar{t})\right|^q \mathrm{d}\tau \right|$$

$$\leq (p-1) \left| \frac{1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau) \left(|\zeta(\tau)|^q - \left|\vartheta(\bar{t})\right|^q \right) \mathrm{d}\tau \right| + (p-1) \left| 1 - \frac{1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau)\, \mathrm{d}\tau \right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Since (see (2.1.55), (2.1.57))

$$(p-1)\vartheta(\bar{t}) + q^{-p} + (p-1) \left|\vartheta(\bar{t})\right|^q = (p-1)\tilde{t} + q^{-p} + (p-1) \left|\tilde{t}\right|^q = 0,$$

we have

$$\left| (p-1)\vartheta(\bar{t}) + q^{-p} + \frac{p-1}{n} \int\limits_{\bar{t}}^{\bar{t}+n} r(\tau)\, |\zeta(\tau)|^q \, \mathrm{d}\tau \right| < \frac{\varepsilon}{2}. \tag{2.1.61}$$

Let T_0 be so large that (see (2.1.27) and also (2.1.48))

$$\left| \frac{1}{n} \int\limits_{t}^{t+n} \frac{s(\tau)}{\tau}\, \mathrm{d}\tau - \frac{1}{n} \int\limits_{t}^{t+n} \frac{s(\tau)}{t+n}\, \mathrm{d}\tau \right| < \frac{\varepsilon}{8(t+n)}, \quad t \geq T_0, \tag{2.1.62}$$

and (see (2.1.2) together with (2.1.58) and also (2.1.50))

$$\left| \frac{p-1}{n} \int_{\bar{t}}^{\bar{t}+n} \frac{r(\tau)\,|\zeta(\tau)|^q}{\tau}\,\mathrm{d}\tau - \frac{p-1}{n} \int_{\bar{t}}^{\bar{t}+n} \frac{r(\tau)\,|\zeta(\tau)|^q}{\bar{t}+n}\,\mathrm{d}\tau \right| < \frac{\varepsilon}{8(\bar{t}+n)}. \tag{2.1.63}$$

Considering (2.1.51), (2.1.61), (2.1.62), and (2.1.63), we obtain

$$\begin{aligned}
\vartheta'(\bar{t}) &= \frac{1}{n}\int_{\bar{t}}^{\bar{t}+n} \zeta'(\tau)\,\mathrm{d}\tau = \frac{1}{n}\int_{\bar{t}}^{\bar{t}+n} \frac{(p-1)\zeta(\tau)+s(\tau)+(p-1)r(\tau)\,|\zeta(\tau)|^q}{\tau}\,\mathrm{d}\tau \\
&< \frac{p-1}{n}\int_{\bar{t}}^{\bar{t}+n} \frac{\zeta(\tau)}{\bar{t}+n}\,\mathrm{d}\tau + \frac{1}{n}\int_{\bar{t}}^{\bar{t}+n} \frac{s(\tau)}{\bar{t}+n}\,\mathrm{d}\tau + \frac{p-1}{n}\int_{\bar{t}}^{\bar{t}+n} r(\tau)\frac{|\zeta(\tau)|^q}{\bar{t}+n}\,\mathrm{d}\tau + \frac{\varepsilon}{4(\bar{t}+n)} \\
&= \frac{1}{\bar{t}+n}\left[(p-1)\vartheta(\bar{t}) + q^{-p} + \frac{p-1}{n}\int_{\bar{t}}^{\bar{t}+n} r(\tau)\,|\zeta(\tau)|^q\,\mathrm{d}\tau \right. \\
&\qquad\qquad \left. + \frac{1}{n}\int_{\bar{t}}^{\bar{t}+n} s(\tau)\,\mathrm{d}\tau - q^{-p} + \frac{\varepsilon}{4}\right] < -\frac{\varepsilon}{4(\bar{t}+n)}.
\end{aligned}$$

This contradiction (see (2.1.57)) means that (2.1.53) is true for $B = K$ and $C = 0$. Since (2.1.52) cannot be valid for any $T_1 < \infty$, the considered solution ζ exists on interval $[T_0, \infty)$. We repeat that the non-oscillation of Eq. (2.1.1) actually follows from Lemma 2.1.1. □

The following theorem is a version of Theorem 2.1.3 which is ready for applications to the half-linear equations written in the form common in the literature.

Theorem 2.1.4. *Let $c : \mathbb{R}_a \to \mathbb{R}$ be a continuous function, for which mean value $M\left(c^{1-q}\right)$ exists and for which it holds*

$$0 < \inf_{t\in\mathbb{R}_a} c(t) \le \sup_{t\in\mathbb{R}_a} c(t) < \infty, \tag{2.1.64}$$

and let $d : \mathbb{R}_a \to \mathbb{R}$ be a continuous function having mean value $M(d)$. Let

$$\Gamma := q^{-p}\left[M\left(c^{1-q}\right)\right]^{1-p} = q^{-p}\left[\lim_{t\to\infty}\frac{1}{t}\int_a^{a+t} c^{1-q}(\tau)\,\mathrm{d}\tau\right]^{1-p}.$$

Consider the equation

$$[c(t)\Phi(x')]' + \frac{d(t)}{t^p}\Phi(x) = 0. \tag{2.1.65}$$

Eq. (2.1.65) is oscillatory if $M(d) > \Gamma$, and non-oscillatory if $M(d) < \Gamma$.

Proof. Let $M(d) > 0$. Eq. (2.1.65) can be rewritten into the form

$$\left[\left[c^{1-q}(t)\right]^{-\frac{p}{q}} \Phi(x')\right]' + \frac{d(t)}{t^p}\Phi(x) = 0,$$

i.e.,

$$\left[\left[\frac{c^{1-q}(t)}{M(c^{1-q})}\right]^{-\frac{p}{q}} \Phi(x')\right]' + \frac{\left[M(c^{1-q})\right]^{\frac{p}{q}} d(t)}{t^p}\Phi(x) = 0. \tag{2.1.66}$$

Eq. (2.1.66) has the form of Eq. (2.1.1) for

$$r(t) = \frac{c^{1-q}(t)}{M(c^{1-q})}, \quad s(t) = \left[M(c^{1-q})\right]^{\frac{p}{q}} d(t), \quad t \geq a.$$

Note that $M(r) = 1$ and $M(s) > 0$ and that (2.1.2) follows from (2.1.64). Thus (see Theorem 2.1.3), Eq. (2.1.65) is oscillatory for

$$M(s) = \left[M(c^{1-q})\right]^{\frac{p}{q}} M(d) = \left[M(c^{1-q})\right]^{p-1} M(d) > q^{-p}, \quad \text{i.e.,} \quad M(d) > \Gamma,$$

and non-oscillatory if the opposite inequality $M(d) < \Gamma$ holds.

It remains to consider the case when $M(d) \leq 0$. Of course, there exists $k > 0$ such that $0 < M(d+k) = M(d) + k < \Gamma$. We know that the equation

$$[c(t)\Phi(x')]' + \frac{d(t)+k}{t^p}\Phi(x) = 0$$

is non-oscillatory. Now, it suffices to use Theorem 2.1.2, (ii). □

Remark 2.1.2. For reader's convenience, we consider Eq. (2.1.65) (instead of Eq. (2.1.1)) in Theorem 2.1.4. The form of Eq. (2.1.65) shows how the presented result improves the known ones. Especially, we get new results in two important cases, when function s changes sign and when it is unbounded. For details, we refer to paper [38].

Remark 2.1.3. For $M(d) = \Gamma$, it is not possible do decide whether Eq. (2.1.65) is oscillatory or non-oscillatory for general functions c, d satisfying the conditions from the statement of Theorem 2.1.4. It follows, e.g., from the main results of [17, 19]. One of the most studied classes of functions which have mean values is formed by almost periodic functions. Based on the constructions from [76], it is conjectured in [38] that the case $M(d) = \Gamma$ is not generally solvable (in the sense whether it is oscillatory or non-oscillatory) even for almost periodic coefficients of Eq. (2.1.65). It means that there exist almost periodic functions c, d such that $M(d) = \Gamma$ and Eq. (2.1.65) is oscillatory. At the same time, there exist another set of almost periodic functions c, d satisfying $M(d) = \Gamma$ such that Eq. (2.1.65) is non-oscillatory. Note that the case of periodic functions c, d with the same period was proved to be non-oscillatory (see [17, 19]).

To illustrate Theorem 2.1.4, we mention at least two examples.

Example 2.1.1. For $A > 1/2$, $B, C > 0$, and $q = 3$, let us consider the equation

$$\left[\frac{\Phi(x')}{A + \cos(Bt)\sin(Bt)}\right]' + \frac{\arcsin(\cos t) + [\cos(Ct)\sin(Ct)]^2}{\sqrt{t^3 + t^2 + t + 1}}\Phi(x) = 0. \tag{2.1.67}$$

Eq. (2.1.67) has the form of Eq. (2.1.65) for

$$c(t) = \frac{1}{A + \cos(Bt)\sin(Bt)}, \quad d(t) = \frac{\arcsin(\cos t) + [\cos(Ct)\sin(Ct)]^2}{\sqrt{1 + \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3}}}.$$

It can be directly verified that

$$M\left(c^{1-q}\right) = M\left([\cos(Bt)\sin(Bt) + A]^2\right) = \frac{1 + 8A^2}{8}$$

and that

$$M(d) = M\left([\cos(Ct)\sin(Ct)]^2\right) = \frac{1}{8}.$$

Hence, if $2^9 < 3^3\left(1 + 8A^2\right)$ then Eq. (2.1.67) is oscillatory and if $2^9 > 3^3\left(1 + 8A^2\right)$ then it is non-oscillatory; i.e., Eq. (2.1.67) is oscillatory for $A \in (\kappa, \infty)$ and non-oscillatory for $A \in (1/2, \kappa)$, where

$$\kappa = \sqrt{(8^3/3^3 - 1)/8} \doteq 1.498\,455\,995.$$

Since d is oscillatory, the other related results in the literature give no conclusion for Eq. (2.1.65).

Example 2.1.2. Let $K \in \mathbb{R}$ and $k \neq 0$ be arbitrarily given. We define the function $d : \mathbb{R}_1 \to \mathbb{R}$ by

$$d(t) := K + k\,(t - n)\,3^n, \quad t \in \left[n, n + \frac{1}{2^n}\right), \; n \in \mathbb{N};$$
$$d(t) := K + k\left(n + \frac{2}{2^n} - t\right)3^n, \quad t \in \left[n + \frac{1}{2^n}, n + \frac{2}{2^n}\right), \; n \in \mathbb{N};$$
$$d(t) := K, \quad t \in \left[n + \frac{2}{2^n}, n + 1\right), \; n \in \mathbb{N}.$$

It is seen that $\limsup_{t\to\infty} |d(t)| = \infty$ and that $M(d) = K$. Analogously, for $L > 0$ and $l > -L$, we define $c : \mathbb{R}_1 \to \mathbb{R}$ as

$$c(t) := L + l\,(t - n)\,2n, \quad t \in \left[n, n + \frac{1}{2n}\right), \; n \in \mathbb{N};$$
$$c(t) := L + l\left(n + \frac{1}{n} - t\right)2n, \quad t \in \left[n + \frac{1}{2n}, n + \frac{1}{n}\right), \; n \in \mathbb{N};$$
$$c(t) := L, \quad t \in \left[n + \frac{1}{n}, n + 1\right), \; n \in \mathbb{N}.$$

We have (2.1.64) and $M(c^{1-q}) = L^{1-q}$. Thus, Eq. (2.1.65) for the above given functions c, d is oscillatory if $K > q^{-p}L$, and non-oscillatory if $K < q^{-p}L$. Since d is not bounded, the other related results in the literature give no conclusion for this equation.

Theorem 2.1.4 gives a new result for linear equations as well. Therefore, we mention the following corollary.

Corollary 2.1.1. *Let* $r, s : \mathbb{R}_a \to \mathbb{R}$ *be continuous functions having mean values* $M(r)$, $M(s)$ *and let* (2.1.2) *hold. Then, the equation*

$$\left[\frac{x'}{r(t)}\right]' + \frac{s(t)}{t^2} x = 0$$

is oscillatory if $4M(r)M(s) > 1$, *and non-oscillatory if* $4M(r)M(s) < 1$.

Example 2.1.3. Using Corollary 2.1.1, we can decide about the oscillation and non-oscillation of several equations. Let $a, b \in \mathbb{R}$ and $m, n \in \mathbb{N}$ be relative prime. Let $f, g : \mathbb{R}_1 \to \mathbb{R}$ be arbitrary positive continuous functions with the property that

$$\lim_{t\to\infty} f(t) = 1, \quad \lim_{t\to\infty} \frac{g(t)}{t^2} = 1.$$

The equations

$$[f(t)x']' + \gamma \frac{|\sin(\sqrt{m}t + a)| + \sin(\sqrt{n}t + b)}{g(t)} x = 0,$$

$$[f(t)x']' + \gamma \frac{|\cos(\sqrt{m}t + a)| + \sin(\sqrt{n}t + b)}{g(t)} x = 0,$$

$$[f(t)x']' + \gamma \frac{|\sin(\sqrt{m}t + a)| + \cos(\sqrt{n}t + b)}{g(t)} x = 0,$$

$$[f(t)x']' + \gamma \frac{|\cos(\sqrt{m}t + a)| + \cos(\sqrt{n}t + b)}{g(t)} x = 0$$

are oscillatory for $8\gamma > \pi$ and non-oscillatory for $8\gamma < \pi$. Note that this simple criterion does not follow from known ones.

In the following corollary, we mention the case of negative coefficient r.

Corollary 2.1.2. *Let* $r, s : \mathbb{R}_a \to \mathbb{R}$ *be continuous functions such that function* s *has mean value* $M(s)$ *and that there exists mean value* $R := M(|r|^{1-q})$. *Let*

$$-\infty < \inf_{t\in\mathbb{R}_a} r(t) \le \sup_{t\in\mathbb{R}_a} r(t) < 0$$

be fulfilled. The equation

$$[r(t)\Phi(x')]' + \frac{s(t)}{t^p}\Phi(x) = 0$$

is oscillatory if $M(s) < -q^{-p}R^{1-p}$, *and non-oscillatory if* $M(s) > -q^{-p}R^{1-p}$.

Proof. To prove the corollary, it suffices to use Theorem 2.1.4 and the fact that the solution space of half-linear equations is homogeneous. $\square$

One can prove a number of consequences using different types of comparison theorems. For example, applying Theorem 2.1.4 and [21, Theorem 2.3.1], we immediately obtain the next new result.

Corollary 2.1.3. *Let $r, s : \mathbb{R}_a \to \mathbb{R}$ be continuous functions having mean values $M(r)$, $M(s)$. Let (2.1.2) hold and let*

$$\Gamma := q^{-p} \left[M\left(r\right)\right]^{1-p} = q^{-p} \left[\lim_{t\to\infty} \frac{1}{t} \int\limits_a^{a+t} r(\tau)\, \mathrm{d}\tau\right]^{1-p}.$$

Let us consider the equation

$$\left[r^{-\frac{p}{q}}(t)\Phi(x')\right]' + y(t)\Phi(x) = 0, \tag{2.1.68}$$

where $y : \mathbb{R}_a \to \mathbb{R}$ is a continuous function satisfying $\int_a^\infty y(\tau)\,\mathrm{d}\tau < \infty$.

(i) *If there exists $t_0 \geq a$ for which*

$$\left| \int\limits_t^\infty \frac{s(\tau)}{\tau^p}\, \mathrm{d}\tau \right| \leq \int\limits_t^\infty y(\tau)\, \mathrm{d}\tau, \quad t \geq t_0,$$

and if $M(s) > \Gamma$, then Eq. (2.1.68) is oscillatory.

(ii) *If there exists $t_0 \geq a$ for which*

$$\int\limits_t^\infty \frac{s(\tau)}{\tau^p}\, \mathrm{d}\tau \geq \left| \int\limits_t^\infty y(\tau)\, \mathrm{d}\tau \right|, \quad t \geq t_0,$$

and if $M(s) < \Gamma$, then Eq. (2.1.68) is non-oscillatory.

In the following example, we demonstrate, that the results are applicable even if $M(s)$ does not exists.

Example 2.1.4. Let $p > 1$ and $X, Y, Z_1, Z_2 > 0$. Putting

$$s(t) := X + Y\left(t - 2^n\right)2^n, \quad t \in \left[2^n, 2^n + \frac{1}{4}\right), \ n \in \mathbb{N};$$

$$s(t) := X + Y\left(2^n + \frac{1}{2} - t\right)2^n, \quad t \in \left[2^n + \frac{1}{4}, 2^n + \frac{1}{2}\right), \ n \in \mathbb{N};$$

$$s(t) := X - Y\left(t - 2^n - \frac{1}{2}\right)2^n, \quad t \in \left[2^n + \frac{1}{2}, 2^n + \frac{3}{4}\right), \ n \in \mathbb{N};$$

$$s(t) := X - Y\left(2^n + 1 - t\right)2^n, \quad t \in \left[2^n + \frac{3}{4}, 2^n + 1\right], \ n \in \mathbb{N};$$

$$s(t) := X, \quad t \in \mathbb{R}_2 \setminus \bigcup_{n\in\mathbb{N}} \left[2^n, 2^n + 1\right],$$

we define the continuous function $s : \mathbb{R}_2 \to \mathbb{R}$. Evidently, mean value $M(s)$ does not exist. Since

$$\begin{aligned} 0 < \int_{2^n}^{2^{n+1}} \frac{s(\tau) - X}{\tau^p} \,\mathrm{d}\tau = \int_{2^n}^{2^n+1} \frac{s(\tau) - X}{\tau^p} \,\mathrm{d}\tau < \frac{Y}{16} \left[\frac{2^n}{(2^n)^p} - \frac{2^n}{(2^n+1)^p} \right] \\ = \frac{Y \cdot 2^n}{16} \left[\frac{(2^n+1)^p - (2^n)^p}{(2^n+1)^p (2^n)^p} \right] < \frac{Y}{16} \cdot \frac{1}{(2^n)^{p-1}}, \quad n \in \mathbb{N}, \end{aligned} \tag{2.1.69}$$

we have

$$0 < \int_2^\infty \frac{s(\tau)}{\tau^p} \,\mathrm{d}\tau < \frac{Y}{16} \sum_{n=1}^\infty \frac{1}{(2^{p-1})^n} + \int_2^\infty \frac{X}{\tau^p} \,\mathrm{d}\tau < \infty. \tag{2.1.70}$$

Considering

$$\lim_{n\to\infty} \frac{\frac{(2^n+1)^p - (2^n)^p}{(2^n+1)^p (2^n)^p}}{\frac{1}{2^{[n+1]p}}} = 0$$

and (2.1.69), there exists a positive continuous function $h : \mathbb{R}_2 \to \mathbb{R}$ with the property that

$$\int_t^{2^{n+1}} \frac{h(\tau)}{\tau^p} \,\mathrm{d}\tau > \left| \int_t^{2^{n+1}} \frac{s(\tau) - X}{\tau^p} \,\mathrm{d}\tau \right|, \quad t \in \left[2^n, 2^{n+1}\right), \, n \in \mathbb{N}, \tag{2.1.71}$$

and that $\lim_{t\to\infty} h(t) = 0$. Using (2.1.71), we obtain

$$\int_t^\infty \frac{X - h(\tau)}{\tau^p} \,\mathrm{d}\tau < \int_t^\infty \frac{s(\tau)}{\tau^p} \,\mathrm{d}\tau < \int_t^\infty \frac{X + h(\tau)}{\tau^p} \,\mathrm{d}\tau, \quad t \geq 2. \tag{2.1.72}$$

In addition, $M(X + h) = M(X - h) = X$ and

$$\int_2^\infty \frac{h(\tau)}{\tau^p} \,\mathrm{d}\tau < \infty. \tag{2.1.73}$$

It is also seen that

$$\int_t^\infty \frac{X - h(\tau)}{\tau^p} \,\mathrm{d}\tau > 0 \tag{2.1.74}$$

for all sufficiently large t. The equations

$$\left[\left(|\sin[Z_1 t]| + |\cos[Z_1 t]| + |\sin[Z_2 t]| + |\cos[Z_2 t]| \right)^{-\frac{p}{q}} \Phi(x') \right]' + \frac{X - h(t)}{t^p} \Phi(x) = 0,$$

$$\left[\left(|\sin[Z_1 t]| + |\cos[Z_1 t]| + |\sin[Z_2 t]| + |\cos[Z_2 t]| \right)^{-\frac{p}{q}} \Phi(x') \right]' + \frac{X + h(t)}{t^p} \Phi(x) = 0$$

are oscillatory for $X > q^{-p}(\pi/8)^{p-1}$ and non-oscillatory for $X < q^{-p}(\pi/8)^{p-1}$ (see Theorem 2.1.4). Indeed,

$$M\left([|\sin(Z_1t)| + |\cos(Z_1t)| + |\sin(Z_2t)| + |\cos(Z_2t)|]^{\frac{p(q-1)}{q}}\right) = \frac{8}{\pi}.$$

Therefore (see Corollary 2.1.3 together with (2.1.70), (2.1.72), (2.1.73), (2.1.74)), we know that the equation

$$\left[(|\sin[Z_1t]| + |\cos[Z_1t]| + |\sin[Z_2t]| + |\cos[Z_2t]|)^{-\frac{p}{q}}\,\Phi(x')\right]' + \frac{s(t)}{t^p}\Phi(x) = 0$$

is oscillatory if $X > q^{-p}(\pi/8)^{p-1}$, and non-oscillatory if $X < q^{-p}(\pi/8)^{p-1}$.

§ 2.1.2 An application

In this paragraph, we use our main result to derive a theorem related to elliptic partial differential equations with p-Laplacian and the power type non-linearity

$$\operatorname{div}\left(A(x)\|\nabla u\|^{p-2}\nabla u\right) + C(x)\Phi(u) = 0, \tag{2.1.75}$$

where $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, A is an elliptic $n \times n$ matrix function with differentiable components, and C is a Hölder continuous function. As a solution of Eq. (2.1.75) in $\Omega \subseteq \mathbb{R}^n$, we understand a differentiable function u such that $A(x)\left\|\nabla u(x)\right\|^{p-2}\nabla u(x)$ is also differentiable and u satisfies (2.1.75) in Ω.

The following notation is used. We consider the usual Euclidean norm

$$\left\|\vec{b}\right\| = \left(\sum_{i=1}^n b_i^2\right)^{\frac{1}{2}},$$

the induced matrix norm

$$\|A\| = \sup_{\|\vec{b}\|\neq 0} \frac{\|A\vec{b}\|}{\|\vec{b}\|},$$

and $\lambda_{\min}(x)$, $\lambda_{\max}(x)$ stands for the smallest and largest eigenvalue of matrix $A(x)$, respectively. From the fact that $A(x)$ is positive definite, it follows that $\|A(x)\| = \lambda_{\max}(x)$.

Denote $\Omega(r_0) := \{x \in \mathbb{R}^n : ||x|| \geq r_0\}$. We say that a solution u of Eq. (2.1.75) is oscillatory if it has a zero in $\Omega(r)$ for every $r \geq r_0$. Eq. (2.1.75) is said to be oscillatory if every solution of this equation is oscillatory. Otherwise, Eq. (2.1.75) is said to be non-oscillatory.

Recall that in general we distinguish two types of oscillation in the theory of (2.1.75): the (weak) oscillation defined in the previous paragraph and the so called (strong) nodal oscillation, which is based on nodal domains (i.e., bounded domains such that the equation possesses a nontrivial solution which vanishes on the boundary of this domain). Concerning this concept of oscillation, Eq. (2.1.75) is said to be nodally oscillatory if every solution has a nodal domain outside of any ball in $\mathbb{R}^n$, and to be nodally non-oscillatory in the opposite case. It is known that the nodal oscillation implies oscillation. The opposite implication has been proved only in the linear case $p = 2$ (see [61]) and remains an open question in the half-linear multidimensional case. We also refer to [72] which relates the weak oscillation of linear PDE's (and thus nodal oscillation) with the finiteness of negative spectrum of the Laplace operator, which is of interest in physical applications.

Since the results of this section are based on the method of Riccati type equation, in the oscillation criteria we essentially prove a nonexistence of eventually positive solution, i.e., we deal with weak oscillation.

In [20, 57], there is proved a theorem which allows to deduce the oscillation of certain half-linear partial differential equations from the oscillation of ordinary differential equations if $A(x)$ is either the identity matrix or a scalar multiple of the identity matrix. This theorem has been later extended in [30] (see also [59]) as follows.

Theorem 2.1.5. *Let us define the functions*

$$\begin{aligned} \tilde{r}(t) &:= \begin{cases} \displaystyle\int_{\|x\|=t} \left(\frac{\lambda_{\max}(x)}{\lambda_{\min}(x)}\right)^{p-1} \lambda_{\max}(x)\,\mathrm{d}S & \text{for } p > 2; \\ \displaystyle\int_{\|x\|=t} \lambda_{\max}(x)\,\mathrm{d}S & \text{for } 1 < p \le 2, \end{cases} \\ \tilde{c}(t) &:= \int_{\|x\|=t} C(x)\,\mathrm{d}S. \end{aligned} \tag{2.1.76}$$

If the equation

$$\Big(\tilde{r}(t)\Phi(u')\Big)' + \tilde{c}(t)\Phi(u) = 0$$

is oscillatory, then Eq. (2.1.75) *is oscillatory as well.*

We can easily apply the oscillation part of Theorem 2.1.4 and Theorem 2.1.5 to obtain the following result.

Theorem 2.1.6. *Let $\tilde{r}(t)$ and $\tilde{c}(t)$ be defined by* (2.1.76)*. Suppose that*

$$0 < \liminf_{t\to\infty} \tilde{r}(t) \le \limsup_{t\to\infty} \tilde{r}(t) < \infty$$

and that $M(\tilde{r}^{1-q})$ and $M(\tilde{c}(t)t^p)$ exist. If

$$M(\tilde{c}(t)t^p) > q^{-p}\left[M(\tilde{r}^{1-q})\right]^{1-p},$$

then Eq. (2.1.75) *is oscillatory.*

Remark 2.1.4. Note that in contrast to Theorem 2.1.4, we lack the non-oscillation part in Theorem 2.1.5 and consequently also in Theorem 2.1.6, because there is a principal problem with non-oscillation criteria for partial differential equations via the Riccati method. A detailed discussion related to the relationship of the Riccati equation and the non-oscillation of second order equations (in the multidimensional case) can be found in [15].

2.2 Non-oscillation of equations with periodic coefficients in critical case

In the previous section, we found the critical oscillation constant for Eq. (2.1.1) with coefficients r and s having mean values. Despite that the critical case cannot be resolved in full generality, there remains still a solvable open problem. It is not known whether Eq. (2.1.1) with positive α-periodic coefficient r and β-periodic coefficient s is oscillatory or not in the critical case (r and s do not need to have any common period, e.g., $\alpha = 1$, $\beta = \sqrt{2}$). In this section, we prove that Eq. (2.1.1) is non-oscillatory in this case. We point out that coefficient s can change its sign (in contrast with the situation common in the literature) and we remark that, according to our best knowledge, the result presented in this section is new in the half-linear case as well as in the linear one (i.e., for $p = 2$). To prove this result, we have to use another method than in Section 2.1.

§ 2.2.1 Preliminaries

In this paragraph, we mention the used form of studied equations together with the corresponding Riccati equation, and the concept of the modified Prüfer angle. These tools will be applied in § 2.2.2 and § 2.2.3.

We study the equation

$$\left[r^{-\frac{p}{q}}(t)\Phi\left(x'\right)\right]' + \frac{s(t)}{t^p}\Phi(x) = 0, \tag{2.2.1}$$

where $r, s : \mathbb{R}_a \to \mathbb{R}$, $\mathbb{R}_a := [a, \infty)$, $a \geq \mathrm{e}$ (e denotes the base of the natural logarithm log). Henceforth, let function r be bounded and positive and s be such that $\limsup_{t\to\infty} |s(t)| < \infty$. For further use, we denote

$$r^+ := \sup\{r(t);\, t \in \mathbb{R}_a\}, \qquad s^+ := \sup\{|s(t)|;\, t \in \mathbb{R}_a\}. \tag{2.2.2}$$

Let us recall that via the Riccati transformation

$$w(t) = r^{-\frac{p}{q}}(t)\Phi\left(\frac{x'(t)}{x(t)}\right),$$

where x is a nontrivial solution of Eq. (2.2.1) and function w is well defined whenever $x(t) \neq 0$, we obtain the Riccati equation

$$w' + \frac{s(t)}{t^p} + (p-1)r(t)|w|^q = 0 \tag{2.2.3}$$

associated to Eq. (2.2.1).

Using the notion of the half-linear trigonometric functions recalled in Section 1.1, we can introduce the modified half-linear Prüfer transformation

$$x(t) = \rho(t)\sin_p\varphi(t), \quad x'(t) = \frac{r(t)\rho(t)}{t}\cos_p\varphi(t). \tag{2.2.4}$$

Denote $v(t) = t^{p-1}w(t)$, where w is a solution of Eq. (2.2.3). Considering the transformation given by (2.2.4), we get

$$v = \Phi\left(\frac{\cos_p \varphi}{\sin_p \varphi}\right). \tag{2.2.5}$$

From the fact that $\sin_p$ solves the equation in (1.1.2), we have

$$v' = (1-p)\left[1 + \left|\frac{\cos_p \varphi}{\sin_p \varphi}\right|^p\right]\varphi'. \tag{2.2.6}$$

On the other hand, applying the Riccati equation (2.2.3), we obtain

$$v' = \left[t^{p-1}w\right]' = (p-1)t^{p-2}w + t^{p-1}w' = \frac{p-1}{t}\left[v - \frac{s(t)}{p-1} - r(t)\left|\frac{\cos_p \varphi}{\sin_p \varphi}\right|^p\right]. \tag{2.2.7}$$

Putting (2.2.6) and (2.2.7) together and using (2.2.5), we have

$$(1-p)\left[1 + \left|\frac{\cos_p \varphi}{\sin_p \varphi}\right|^p\right]\varphi' = \frac{p-1}{t}\left[\Phi\left(\frac{\cos_p \varphi}{\sin_p \varphi}\right) - \frac{s(t)}{p-1} - r(t)\left|\frac{\cos_p \varphi}{\sin_p \varphi}\right|^p\right]. \tag{2.2.8}$$

Then, by a direct calculation starting with (2.2.8) and taking into account the Pythagorean identity (1.1.5), we obtain the equation for the Prüfer angle φ associated to Eq. (2.2.1) as

$$\varphi' = \frac{1}{t}\left[r(t)|\cos_p \varphi|^p - \Phi(\cos_p \varphi)\sin_p \varphi + s(t)\frac{|\sin_p \varphi|^p}{p-1}\right]. \tag{2.2.9}$$

For details, we can also refer to [19].

§ 2.2.2 Auxiliary results

To prove the announced result, we will use the following lemmata. The first four of them deal with Eq. (2.2.9).

Lemma 2.2.1. *For a solution φ of Eq.* (2.2.9) *on* $[a, \infty)$, *it holds*

$$\limsup_{t\to\infty}\left|\frac{\varphi(t)}{\log t}\right| < \infty,$$

i.e., there exists $N > 0$ with the property that

$$|\varphi(t)| < N \log t, \quad t \geq a.$$

Proof. Considering (2.2.2) and (1.1.6), one can directly calculate

$$\limsup_{t\to\infty}\left|\frac{\varphi(t)-\varphi(t_0)}{\log t}\right| \le \limsup_{t\to\infty}\left[\frac{1}{\log t}\int\limits_{t_0}^{t}|\varphi'(\tau)|\,\mathrm{d}\tau\right]$$

$$\le \limsup_{t\to\infty}\left[\frac{1}{\log t}\int\limits_{t_0}^{t}\frac{1}{\tau}\Big(r(\tau)|\cos_p\varphi(\tau)|^p\right.$$

$$\left.+|\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)|+|s(\tau)|\,\frac{|\sin_p\varphi(\tau)|^p}{p-1}\Big)\,\mathrm{d}\tau\right]$$

$$\le \limsup_{t\to\infty}\left[\frac{1}{\log t}\int\limits_{t_0}^{t}\frac{1}{\tau}\left(r^+L+L+\frac{s^+L}{p-1}\right)\mathrm{d}\tau\right] = K\limsup_{t\to\infty}\frac{\log t-\log t_0}{\log t}=K,$$

where $t_0\in\mathbb{R}_a$ is arbitrarily given and

$$K := r^+L+L+\frac{s^+L}{p-1}. \tag{2.2.10}$$

□

Lemma 2.2.2. *If φ is a solution of Eq.* (2.2.9) *on* $[a,\infty)$*, then the function $\psi:\mathbb{R}_a\to\mathbb{R}$ defined by*

$$\psi(t) := \int\limits_{t}^{t+\sqrt{t}}\frac{\varphi(\tau)}{\sqrt{\tau}}\,\mathrm{d}\tau, \quad t\ge a, \tag{2.2.11}$$

satisfies

$$|\varphi(t+s)-\psi(t)|\le\frac{C\log t}{\sqrt{t}}, \quad t\ge a,\ s\in\left[0,\sqrt{t}\right], \tag{2.2.12}$$

for some $C>0$.

Proof. At first, we consider the function

$$\tilde{\psi}(t) := \frac{1}{\sqrt{t}}\int\limits_{t}^{t+\sqrt{t}}\varphi(\tau)\,\mathrm{d}\tau, \quad t\ge a,$$

and we estimate its difference from ψ. For $t \geq a$, we have

$$\begin{aligned}
\left|\tilde{\psi}(t) - \psi(t)\right| &= \left| \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} \varphi(\tau)\,\mathrm{d}\tau - \int\limits_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{\sqrt{\tau}}\,\mathrm{d}\tau \right| \\
&\leq \int\limits_t^{t+\sqrt{t}} |\varphi(\tau)| \left(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{\tau}} \right) \mathrm{d}\tau \\
&\leq \frac{\sqrt{t+\sqrt{t}} - \sqrt{t}}{t} \int\limits_t^{t+\sqrt{t}} N \log \tau \,\mathrm{d}\tau \leq \frac{\sqrt{t+\sqrt{t}} - \sqrt{t}}{\sqrt{t}} N \log\left(t + \sqrt{t}\right),
\end{aligned}$$

where N is taken from the statement of Lemma 2.2.1. Evidently, it holds

$$\lim_{t\to\infty} \sqrt{t+\sqrt{t}} - \sqrt{t} = \frac{1}{2}, \qquad \lim_{t\to\infty} \frac{\log\left(t+\sqrt{t}\right)}{\log t} = 1.$$

Thus, there exists $\widetilde{K} > 0$ for which

$$\left|\psi(t) - \tilde{\psi}(t)\right| \leq \frac{\widetilde{K}\log t}{\sqrt{t}}, \quad t \geq a. \tag{2.2.13}$$

Since φ is continuous, we have that, for any $t \geq a$, there exists $t_0 \in \left[t, t+\sqrt{t}\right]$ such that $\tilde{\psi}(t) = \varphi(t_0)$. Hence, we have

$$\begin{aligned}
\left|\varphi(t+s) - \tilde{\psi}(t)\right| &= |\varphi(t+s) - \varphi(t_0)| \leq \int\limits_t^{t+\sqrt{t}} |\varphi'(\tau)|\,\mathrm{d}\tau \\
&\leq \frac{1}{t} \left[\int\limits_t^{t+\sqrt{t}} r(\tau) |\cos_p \varphi(\tau)|^p + |\Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau)|\,\mathrm{d}\tau \right. \\
&\qquad \left. + \int\limits_t^{t+\sqrt{t}} \frac{|\sin_p \varphi(\tau)|^p}{p-1} |s(\tau)|\,\mathrm{d}\tau \right] \qquad (2.2.14) \\
&\leq \frac{1}{t} \int\limits_t^{t+\sqrt{t}} \left(Lr^+ + L + \frac{Ls^+}{p-1} \right) \mathrm{d}\tau \leq \frac{K}{\sqrt{t}}, \quad t \geq a,\ s \in \left[0, \sqrt{t}\right],
\end{aligned}$$

where K is given in (2.2.10) (r^+, s^+ are defined in (2.2.2) and L is from (1.1.6)). Combining (2.2.13) and (2.2.14), we obtain (2.2.12) for $C = \widetilde{K} + K$. $\square$

Remark 2.2.1. From the above lemmata, it follows that there exists $U > 0$ for which

$$|\psi(t)| < U \log t, \quad t \geq a, \tag{2.2.15}$$

where ψ is defined in (2.2.11) for a solution φ of Eq. (2.2.9) on $[a, \infty)$.

Lemma 2.2.3. *Let φ be a solution of Eq.* (2.2.9) *on $[a, \infty)$. Then, there exist $P, \varrho > 0$ such that the function $\psi : \mathbb{R}_a \to \mathbb{R}$ defined in* (2.2.11) *satisfies the inequalities*

$$\begin{aligned}\psi' \leq \frac{1}{t} \Bigg[& \frac{|\cos_p \psi|^p}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} r(\tau)\, \mathrm{d}\tau - \Phi(\cos_p \psi) \sin_p \psi \\ & + \frac{|\sin_p \psi|^p}{(p-1)\sqrt{t}} \int\limits_t^{t+\sqrt{t}} s(\tau)\, \mathrm{d}\tau + \frac{P}{t^\varrho} \Bigg]\end{aligned} \tag{2.2.16}$$

and

$$\begin{aligned}\psi' \geq \frac{1}{t} \Bigg[& \frac{|\cos_p \psi|^p}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} r(\tau)\, \mathrm{d}\tau - \Phi(\cos_p \psi) \sin_p \psi \\ & + \frac{|\sin_p \psi|^p}{(p-1)\sqrt{t}} \int\limits_t^{t+\sqrt{t}} s(\tau)\, \mathrm{d}\tau - \frac{P}{t^\varrho} \Bigg].\end{aligned} \tag{2.2.17}$$

Proof. For arbitrarily given $t > a$, we have

$$\begin{aligned}\psi'(t) &= \left(1 + \frac{1}{2\sqrt{t}}\right) \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t + \sqrt{t}}} - \frac{\varphi(t)}{\sqrt{t}} = \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t + \sqrt{t}}} + \int\limits_t^{t+\sqrt{t}} \left[\frac{\varphi(\tau)}{\sqrt{\tau}}\right]' \mathrm{d}\tau \\ &= \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t + \sqrt{t}}} + \int\limits_t^{t+\sqrt{t}} \frac{\varphi'(\tau)}{\tau^{\frac{1}{2}}} - \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}}\, \mathrm{d}\tau \\ &= \int\limits_t^{t+\sqrt{t}} \frac{1}{\tau^{\frac{3}{2}}} \Bigg[r(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) \\ &\qquad + \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau) \Bigg] \mathrm{d}\tau + \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t + \sqrt{t}}} - \int\limits_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}}\, \mathrm{d}\tau.\end{aligned} \tag{2.2.18}$$

Since

$$\lim_{t \to \infty} \frac{\left(t + \sqrt{t}\right)^{\frac{3}{2}} - t^{\frac{3}{2}}}{t} = \frac{3}{2},$$

there exists $V > 0$ for which

$$\frac{\left(t+\sqrt{t}\right)^{\frac{3}{2}} - t^{\frac{3}{2}}}{t^{\frac{5}{2}}} < \frac{V}{t^{\frac{3}{2}}}, \quad t \geq a. \tag{2.2.19}$$

Thus, it holds (see again (2.2.2), (1.1.6), (2.2.10))

$$\begin{aligned}
&\left| \int\limits_{t}^{t+\sqrt{t}} \frac{1}{\tau^{\frac{3}{2}}} \left[r(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau) \right] \mathrm{d}\tau \right. \\
&\left. - \frac{1}{t^{\frac{3}{2}}} \int\limits_{t}^{t+\sqrt{t}} \left[r(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau) \right] \mathrm{d}\tau \right| \\
&\leq \int\limits_{t}^{t+\sqrt{t}} \left[r^+ L + L + \frac{s^+ L}{p-1} \right] \left[\frac{1}{t^{\frac{3}{2}}} - \frac{1}{\tau^{\frac{3}{2}}} \right] \mathrm{d}\tau \leq K \frac{\left(t+\sqrt{t}\right)^{\frac{3}{2}} - t^{\frac{3}{2}}}{t^{\frac{5}{2}}} \leq \frac{KV}{t^{\frac{3}{2}}}, \quad t \geq a.
\end{aligned} \tag{2.2.20}$$

We have (see (2.2.12) in Lemma 2.2.2 and (2.2.15) in Remark 2.2.1)

$$\begin{aligned}
&\left| \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t+\sqrt{t}\right)}{\sqrt{t+\sqrt{t}}} - \frac{\psi(t)}{2t} \right| = \frac{1}{2t} \left| \frac{\varphi\left(t+\sqrt{t}\right)}{\sqrt{1+\frac{1}{\sqrt{t}}}} - \psi(t) \right| \\
&\leq \frac{1}{2t} \left[\left| \frac{\varphi\left(t+\sqrt{t}\right) - \psi(t)}{\sqrt{1+\frac{1}{\sqrt{t}}}} \right| + \left| \psi(t) \left(1 - \frac{1}{\sqrt{1+\frac{1}{\sqrt{t}}}} \right) \right| \right] \\
&\leq \frac{1}{2t} \left[\left| \varphi\left(t+\sqrt{t}\right) - \psi(t) \right| + |\psi(t)| \frac{\sqrt{1+\frac{1}{\sqrt{t}}} - 1}{\sqrt{1+\frac{1}{\sqrt{t}}}} \right] \\
&\leq \frac{1}{2t} \left[\frac{C \log t}{\sqrt{t}} + \frac{U \log t}{\sqrt{t}} \cdot \frac{1}{\sqrt{1+\frac{1}{\sqrt{t}}} \left(\sqrt{1+\frac{1}{\sqrt{t}}} + 1 \right)} \right] \leq \frac{Q_1}{t^{\frac{4}{3}}}
\end{aligned} \tag{2.2.21}$$

for some $Q_1 > 0$ and for all $t \geq a$. We also have (see again (2.2.12), (2.2.15) with (2.2.19))

$$\begin{aligned}
&\left| \frac{\psi(t)}{2t} - \int_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}}\, \mathrm{d}\tau \right| = \left| \int_t^{t+\sqrt{t}} \frac{\psi(t)}{2t^{\frac{3}{2}}} - \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}}\, \mathrm{d}\tau \right| \\
&\quad \leq \left| \int_t^{t+\sqrt{t}} \frac{\psi(t)}{2t^{\frac{3}{2}}} - \frac{\psi(t)}{2\tau^{\frac{3}{2}}}\, \mathrm{d}\tau \right| + \left| \int_t^{t+\sqrt{t}} \frac{\psi(t)}{2\tau^{\frac{3}{2}}} - \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}}\, \mathrm{d}\tau \right| \qquad (2.2.22) \\
&\quad \leq \frac{U \log t}{2} \int_t^{t+\sqrt{t}} \left(\frac{1}{t^{\frac{3}{2}}} - \frac{1}{\tau^{\frac{3}{2}}} \right) \mathrm{d}\tau + \int_t^{t+\sqrt{t}} \frac{C \log t}{\sqrt{t}\, 2\tau^{\frac{3}{2}}}\, \mathrm{d}\tau \\
&\quad \leq \frac{U \log t}{2} \cdot \frac{\left(t + \sqrt{t}\right)^{\frac{3}{2}} - t^{\frac{3}{2}}}{t^{\frac{5}{2}}} + \frac{C \log t}{2t^{\frac{3}{2}}} \leq \frac{(VU + C) \log t}{2t^{\frac{3}{2}}} \leq \frac{Q_2}{t^{\frac{4}{3}}}
\end{aligned}$$

for a number $Q_2 > 0$ and for all $t \geq a$. Considering (2.2.21) and (2.2.22), we get

$$\left| \frac{1}{2\sqrt{t}} \cdot \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t + \sqrt{t}}} - \int_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{2\tau^{\frac{3}{2}}}\, \mathrm{d}\tau \right| \leq \frac{Q_1 + Q_2}{t^{\frac{4}{3}}}, \quad t \geq a. \tag{2.2.23}$$

It means (see also (2.2.18) and (2.2.20)) that it suffices to consider the expression

$$\frac{1}{t} \left[\frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau)\ \mathrm{d}\tau \right]$$

and that, to prove the statement of the lemma, it suffices to obtain the following set of inequalities

$$\left| \frac{|\cos_p \psi(t)|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau)\, \mathrm{d}\tau - \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) |\cos_p \varphi(\tau)|^p\, \mathrm{d}\tau \right| \leq \frac{A_1 \log t}{\sqrt{t}}, \tag{2.2.24}$$

$$\left| \Phi(\cos_p \psi(t)) \sin_p \psi(t) - \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau)\, \mathrm{d}\tau \right| \leq \frac{A_2}{t^{\varrho}}, \tag{2.2.25}$$

$$\left| \frac{|\sin_p \psi(t)|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau)\, \mathrm{d}\tau - \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau) |\sin_p \varphi(\tau)|^p\, \mathrm{d}\tau \right| \leq \frac{A_3 \log t}{\sqrt{t}} \tag{2.2.26}$$

for some constants $A_1, A_2, A_3 > 0$, for a number $\varrho > 0$, and for all $t \geq a$.

Since the half-linear trigonometric functions are continuously differentiable and periodic, there exists $B > 0$ with the property that

$$\big| |\cos_p y|^p - |\cos_p z|^p \big| \le B|y - z|, \quad y, z \in \mathbb{R}, \tag{2.2.27}$$

$$\big| \cos_p y - \cos_p z \big| \le B|y - z|, \quad y, z \in \mathbb{R}, \tag{2.2.28}$$

$$\big| |\sin_p y|^p - |\sin_p z|^p \big| \le B|y - z|, \quad y, z \in \mathbb{R}, \tag{2.2.29}$$

$$\big| \sin_p y - \sin_p z \big| \le B|y - z|, \quad y, z \in \mathbb{R}. \tag{2.2.30}$$

If $p \ge 2$, then function Φ has the Lipschitz property, i.e., there exists $\widetilde{B} \ge 2$ for which

$$|\Phi(y) - \Phi(z)| \le \widetilde{B}|y - z|, \quad y, z \in (-L, L). \tag{2.2.31}$$

If $p \in (1, 2)$, then

$$\big| y^{p-1} - z^{p-1} \big| \le |y - z|^{p-1}, \quad y, z \in [0, L), \tag{2.2.32}$$

and

$$|y|^{p-1} + |z|^{p-1} \le 2|y + z|^{p-1}, \quad y, z \in [0, L). \tag{2.2.33}$$

Considering (2.2.32) and (2.2.33), for $p \in (1, 2)$, we have

$$|\Phi(y) - \Phi(z)| \le 2|y - z|^{p-1}, \quad y, z \in (-L, L). \tag{2.2.34}$$

Thus, for all $p > 1$, (2.2.31) and (2.2.34) give

$$|\Phi(y) - \Phi(z)| \le \widetilde{B}2L|y - z|^\rho, \quad y, z \in (-L, L), \tag{2.2.35}$$

where $\rho := \min\{1, p - 1\}$ and where we use

$$|y - z| \le 2L|y - z|^\rho, \quad y, z \in (-L, L). \tag{2.2.36}$$

Altogether, it holds (see (1.1.6), (2.2.28), (2.2.30), (2.2.35), and (2.2.36))

$$\begin{aligned} |\Phi(\cos_p y) \sin_p y - \Phi(\cos_p z) \sin_p z\,| &\le |\Phi(\cos_p y) \sin_p y - \Phi(\cos_p z) \sin_p y| \\ + |\Phi(\cos_p z) \sin_p y - \Phi(\cos_p z) \sin_p z| &\le L\, |\Phi(\cos_p y) - \Phi(\cos_p z)| \\ + L^{p-1} |\sin_p y - \sin_p z| &\le 2L^2 \widetilde{B} B^\rho\, |y - z|^\rho + 2L^p B|y - z|^\rho \end{aligned} \tag{2.2.37}$$

for all $y, z \in \mathbb{R}$ and $p > 1$. Of course, (2.2.37) guarantees the existence of $\overline{B} > 0$ such that

$$|\Phi(\cos_p y) \sin_p y - \Phi(\cos_p z) \sin_p z| \le \overline{B}\, |y - z|^\rho\,, \quad y, z \in \mathbb{R}. \tag{2.2.38}$$

Inequality (2.2.24) follows directly from (see (2.2.2), (2.2.12), and (2.2.27))

$$\begin{aligned} &\left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) \left(|\cos_p \psi(t)|^p - |\cos_p \varphi(\tau)|^p \right) \mathrm{d}\tau \right| \\ &\le \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau) B |\psi(t) - \varphi(\tau)|\, \mathrm{d}\tau \le \frac{r^+ BC \log t}{\sqrt{t}}, \quad t \ge a. \end{aligned} \tag{2.2.39}$$

Applying (2.2.12) and (2.2.38), we have

$$\left|\Phi(\cos_p\psi(t))\sin_p\psi(t) - \frac{1}{\sqrt{t}}\int\limits_t^{t+\sqrt{t}}\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)\,\mathrm{d}\tau\right|$$

$$\leq \frac{1}{\sqrt{t}}\int\limits_t^{t+\sqrt{t}}|\Phi(\cos_p\psi(t))\sin_p\psi(t) - \Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)|\,\mathrm{d}\tau$$

$$\leq \frac{1}{\sqrt{t}}\int\limits_t^{t+\sqrt{t}}\overline{B}\,|\psi(t)-\varphi(\tau)|^\rho\,\mathrm{d}\tau \leq \frac{1}{\sqrt{t}}\int\limits_t^{t+\sqrt{t}}\frac{\overline{B}C^\rho\log^\rho t}{t^{\frac{\varrho}{2}}}\,\mathrm{d}\tau = \frac{\overline{B}C^\rho\log^\rho t}{t^{\frac{\varrho}{2}}}, \quad t \geq a,$$

i.e., (2.2.25) is true for some $A_2 > 0$ and $\varrho \in (0, \rho/2)$. Analogously as in (2.2.39) (consider (2.2.2), (2.2.12), and (2.2.29)), one can obtain (2.2.26) using

$$\left|\frac{|\sin_p\psi(t)|^p}{\sqrt{t}}\int\limits_t^{t+\sqrt{t}}s(\tau)\,\mathrm{d}\tau - \frac{1}{\sqrt{t}}\int\limits_t^{t+\sqrt{t}}s(\tau)|\sin_p\varphi(\tau)|^p\,\mathrm{d}\tau\right|$$

$$\leq \frac{1}{\sqrt{t}}\int\limits_t^{t+\sqrt{t}}|s(\tau)|B|\psi(t)-\varphi(\tau)|\,\mathrm{d}\tau \leq \frac{s^+BC\log t}{\sqrt{t}}, \quad t \geq a.$$

From the above calculations, we get (2.2.16) and (2.2.17) for a number $P > 0$ and any ϱ such that $\varrho \in (0, \rho/2) = (0, \min\{p-1, 1\}/2)$ and $\varrho < 1/3$ (see (2.2.23)). □

Lemma 2.2.4. *Let function r be α-periodic and s be β-periodic for arbitrary $\alpha, \beta > 0$. Let φ be a solution of Eq. (2.2.9) on $[a, \infty)$. Then, there exist $\widetilde{P} > 0$ and $\tilde{\varrho} > 0$ such that the function $\psi : \mathbb{R}_a \to \mathbb{R}$ defined by (2.2.11) satisfies the inequality*

$$\psi' \leq \frac{1}{t}\left[|\cos_p\psi|^p M(r) - \Phi(\cos_p\psi)\sin_p\psi + M(s)\frac{|\sin_p\psi|^p}{p-1} + \frac{\widetilde{P}}{t^{\tilde{\varrho}}}\right]. \tag{2.2.40}$$

Proof. From Lemma 2.2.3 (see (2.2.16)), we know that ψ satisfies the inequality

$$\begin{aligned}\psi' \leq \frac{1}{t}\Bigg[&\frac{|\cos_p\psi|^p}{\sqrt{t}}\int\limits_t^{t+\sqrt{t}}r(\tau)\,\mathrm{d}\tau - \Phi(\cos_p\psi)\sin_p\psi \\ &+ \frac{|\sin_p\psi|^p}{(p-1)\sqrt{t}}\int\limits_t^{t+\sqrt{t}}s(\tau)\,\mathrm{d}\tau + \frac{P}{t^\varrho}\Bigg]\end{aligned} \tag{2.2.41}$$

for some $P > 0$ and $\varrho \in (0, 1/3)$. Let $t \geq a$ be arbitrarily given. Let $n \in \mathbb{N} \cup \{0\}$ be such that $n\alpha \leq \sqrt{t} < (n+1)\alpha$. Using the periodicity of function r and (2.2.2), we obtain

$$\begin{aligned}\left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau)\,\mathrm{d}\tau - M(r) \right| &\leq \left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau)\,\mathrm{d}\tau - \frac{1}{\sqrt{t}} \int_t^{t+n\alpha} r(\tau)\,\mathrm{d}\tau \right| \\ &\quad + \left| \frac{1}{\sqrt{t}} \int_t^{t+n\alpha} r(\tau)\,\mathrm{d}\tau - \frac{1}{n\alpha} \int_t^{t+n\alpha} r(\tau)\,\mathrm{d}\tau \right| \\ &\leq \frac{r^+\alpha}{\sqrt{t}} + \left(\frac{1}{n\alpha} - \frac{1}{\sqrt{t}} \right) n\alpha M(r) \leq \frac{[r^+ + M(r)]\,\alpha}{\sqrt{t}}.\end{aligned} \tag{2.2.42}$$

Analogously, we can obtain

$$\left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau)\,\mathrm{d}\tau - M(s) \right| \leq \frac{[s^+ + M(s)]\,\beta}{\sqrt{t}}. \tag{2.2.43}$$

Obviously, inequalities (2.2.41), (2.2.42), and (2.2.43) give the statement of the lemma. $\square$

Next, we deal with a perturbed equation and we state the equation for its Prüfer angle. We also mention a consequence of Lemma 2.2.3, as the below given Lemma 2.2.6, which will be essential in § 2.2.3.

Lemma 2.2.5. *There exists $\varepsilon > 0$ such that the equation*

$$\left[\left(1 + \frac{\varepsilon}{\log^2 t} \right)^{-\frac{p}{q}} \Phi(x') \right]' + \frac{\Phi(x)}{t^p} \left(q^{-p} + \frac{\varepsilon}{\log^2 t} \right) = 0 \tag{2.2.44}$$

is non-oscillatory.

Proof. The lemma follows from [16, Theorem 4.1] (see also [17]). $\square$

Considering (2.2.9), the equation for the Prüfer angle η associated to Eq. (2.2.44) is

$$\begin{aligned}\eta' = \frac{1}{t} \Bigg[&\left(1 + \frac{\varepsilon}{\log^2 t} \right) |\cos_p \eta|^p - \Phi(\cos_p \eta) \sin_p \eta \\ &+ \left(q^{-p} + \frac{\varepsilon}{\log^2 t} \right) \frac{|\sin_p \eta|^p}{p-1} \Bigg].\end{aligned} \tag{2.2.45}$$

Lemma 2.2.6. *Let η be a solution of Eq. (2.2.45) on $[a, \infty)$. Then, there exist $\widehat{P} > 0$ and $\hat{\varrho} > 0$ such that the function $\zeta : \mathbb{R}_a \to \mathbb{R}$ defined as*

$$\zeta(t) := \int_t^{t+\sqrt{t}} \frac{\eta(\tau)}{\sqrt{\tau}}\,\mathrm{d}\tau, \quad t \geq a,$$

satisfies the inequality

$$\begin{aligned}\zeta' \geq \frac{1}{t}\Bigg[|\cos_p \zeta|^p \left(1 + \frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \Phi(\cos_p \zeta)\sin_p \zeta \\ + \frac{|\sin_p \zeta|^p}{p-1}\left(q^{-p} + \frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \frac{\widehat{P}}{t^{\hat{\varrho}}}\Bigg].\end{aligned} \tag{2.2.46}$$

Proof. Since Eq. (2.2.44) is a special case of Eq. (2.2.1) for

$$r(t) = 1 + \frac{\varepsilon}{\log^2 t}, \qquad s(t) = q^{-p} + \frac{\varepsilon}{\log^2 t},$$

we can use the above lemmata for ζ which corresponds to ψ.

Especially, from Lemma 2.2.3 (see (2.2.17)), we have

$$\begin{aligned}\zeta' &\geq \frac{1}{t}\Bigg[\frac{|\cos_p \zeta|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} \left(1 + \frac{\varepsilon}{\log^2 \tau}\right) \mathrm{d}\tau - \Phi(\cos_p \zeta)\sin_p \zeta \\ &\qquad + \frac{|\sin_p \zeta|^p}{(p-1)\sqrt{t}} \int_t^{t+\sqrt{t}} \left(q^{-p} + \frac{\varepsilon}{\log^2 \tau}\right) \mathrm{d}\tau - \frac{P}{t^{\varrho}}\Bigg] \\ &\geq \frac{1}{t}\Bigg[|\cos_p \zeta|^p \left(1 + \frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \Phi(\cos_p \zeta)\sin_p \zeta \\ &\qquad + \frac{|\sin_p \zeta|^p}{p-1}\left(q^{-p} + \frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right) - \frac{P}{t^{\varrho}}\Bigg].\end{aligned}$$

It means that it suffices to put $\widehat{P} = P$ and $\hat{\varrho} = \varrho$ in (2.2.46). □

§ 2.2.3 Main results and examples

Now, we can prove the announced result.

Theorem 2.2.1. *If function r is α-periodic and has mean value $M(r) = 1$ and if function s is β-periodic and has mean value $M(s) = q^{-p}$, then Eq. (2.2.1) is non-oscillatory.*

Proof. Taking into account the half-linear Pythagorean identity (see (1.1.5)), we observe

$$\max\{|\sin_p y|^p, |\cos_p y|^p\} \geq \frac{1}{2}, \quad y \in \mathbb{R}.$$

Hence, for $\varepsilon > 0$ from the statement of Lemma 2.2.5, there exists $\delta > 0$ with the property that

$$\varepsilon\,|\cos_p y|^p + \frac{\varepsilon\,|\sin_p y|^p}{p-1} > \delta, \quad y \in \mathbb{R};$$

i.e., it holds

$$\frac{\varepsilon\,|\cos_p y|^p}{\log^2\left[t+\sqrt{t}\right]}+\frac{\varepsilon\,|\sin_p y|^p}{(p-1)\log^2\left[t+\sqrt{t}\right]}>\frac{D}{t^{\varrho}},\quad y\in\mathbb{R}, \tag{2.2.47}$$

for any constant $D>0$ and $\varrho>0$ and for all sufficiently large t.

Let φ be a solution of Eq. (2.2.9) which is associated to Eq. (2.2.1). Lemma 2.2.4 says that the function ψ defined by (2.2.11) satisfies inequality (2.2.40). Thus, considering (2.2.47), where $D=\widetilde{P}+\widehat{P}$ and $\varrho=\min\{\tilde{\varrho},\hat{\varrho}\}$, we have

$$\begin{aligned}\psi' &\leq \frac{1}{t}\left[|\cos_p\psi|^p-\Phi(\cos_p\psi)\sin_p\psi+q^{-p}\frac{|\sin_p\psi|^p}{p-1}+\frac{\widetilde{P}}{t^{\tilde{\varrho}}}\right]\\ &< \frac{1}{t}\left[|\cos_p\psi|^p\left(1+\frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right)-\Phi(\cos_p\psi)\sin_p\psi\right.\\ &\qquad\left.+\frac{|\sin_p\psi|^p}{p-1}\left(q^{-p}+\frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right)-\frac{\widehat{P}}{t^{\hat{\varrho}}}\right]\end{aligned} \tag{2.2.48}$$

for sufficiently large t. It is well-known that the non-oscillation of Eq. (2.2.1) is equivalent to the boundedness from above of the Prüfer angle φ (given by (2.2.9)). We can refer, e.g., to [21, Section 1.1.3], [19], [71] (or consider directly (2.2.4) together with Eq. (2.2.9) when $\sin_p\varphi=0$). We remark that the space of all values of φ is unbounded if and only if $\lim_{t\to\infty}\varphi(t)=\infty$. It follows from the periodicity of the half-linear sine function and the right-hand side of Eq. (2.2.9) for values φ satisfying $\sin_p\varphi=0$ (when the derivative is positive).

Considering Lemma 2.2.5, we know that the Prüfer angle η given by Eq. (2.2.45) is bounded. Lemma 2.2.2 says that φ is bounded if and only if ψ is bounded. In particular, ζ is bounded, because η, ζ are special cases of φ, ψ, respectively. Thus, Lemma 2.2.6 together with (2.2.48) guarantees that the considered solution φ (given by Eq. (2.2.9)) is bounded, i.e., Eq. (2.2.1) is non-oscillatory. Indeed, it suffices to consider the solutions μ, ν of the equations

$$\begin{aligned}\mu' &= \frac{1}{t}\left[|\cos_p\mu|^p-\Phi(\cos_p\mu)\sin_p\mu+q^{-p}\frac{|\sin_p\mu|^p}{p-1}+\frac{\widetilde{P}}{t^{\tilde{\varrho}}}\right],\\ \nu' &= \frac{1}{t}\left[|\cos_p\nu|^p\left(1+\frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right)-\Phi(\cos_p\nu)\sin_p\nu\right.\\ &\qquad\left.+\frac{|\sin_p\nu|^p}{p-1}\left(q^{-p}+\frac{\varepsilon}{\log^2\left[t+\sqrt{t}\right]}\right)-\frac{\widehat{P}}{t^{\hat{\varrho}}}\right]\end{aligned}$$

determined by the same initial condition $\mu(T)=\nu(T)=0$, where T is sufficiently large. We have $\nu(t)\geq\mu(t)$, $t\geq T$. Therefore (see again (2.2.12)),

$$\limsup_{t\to\infty}\zeta(t)=\limsup_{t\to\infty}\eta(t)<\infty$$

gives

$$\limsup_{t\to\infty} \varphi(t) = \limsup_{t\to\infty} \psi(t) < \infty.$$

$\square$

We formulate the following direct consequence of Theorem 2.1.4.

Corollary 2.2.1. *Let r be an α-periodic function having mean value $M(r) = 1$ and let s be a β-periodic function. Eq.* (2.2.1) *is oscillatory if $M(s) > q^{-p}$; and Eq.* (2.2.1) *is non-oscillatory if $M(s) < q^{-p}$.*

Remark 2.2.2. In fact, the non-oscillatory part of Corollary 2.2.1 is also a consequence of our Theorem 2.2.1 and the half-linear Sturm comparison theorem 1.1.3.

Using Corollary 2.2.1, we can improve Theorem 2.2.1 in the next form common in the literature.

Theorem 2.2.2. *Let function f be α-periodic, positive, and continuous and let function h be β-periodic and continuous for arbitrary $\alpha, \beta > 0$. Consider the half-linear equation*

$$[f(t)\Phi(x')]' + \frac{h(t)}{t^p}\Phi(x) = 0. \tag{2.2.49}$$

Let

$$\gamma := q^{-p}\left[M\left(f^{1-q}\right)\right]^{1-p} = q^{-p}\left[\frac{1}{\alpha}\int_0^\alpha f^{1-q}(\tau)\,\mathrm{d}\tau\right]^{1-p}. \tag{2.2.50}$$

(i) If $M(h) > \gamma$, then Eq. (2.2.49) *is oscillatory.*

(ii) If $M(h) \le \gamma$, then Eq. (2.2.49) *is non-oscillatory.*

Proof. We rewrite Eq. (2.2.49) as

$$\left[\left[f^{1-q}(t)\right]^{-\frac{p}{q}}\Phi(x')\right]' + \frac{h(t)}{t^p}\Phi(x) = 0,$$

i.e., it takes the form of Eq. (2.2.1) for

$$r(t) = \frac{f^{1-q}(t)}{M\left(f^{1-q}\right)}, \qquad s(t) = \left[M\left(f^{1-q}\right)\right]^{\frac{p}{q}} h(t).$$

Theorem 2.2.1 and Corollary 2.2.1 give that Eq. (2.2.49) is non-oscillatory if and only if

$$M(s) = \left[M\left(f^{1-q}\right)\right]^{\frac{p}{q}} M(h) = \left[M\left(f^{1-q}\right)\right]^{p-1} M(h) \le q^{-p}.$$

Using γ given in (2.2.50), we can reformulate this observation as follows. Eq. (2.2.49) is non-oscillatory if and only if $M(h) \le \gamma$. $\square$

Remark 2.2.3. Let us consider the case when $M(h) = \gamma$. Note that it is not possible to generalize the above obtained result (see Theorem 2.2.2 or directly Theorem 2.2.1) for general function h having mean value. It follows, e.g., from the main result of [19]. We conjecture that such a generalization is not true even for limit periodic and almost periodic functions on the position of h. Our conjecture is based on the constructions mentioned in [76] (or see [75, Theorem 3.5] together with [11, Theorem 1.27]).

Immediately, Theorem 2.2.2 guarantees the conditional oscillation of general periodic linear equations which is explicitly embodied in the below mentioned corollary.

Corollary 2.2.2. *Let g_1, g_2 be periodic and continuous functions and let g_1 be positive. The equation*

$$\left[\frac{x'}{g_1(t)}\right]' + \frac{g_2(t)}{t^2}x = 0$$

is oscillatory if and only if $M(g_1)\,M(g_2) > 1/4$.

Proof. It suffices to put $p = 2$ in Theorem 2.2.2. □

Remark 2.2.4. If $M(g_1)\,M(g_2) \neq 1/4$ and if g_2 is positive, then the statement of Corollary 2.2.2 follows from many known results.

To illustrate Theorem 2.2.2 and Corollary 2.2.2, we give the following two examples which are not generally solvable using known oscillatory criteria. We recall that the most general result concerning the conditional oscillation of Eq. (2.2.1) is proved in Section 2.1 (see [35]). There is analyzed the conditional oscillation of equations with coefficients having mean values. The critical constant is found, but the critical case remains unsolved. Remark 2.2.3 is devoted to the description of this problem.

On the other hand, the critical case is studied in papers [17, 19], where the coefficients of the considered equations have the same period. The critical case with different periods of coefficients has not been analyzed in the literature.

Example 2.2.1. Let $\alpha > 1/2$, $\beta_1, \beta_2 \neq 0$, $p = 3/2$. The coefficients of the half-linear equation

$$\left[\frac{\Phi(x')}{\alpha + \cos[\beta_1 t]\sin[\beta_1 t]}\right]' + \frac{(\cos[\beta_2 t]\sin[\beta_2 t])^2}{t^{\frac{3}{2}}}\Phi(x) = 0 \tag{2.2.51}$$

satisfy the conditions of Theorem 2.2.2. Since

$$M\left((\cos[\beta_2 t]\sin[\beta_2 t])^2\right) = \frac{1}{8}$$

and (see (2.2.50))

$$\gamma = 3^{-\frac{3}{2}}\left[M\left((\alpha + \cos[\beta_1 t]\sin[\beta_1 t])^2\right)\right]^{-\frac{1}{2}} = \frac{1}{\sqrt{27\left(\alpha^2 + \frac{1}{8}\right)}},$$

Eq. (2.2.51) is non-oscillatory if and only if $1 + 8\alpha^2 \leq (8/3)^3$. We remark that this equivalence is new for all $\beta_1, \beta_2 \neq 0$ satisfying $\beta_1/\beta_2 \notin \mathbb{Q}$, because, in this case, the coefficient in the differential term and the coefficient in the potential of Eq. (2.2.51) do not have any common period.

Example 2.2.2. Let $\sigma(1), \sigma(2) > 1$ be arbitrary. The linear equations

$$\left[\frac{x'}{2+\sin_{\sigma(1)} t}\right]' + \frac{1+\sin_{\sigma(2)} t}{8t^2} x = 0, \tag{2.2.52}$$

$$\left[\frac{x'}{2+\sin_{\sigma(1)} t}\right]' + \frac{1+\cos_{\sigma(2)} t}{8t^2} x = 0, \tag{2.2.53}$$

$$\left[\frac{x'}{2+\cos_{\sigma(1)} t}\right]' + \frac{1+\sin_{\sigma(2)} t}{8t^2} x = 0, \tag{2.2.54}$$

$$\left[\frac{x'}{2+\cos_{\sigma(1)} t}\right]' + \frac{1+\cos_{\sigma(2)} t}{8t^2} x = 0 \tag{2.2.55}$$

are in the so-called border case $M(g_1)\, M(g_2) = 1/4$ (see Corollary 2.2.2), because

$$M(c + d \sin_\sigma t) = M(c + d\cos_\sigma t) = c, \quad c, d \in \mathbb{R},\ \sigma > 1.$$

Nevertheless, we actually know that these equations are non-oscillatory. This fact does not follow from any previous result for, e.g., $\sigma(1) = 2$, $\sigma(2) = 3$. Indeed, in this case, the coefficients in the differential terms of Eq. (2.2.52), (2.2.53), (2.2.54), and (2.2.55) have the period $2\pi_2 = 2\pi$ and the coefficients in the potentials have the period $2\pi_3 = 8\pi\sqrt{3}/9$ (see (1.1.3)). Since $\pi_3/\pi_2 = 4\sqrt{3}/9 \notin \mathbb{Q}$, the coefficients do not have any common period for $\sigma(1) = 2$, $\sigma(2) = 3$.

Applying known comparison theorems, we can obtain several new results which follow from Theorem 2.2.2. We mention at least one known comparison theorem and a new result as Corollary 2.2.3 with the below given Example 2.2.3.

Theorem 2.2.3. *Let $r : \mathbb{R}_a \to \mathbb{R}$ be a continuous positive function satisfying*

$$\int_a^\infty r^{1-q}(\tau)\,\mathrm{d}\tau = \infty \tag{2.2.56}$$

and $s_1, s_2 : \mathbb{R}_a \to \mathbb{R}$ be continuous functions satisfying

$$\int_t^\infty s_2(\tau)\,\mathrm{d}\tau \ge \left| \int_t^\infty s_1(\tau)\,\mathrm{d}\tau \right|, \quad t \ge T, \tag{2.2.57}$$

for some $T \ge a$, where the integrals $\int_T^\infty s_1(\tau)\,\mathrm{d}\tau$, $\int_T^\infty s_2(\tau)\,\mathrm{d}\tau$ are convergent. Consider the equations

$$[r(t)\Phi(x')]' + s_1(t)\Phi(x) = 0, \tag{2.2.58}$$

$$[r(t)\Phi(x')]' + s_2(t)\Phi(x) = 0. \tag{2.2.59}$$

If Eq. (2.2.59) is non-oscillatory, then Eq. (2.2.58) is non-oscillatory as well.

Proof. See [21, Theorem 2.3.1]. □

Corollary 2.2.3. *Let function r be α-periodic, positive, and continuous and let function s be β-periodic and continuous for arbitrary $\alpha, \beta > 0$. Consider the equation*

$$\left[r^{-\frac{p}{q}}(t)\Phi\left(x'\right)\right]' + z(t)\Phi(x) = 0, \tag{2.2.60}$$

where $z : \mathbb{R}_a \to \mathbb{R}$ is a continuous function satisfying

$$\left|\int_a^\infty z(\tau)\,\mathrm{d}\tau\right| < \infty. \tag{2.2.61}$$

If

$$M(s) = \frac{1}{\beta}\int_0^\beta s(\tau)\,\mathrm{d}\tau \le q^{-p}\left[M\left(r\right)\right]^{1-p} = q^{-p}\left[\frac{1}{\alpha}\int_0^\alpha r(\tau)\,\mathrm{d}\tau\right]^{1-p} \tag{2.2.62}$$

and if there exists $t_0 \ge a$ for which

$$\int_t^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \ge \left|\int_t^\infty z(\tau)\,\mathrm{d}\tau\right|, \quad t \ge t_0, \tag{2.2.63}$$

then Eq. (2.2.60) *is non-oscillatory.*

Proof. The corollary follows from Theorem 2.2.2, (ii) and Theorem 2.2.3. At first, we discuss the assumptions of Theorem 2.2.3. Putting $s_1(t) = z(t)$, $s_2(t) = s(t)/t^p$ for $t \ge a$, we consider Eq. (2.2.60) as Eq. (2.2.58) and the equation

$$\left[r^{-\frac{p}{q}}(t)\Phi\left(x'\right)\right]' + \frac{s(t)}{t^p}\Phi(x) = 0 \tag{2.2.64}$$

as Eq. (2.2.59), i.e., we replace function r by $r^{-\frac{p}{q}}$. Since

$$\int_a^\infty \left[r^{-\frac{p}{q}}(\tau)\right]^{1-q}\mathrm{d}\tau = \int_a^\infty r(\tau)\,\mathrm{d}\tau = \lim_{n\to\infty} n\int_a^{a+\alpha} r(\tau)\,\mathrm{d}\tau = \infty,$$

condition (2.2.56) from Theorem 2.2.3 is fulfilled. The integral $\int_a^\infty s_1(\tau)\,\mathrm{d}\tau$ is convergent due to (2.2.61). The periodicity together with the continuity of function s implies its boundedness. Therefore (consider that $p > 1$), we have

$$\left|\int_a^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau\right| \le \int_a^\infty \frac{|s(\tau)|}{\tau^p}\,\mathrm{d}\tau < \infty.$$

Hence, the integral $\int_a^\infty s_2(\tau)\,\mathrm{d}\tau$ is convergent as well. Moreover, (2.2.63) gives (2.2.57).

To finish the proof, it suffices to show that Eq. (2.2.64) is non-oscillatory which implies the non-oscillation of Eq. (2.2.60) (consider Theorem 2.2.3). Putting $f(t) = r^{-\frac{p}{q}}(t)$ and $h(t) = s(t)$ in Theorem 2.2.2, we can see that (2.2.62) ensures the validity of the inequality in Theorem 2.2.2, (ii). Indeed, it holds

$$\frac{1}{\alpha}\int_0^\alpha f^{1-q}(\tau)\,\mathrm{d}\tau = \frac{1}{\alpha}\int_0^\alpha \left[r^{-\frac{p}{q}}(\tau)\right]^{1-q}\mathrm{d}\tau = \frac{1}{\alpha}\int_0^\alpha r(\tau)\,\mathrm{d}\tau.$$

Thus, Eq. (2.2.64) is non-oscillatory and, consequently, Eq. (2.2.60) is non-oscillatory as well. □

Example 2.2.3. Let $a, b \neq 0$ be arbitrarily given. We define

$$\overline{z}(t) := \left(\frac{\pi}{4q}\right)^p \left(|\sin[bt]| + |\cos[bt]| + \tilde{z}(t)\right), \quad t \in \mathbb{R}_3,$$

where

$$\tilde{z}(t) := \begin{cases} (t - 2^n)\,\dfrac{n-1}{n}, & t \in \left[2^n, 2^n + \dfrac{1}{4}\right), \ n \in \mathbb{N} \setminus \{1\}; \\ \left(2^n + \dfrac{1}{2} - t\right)\dfrac{n-1}{n}, & t \in \left[2^n + \dfrac{1}{4}, 2^n + \dfrac{1}{2}\right), \ n \in \mathbb{N} \setminus \{1\}; \\ -2\left(t - 2^n - \dfrac{1}{2}\right)\dfrac{n-1}{n}, & t \in \left[2^n + \dfrac{1}{2}, 2^n + \dfrac{3}{4}\right), \ n \in \mathbb{N} \setminus \{1\}; \\ -2\left(2^n + 1 - t\right)\dfrac{n-1}{n}, & t \in \left[2^n + \dfrac{3}{4}, 2^n + 1\right], \ n \in \mathbb{N} \setminus \{1\}; \\ 0, & t \in \mathbb{R}_3 \setminus \bigcup\limits_{n\in\mathbb{N}\setminus\{1\}} \left[2^n, 2^n + 1\right]. \end{cases}$$

We consider the equation

$$\left[\left(|\sin[at]| + |\cos[at]|\right)^{-\frac{p}{q}} \Phi(x')\right]' + \frac{\overline{z}(t)}{t^p}\,\Phi(x) = 0 \tag{2.2.65}$$

which is in the form of Eq. (2.2.60) for $z(t) = \overline{z}(t)/t^p$. It is seen that

$$0 \leq \int_t^\infty z(\tau)\,\mathrm{d}\tau = \int_t^\infty |z(\tau)|\,\mathrm{d}\tau \leq \int_t^\infty \frac{H}{\tau^p}\,\mathrm{d}\tau < \infty, \quad t \geq 3, \tag{2.2.66}$$

for some $H > 0$. We put

$$s(t) := \left(\frac{\pi}{4q}\right)^p \left(|\sin[bt]| + |\cos[bt]|\right), \quad t \in \mathbb{R}_3.$$

Directly from $\lim_{t\to\infty}\left(\frac{t}{t+1}\right)^p = 1$, we get

$$\int_t^\infty \frac{\tilde{z}(\tau)}{\tau^p}\,\mathrm{d}\tau < 0, \quad \text{i.e.,} \quad \int_t^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau > \int_t^\infty z(\tau)\,\mathrm{d}\tau,$$

for all sufficiently large t. Hence (see also (2.2.66)), we have (2.2.63). Since

$$M(s) = \left(\frac{\pi}{4q}\right)^p \frac{4}{\pi} = q^{-p}\left[\frac{4}{\pi}\right]^{1-p} = q^{-p}\left[M\left(|\sin[at]| + |\cos[at]|\right)\right]^{1-p},$$

inequality (2.2.62) is satisfied as well. Finally, applying Corollary 2.2.3, we obtain the non-oscillation of Eq. (2.2.65) which does not follow from any known theorem.

2.3 Perturbed equations with sums of periodic coefficients

In this section, we continue in the study of Eq. (2.1.1). We are interested in perturbations of both terms of this equation when both of the perturbations are the sums of periodic functions. In contrast with the situation common in the literature, the functions in the perturbations do not need to have any common period and can change sign. We prove that all considered equations are non-oscillatory in the critical case. According to our best knowledge, this result is new also in the linear case (i.e., for $p = 2$).

§ 2.3.1 Preliminaries

This paragraph is devoted to the description of the considered equations and to the modified Prüfer angle which is the main method in our processes.

Our main objective is to give a non-oscillation criterion for the half-linear differential equations in the form

$$\left[\left(r_0(t) + \frac{r_1(t)}{\log^2 t}\right)^{-\frac{p}{q}} \Phi\left(x'\right)\right]' + \left(s_0(t) + \frac{s_1(t)}{\log^2 t}\right)\frac{\Phi(x)}{t^p} = 0, \tag{2.3.1}$$

where $r_0, r_1, s_0, s_1 : \mathbb{R}_a \to \mathbb{R}$, $a \geq \mathrm{e}$, (e stands for the base of the natural logarithm log and $\mathbb{R}_a := [a, \infty)$) are continuous functions such that r_0 is positive and α-periodic, s_0 is α-periodic, and

$$r_1(t) = \sum_{i=1}^n R_i(t), \quad s_1(t) = \sum_{i=1}^n S_i(t), \quad t \in \mathbb{R}_a, \tag{2.3.2}$$

for arbitrarily given periodic continuous functions R_i and S_i with periods α_i and β_i, respectively. Of course, we can assume that all considered periods $\alpha, \alpha_i, \beta_i$ are positive and that some of functions R_i, S_i are identically zero.

At this place, we recall that we use the definition of mean values 2.1.1 for continuous functions as a tool that helps us to identify the critical case for studied equations. It is seen that functions r_1, s_1 given in (2.3.2) have mean values

$$M(r_1) = \sum_{i=1}^{n} M(R_i), \qquad M(s_1) = \sum_{i=1}^{n} M(S_i). \tag{2.3.3}$$

Concerning the presented results, we will assume that $M(r_1), M(s_1) \geq 0$.

In fact, we study oscillatory properties of the equation

$$\left[\left(r_0(t) + \frac{\sum_{i=1}^{m} R_i(t)}{\log^2 t}\right)^{-\frac{p}{q}} \Phi\left(x'\right)\right]' + \left(s_0(t) + \frac{\sum_{i=1}^{n} S_i(t)}{\log^2 t}\right) \frac{\Phi(x)}{t^p} = 0 \tag{2.3.4}$$

with periodic coefficients $R_1, \ldots, R_m$, $S_1, \ldots, S_n$ on $\mathbb{R}_a$ at infinity (i.e., value a is large enough) when $\sum_{i=1}^{m} M(R_i) \geq 0$, $\sum_{i=1}^{n} M(S_i) \geq 0$. For simplicity, we will consider Eq. (2.3.1) only in the critical case (see the below given Theorem 2.3.3 and [16, 17, 22, 35]) given by

$$[M(r_0)]^{\frac{p}{q}} M(s_0) = \frac{1}{\alpha^p} \left(\int_a^{a+\alpha} r_0(\tau)\,\mathrm{d}\tau\right)^{\frac{p}{q}} \left(\int_a^{a+\alpha} s_0(\tau)\,\mathrm{d}\tau\right) = q^{-p}, \tag{2.3.5}$$

$$M(s_1)\,[M(r_0)]^{\frac{p}{q}} + \frac{p}{q^{p+1}}\, M(r_1)\,[M(r_0)]^{-1} = \frac{q^{1-p}}{2}, \tag{2.3.6}$$

i.e.,

$$\lim_{t\to\infty} \left[\frac{1}{t^p}\left(\int_a^{a+t} s_1(\tau)\,\mathrm{d}\tau\right)\left(\int_a^{a+t} r_0(\tau)\,\mathrm{d}\tau\right)^{\frac{p}{q}} + \frac{p \int_a^{a+t} r_1(\tau)\,\mathrm{d}\tau}{q^{p+1} \int_a^{a+t} r_0(\tau)\,\mathrm{d}\tau}\right] = \frac{q^{1-p}}{2}.$$

Then (see the below given Theorem 2.3.4), we formulate the general result about the oscillation and non-oscillation of Eq. (2.3.4).

To study Eq. (2.3.1), we will consider the equation

$$\begin{aligned}\left[\left(M(r_0) + \frac{M(r_1)}{\log^2 t} + \frac{1}{\log^4 t}\right)^{-\frac{p}{q}} \Phi\left(x'\right)\right]'&\\ + \left(M(s_0) + \frac{M(s_1)}{\log^2 t} + \frac{1}{\log^4 t}\right)\frac{\Phi(x)}{t^p} &= 0\end{aligned} \tag{2.3.7}$$

with constant coefficients and we will also use the notations

$$\begin{aligned} r_0^+ &:= \max\{r_0(t);\, t\in\mathbb{R}_a\}, & r_1^+ &:= \sup\{|r_1(t)|;\, t\in\mathbb{R}_a\},\\ s_0^+ &:= \max\{|s_0(t)|;\, t\in\mathbb{R}_a\}, & s_1^+ &:= \sup\{|s_1(t)|;\, t\in\mathbb{R}_a\},\end{aligned} \tag{2.3.8}$$

and

$$R_i^+ := \max\{|R_i(t)|;\, t\in\mathbb{R}_a\}, \qquad S_i^+ := \max\{|S_i(t)|;\, t\in\mathbb{R}_a\} \tag{2.3.9}$$

for each i.

The main method used in this section is the analogy of the one used in the previous section, i.e., the modified Prüfer transformation (2.2.4) and the equation for the Prüfer angle (2.2.9). In particular, the Prüfer angle φ associated to Eq. (2.3.1) via (2.2.4) satisfies the equation

$$\begin{aligned}\varphi'(t) = \frac{1}{t}\Bigg[&\left(r_0(t) + \frac{r_1(t)}{\log^2 t}\right)|\cos_p \varphi(t)|^p \\ &- \Phi(\cos_p \varphi(t)) \sin_p \varphi(t) + \left(s_0(t) + \frac{s_1(t)}{\log^2 t}\right)\frac{|\sin_p \varphi(t)|^p}{p-1}\Bigg].\end{aligned} \tag{2.3.10}$$

The equation for the Prüfer angle φ associated to Eq. (2.3.7) is (see Eq. (2.2.9))

$$\begin{aligned}\varphi'(t) = \frac{1}{t}\Bigg[&\left(M(r_0) + \frac{M(r_1)}{\log^2 t} + \frac{1}{\log^4 t}\right)|\cos_p \varphi(t)|^p \\ &- \Phi(\cos_p \varphi(t)) \sin_p \varphi(t) + \left(M(s_0) + \frac{M(s_1)}{\log^2 t} + \frac{1}{\log^4 t}\right)\frac{|\sin_p \varphi(t)|^p}{p-1}\Bigg].\end{aligned} \tag{2.3.11}$$

Similarly as in Lemma 2.2.2, for any solution φ of Eq. (2.2.9) on $\mathbb{R}_a$, we define the function $\psi : \mathbb{R}_a \to \mathbb{R}$ by the formula

$$\psi(t) := \int_t^{t+\sqrt{t}} \frac{\varphi(\tau)}{\sqrt{\tau}}\,\mathrm{d}\tau, \quad t \geq a. \tag{2.3.12}$$

This auxiliary function ψ will play an important role in the rest of this section. Note that Eq. (2.3.10) and (2.3.11) are special cases of Eq. (2.2.9). Thus, the above function ψ is introduced also for solutions of Eq. (2.3.10) and (2.3.11).

§ 2.3.2 Lemmata

In this paragraph, we complete necessary statements which we will use to prove the main result. We will need Lemma 2.2.2 from the previous section and a consequence of Lemma 2.2.3.

Lemma 2.3.1. *Let φ be a solution of Eq.* (2.2.9) *on $\mathbb{R}_a$. Then, there exist $A, c > 0$ such that the function $\psi : \mathbb{R}_a \to \mathbb{R}$ defined in* (2.3.12) *satisfies the inequality*

$$\begin{aligned}\Bigg|\psi'(t) - \frac{1}{t}\Bigg[&\frac{|\cos_p \psi(t)|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} r(\tau)\,\mathrm{d}\tau - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \\ &+ \frac{|\sin_p \psi(t)|^p}{(p-1)\sqrt{t}} \int_t^{t+\sqrt{t}} s(\tau)\,\mathrm{d}\tau\Bigg]\Bigg| \leq \frac{A}{t^{1+c}}\end{aligned}$$

for all $t > a$.

Proof. The lemma comes directly from Lemma 2.2.3. □

Next, we will need the following results.

Lemma 2.3.2. *Let φ be a solution of Eq.* (2.3.11) *on $\mathbb{R}_a$. Then, the function $\psi : \mathbb{R}_a \to \mathbb{R}$ defined in* (2.3.12) *satisfies the inequality*

$$\begin{aligned}\psi'(t) \geq \frac{1}{t}\Bigg[&\left(M(r_0) + \frac{M(r_1)}{\log^2 t}\right)|\cos_p \psi(t)|^p - \Phi(\cos_p \psi(t))\sin_p \psi(t)\\ &+ \left(M(s_0) + \frac{M(s_1)}{\log^2 t}\right)\frac{|\sin_p \psi(t)|^p}{p-1} + \frac{1}{\log^5 t}\Bigg]\end{aligned} \tag{2.3.13}$$

for all sufficiently large t.

Proof. From Lemma 2.3.1, we have

$$\begin{aligned}\psi'(t) &\geq \frac{1}{t}\Bigg[\frac{|\cos_p \psi(t)|^p}{\sqrt{t}}\int_t^{t+\sqrt{t}}\left(M(r_0) + \frac{M(r_1)}{\log^2 \tau} + \frac{1}{\log^4 \tau}\right)\mathrm{d}\tau - \Phi(\cos_p \psi(t))\sin_p \psi(t)\\ &\qquad + \frac{|\sin_p \psi(t)|^p}{(p-1)\sqrt{t}}\int_t^{t+\sqrt{t}}\left(M(s_0) + \frac{M(s_1)}{\log^2 \tau} + \frac{1}{\log^4 \tau}\right)\mathrm{d}\tau - \frac{A}{t^c}\Bigg]\\ &\geq \frac{1}{t}\Bigg[|\cos_p \psi(t)|^p\left(M(r_0) + \frac{M(r_1)}{\log^2(t+\sqrt{t}\,)} + \frac{1}{\log^4(t+\sqrt{t}\,)}\right) - \Phi(\cos_p \psi(t))\sin_p \psi(t)\\ &\qquad + \frac{|\sin_p \psi(t)|^p}{p-1}\left(M(s_0) + \frac{M(s_1)}{\log^2(t+\sqrt{t}\,)} + \frac{1}{\log^4(t+\sqrt{t}\,)}\right) - \frac{A}{t^c}\Bigg]\end{aligned}$$

for all $t > a$. Via the mean value theorem, one can directly compute

$$0 \leq \limsup_{t\to\infty} \log^2 t\left[\log^2\left(t+\sqrt{t}\right) - \log^2 t\right] \leq \lim_{t\to\infty} \log^2 t\,\frac{2\log t}{t}\,\sqrt{t} = 0. \tag{2.3.14}$$

Thus, we have

$$\left|\frac{M(r_1)}{\log^2(t+\sqrt{t}\,)} - \frac{M(r_1)}{\log^2 t}\right| \leq M(r_1)\frac{\log^2(t+\sqrt{t}\,) - \log^2 t}{\log^4 t} \leq \frac{1}{\log^6 t}, \tag{2.3.15}$$

$$\left|\frac{M(s_1)}{\log^2(t+\sqrt{t}\,)} - \frac{M(s_1)}{\log^2 t}\right| \leq M(s_1)\frac{\log^2(t+\sqrt{t}\,) - \log^2 t}{\log^4 t} \leq \frac{p-1}{\log^6 t} \tag{2.3.16}$$

for all large t. From (1.1.5), it is seen that

$$\max\{|\sin_p y|^p, |\cos_p y|^p\} \geq \frac{1}{2}, \quad y \in \mathbb{R}.$$

Hence, for large t, it holds

$$\frac{|\cos_p y|^p}{\log^4(t+\sqrt{t}\,)} + \frac{|\sin_p y|^p}{(p-1)\log^4(t+\sqrt{t}\,)} > \frac{2}{\log^5 t}, \quad y \in \mathbb{R}. \tag{2.3.17}$$

Altogether, using (2.3.15), (2.3.16), and (2.3.17), we obtain

$$\psi'(t) \geq \frac{1}{t}\left[\left(M(r_0) + \frac{M(r_1)}{\log^2 t}\right)|\cos_p \psi(t)|^p - \Phi(\cos_p \psi(t))\sin_p \psi(t) \right.$$
$$\left. + \left(M(s_0) + \frac{M(s_1)}{\log^2 t}\right)\frac{|\sin_p \psi(t)|^p}{p-1} + \frac{2}{\log^5 t} - \frac{2}{\log^6 t} - \frac{A}{t^c}\right]$$

for large t which gives (2.3.13). □

Lemma 2.3.3. *Let φ be a solution of Eq.* (2.3.10) *on $\mathbb{R}_a$. Then, there exists $B > 0$ such that the function $\psi : \mathbb{R}_a \to \mathbb{R}$ defined by* (2.3.12) *satisfies the inequality*

$$\psi'(t) \leq \frac{1}{t}\left[\left(M(r_0) + \frac{M(r_1)}{\log^2 t}\right)|\cos_p \psi(t)|^p - \Phi(\cos_p \psi(t))\sin_p \psi(t) \right.$$
$$\left. + \left(M(s_0) + \frac{M(s_1)}{\log^2 t}\right)\frac{|\sin_p \psi(t)|^p}{p-1} + \frac{B}{\log^6 t}\right]$$

for all sufficiently large t.

Proof. From Lemma 2.3.1, we know that the inequality

$$\psi'(t) \leq \frac{1}{t}\left[\frac{|\cos_p \psi(t)|^p}{\sqrt{t}} \int_t^{t+\sqrt{t}} \left(r_0(\tau) + \frac{r_1(\tau)}{\log^2 \tau}\right) \mathrm{d}\tau - \Phi(\cos_p \psi(t))\sin_p \psi(t) \right.$$
$$\left. + \frac{|\sin_p \psi(t)|^p}{(p-1)\sqrt{t}} \int_t^{t+\sqrt{t}} \left(s_0(\tau) + \frac{s_1(\tau)}{\log^2 \tau}\right) \mathrm{d}\tau + \frac{A}{t^c}\right]$$

holds for all $t > a$. It means that it suffices to prove

$$\left|\frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r_0(\tau)\,\mathrm{d}\tau - M(r_0)\right| \leq \frac{A_0}{\sqrt{t}}, \qquad \left|\frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} s_0(\tau)\,\mathrm{d}\tau - M(s_0)\right| \leq \frac{B_0}{\sqrt{t}}, \tag{2.3.18}$$

$$\left|\frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \frac{r_1(\tau)}{\log^2 \tau}\,\mathrm{d}\tau - \frac{M(r_1)}{\log^2 t}\right| \leq \frac{A_1}{\log^6 t}, \tag{2.3.19}$$

and
$$\left| \frac{1}{\sqrt{t}} \int_{t}^{t+\sqrt{t}} \frac{s_1(\tau)}{\log^2 \tau} \, \mathrm{d}\tau - \frac{M(s_1)}{\log^2 t} \right| \le \frac{B_1}{\log^6 t} \tag{2.3.20}$$
for some $A_0, B_0, A_1, B_1 > 0$ and for all large t.

Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary continuous periodic function with period $\delta > 0$. Let a given number t be sufficiently large and $l \in \mathbb{N}$ be such that $\sqrt{t} \in [l\delta, (l+1)\delta)$. We have
$$\begin{aligned}
&\left| \frac{1}{\sqrt{t}} \int_{t}^{t+\sqrt{t}} f(\tau) \, \mathrm{d}\tau - M(f) \right| \\
&\le \left| \frac{1}{\sqrt{t}} \int_{t}^{t+\sqrt{t}} f(\tau) \, \mathrm{d}\tau - \frac{1}{\sqrt{t}} \int_{t}^{t+l\delta} f(\tau) \, \mathrm{d}\tau \right| + \left| \frac{1}{\sqrt{t}} \int_{t}^{t+l\delta} f(\tau) \, \mathrm{d}\tau - M(f) \right| \\
&\quad \le \left| \frac{1}{\sqrt{t}} \int_{t+l\delta}^{t+\sqrt{t}} f(\tau) \, \mathrm{d}\tau \right| + \left| \frac{1}{\sqrt{t}} \int_{t}^{t+l\delta} f(\tau) \, \mathrm{d}\tau - \frac{1}{l\delta} \int_{t}^{t+l\delta} f(\tau) \, \mathrm{d}\tau \right| \\
&\qquad \le \frac{\delta \max\limits_{t \in [0,\delta)} |f(t)|}{\sqrt{t}} + \left(\frac{1}{l\delta} - \frac{1}{\sqrt{t}} \right) l\delta M(f) \le \frac{\delta \max\limits_{t \in [0,\delta)} |f(t)| + \delta M(f)}{\sqrt{t}}.
\end{aligned} \tag{2.3.21}$$
Thus, (2.3.18) is valid for (see (2.3.8))
$$A_0 = \alpha \left[r_0^+ + M(r_0) \right], \qquad B_0 = \alpha \left[s_0^+ + M(s_0) \right].$$
Since (2.3.21) is true for any periodic continuous function f, we obtain (see (2.3.2), (2.3.3), (2.3.9))
$$\begin{aligned}
\left| \frac{1}{\sqrt{t}} \int_{t}^{t+\sqrt{t}} r_1(\tau) \, \mathrm{d}\tau - M(r_1) \right| &= \left| \frac{1}{\sqrt{t}} \int_{t}^{t+\sqrt{t}} \sum_{i=1}^{n} R_i(\tau) \, \mathrm{d}\tau - M\left(\sum_{i=1}^{n} R_i \right) \right| \\
&= \left| \sum_{i=1}^{n} \left(\frac{1}{\sqrt{t}} \int_{t}^{t+\sqrt{t}} R_i(\tau) \, \mathrm{d}\tau - M(R_i) \right) \right| \\
&\le \sum_{i=1}^{n} \left| \frac{1}{\sqrt{t}} \int_{t}^{t+\sqrt{t}} R_i(\tau) \, \mathrm{d}\tau - M(R_i) \right| \le \sum_{i=1}^{n} \frac{\alpha_i R_i^+ + \alpha_i M(R_i)}{\sqrt{t}}.
\end{aligned} \tag{2.3.22}$$
Analogously,
$$\left| \frac{1}{\sqrt{t}} \int_{t}^{t+\sqrt{t}} s_1(\tau) \, \mathrm{d}\tau - M(s_1) \right| \le \sum_{i=1}^{n} \frac{\beta_i S_i^+ + \beta_i M(S_i)}{\sqrt{t}}. \tag{2.3.23}$$

Using (2.3.22), we have

$$\begin{aligned}
&\left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \frac{r_1(\tau)}{\log^2 \tau} \,\mathrm{d}\tau - \frac{M(r_1)}{\log^2 t} \right| \\
&\leq \left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \frac{r_1(\tau)}{\log^2 \tau} \,\mathrm{d}\tau - \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \frac{r_1(\tau)}{\log^2 t} \,\mathrm{d}\tau \right| + \left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \frac{r_1(\tau)}{\log^2 t} \,\mathrm{d}\tau - \frac{M(r_1)}{\log^2 t} \right| \\
&\qquad \leq \frac{r_1^+}{\sqrt{t}} \int_t^{t+\sqrt{t}} \left| \frac{1}{\log^2 \tau} - \frac{1}{\log^2 t} \right| \mathrm{d}\tau + \frac{1}{\log^2 t} \left| \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} r_1(\tau) \,\mathrm{d}\tau - M(r_1) \right| \\
&\qquad\qquad \leq r_1^+ \left[\frac{\log^2 \left(t+\sqrt{t}\right) - \log^2 t}{\log^4 t} \right] + \frac{1}{\log^2 t} \sum_{i=1}^{n} \frac{\alpha_i R_i^+ + \alpha_i M(R_i)}{\sqrt{t}}
\end{aligned} \tag{2.3.24}$$

for large t. Considering (2.3.14), we obtain (2.3.19) from (2.3.24). Analogously, one can obtain (2.3.20) applying (2.3.23). Hence, the proof is complete. □

Lemma 2.3.4. *Eq.* (2.3.7) *is non-oscillatory.*

Proof. The non-oscillation of Eq. (2.3.7) follows from [16, Theorem 4.1] (see also [17]) and the Sturm half-linear comparison theorem 1.1.3. More precisely, from [16, Theorem 4.1] it follows that the equation

$$\begin{aligned}
&\left[\left(M(r_0) + \frac{M(r_1)}{\log^2 t} + \frac{\varepsilon}{[\log t \cdot \log(\log t)]^2} \right)^{-\frac{p}{q}} \Phi\left(x'\right) \right]' \\
&\qquad + \left(M(s_0) + \frac{M(s_1)}{\log^2 t} + \frac{\varepsilon}{[\log t \cdot \log(\log t)]^2} \right) \frac{\Phi(x)}{t^p} = 0
\end{aligned} \tag{2.3.25}$$

is non-oscillatory for any sufficiently small $\varepsilon > 0$ (it is described in [17]) and Eq. (2.3.25) is a non-oscillatory majorant of Eq. (2.3.7). □

Lemma 2.3.5. *For a solution φ of Eq.* (2.3.11) *on $\mathbb{R}_a$, it holds*

$$\limsup_{t\to\infty} \varphi(t) = \limsup_{t\to\infty} \psi(t) < \infty, \tag{2.3.26}$$

where ψ is introduced in (2.3.12).

Proof. Lemma 2.3.4 says that any considered solution φ is bounded from above. Indeed, it suffices to consider (2.2.4) and Eq. (2.3.11) when $\sin_p \varphi(t) = 0$. For details, we can refer, e.g., to [21, Section 1.1.3], [19], [39], [71]. Finally, the equality in (2.3.26) follows from Lemma 2.2.2. □

§ 2.3.3 Results and examples

Now, we can prove the announced result.

Theorem 2.3.1. *Eq.* (2.3.1) *with* (2.3.5) *and* (2.3.6) *is non-oscillatory.*

Proof. We recall that the non-oscillation of Eq. (2.3.1) is equivalent to the boundedness of a solution φ of Eq. (2.3.10) on $\mathbb{R}_a$ (see again each one of papers [19], [39]) or [71]. In addition, a solution φ of Eq. (2.3.10) on $\mathbb{R}_a$ is bounded if and only if $\limsup_{t\to\infty} \varphi(t) < \infty$. It is seen from the right-hand side of Eq. (2.3.10) when $\sin_p \varphi(t) = 0$.

Let sufficiently large $T > a$ be given. Let us consider an arbitrary solution φ of Eq. (2.3.10) on $\mathbb{R}_T$ and the corresponding function $\psi : \mathbb{R}_T \to \mathbb{R}$ given by (2.3.12). Lemma 2.3.3 ensures

$$\psi'(t) \le \frac{1}{t}\Bigg[\left(M(r_0) + \frac{M(r_1)}{\log^2 t}\right) |\cos_p \psi(t)|^p - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \\ + \left(M(s_0) + \frac{M(s_1)}{\log^2 t}\right) \frac{|\sin_p \psi(t)|^p}{p-1} + \frac{B}{\log^6 t}\Bigg], \quad t > T.$$

Thus, we have

$$\psi'(t) < \frac{1}{t}\Bigg[\left(M(r_0) + \frac{M(r_1)}{\log^2 t}\right) |\cos_p \psi(t)|^p - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \\ + \left(M(s_0) + \frac{M(s_1)}{\log^2 t}\right) \frac{|\sin_p \psi(t)|^p}{p-1} + \frac{1}{\log^5 t}\Bigg], \quad t > T, \tag{2.3.27}$$

because T can be chosen arbitrarily.

We consider the solution $\tilde\varphi$ of Eq. (2.3.11) given by the initial condition (see (1.1.3))

$$\tilde\varphi(T) = \max\left\{\varphi\left(T + \sqrt{t}\right); t \in [0, T]\right\} + \pi_p \tag{2.3.28}$$

and the corresponding function $\tilde\psi$ given by (2.3.12). Considering the form of Eq. (2.3.11) and (2.3.28), one can show that

$$\psi(T) < \tilde\psi(T). \tag{2.3.29}$$

Lemma 2.3.5 says that (2.3.26) is valid for $\tilde\varphi$ and $\tilde\psi$, i.e., it holds

$$\limsup_{t\to\infty} \tilde\varphi(t) = \limsup_{t\to\infty} \tilde\psi(t) < \infty. \tag{2.3.30}$$

Lemma 2.3.2 gives

$$\tilde\psi'(t) \ge \frac{1}{t}\Bigg[\left(M(r_0) + \frac{M(r_1)}{\log^2 t}\right) \left|\cos_p \tilde\psi(t)\right|^p - \Phi\left(\cos_p \tilde\psi(t)\right) \sin_p \tilde\psi(t) \\ + \left(M(s_0) + \frac{M(s_1)}{\log^2 t}\right) \frac{\left|\sin_p \tilde\psi(t)\right|^p}{p-1} + \frac{1}{\log^5 t}\Bigg], \quad t > T. \tag{2.3.31}$$

Considering (2.3.27), (2.3.29), (2.3.30), and (2.3.31), we obtain

$$\limsup_{t\to\infty} \psi(t) \le \limsup_{t\to\infty} \tilde{\psi}(t) < \infty.$$

Indeed, it suffices to consider the case when $\psi(t_0) = \tilde{\psi}(t_0)$ for any $t_0 > T$. Using Lemma 2.2.2, we know that φ is bounded from above which implies the non-oscillation of Eq. (2.3.1). $\square$

To illustrate our results, we mention examples. We remark that all given examples are not generally solvable using any previously known non-oscillation criteria.

Example 2.3.1. Immediately, Theorem 2.3.1 gives the non-oscillation of several equations. For example, the equations

$$\left[\left(1+\frac{\sin t}{p}+\frac{q^2+\sin\left(\sqrt{2}t\right)}{2p\log^2 t}\right)^{-\frac{p}{q}}\Phi\left(x'\right)\right]'+\left(q^{-p}+\sin\left(5t\right)\right)\frac{\Phi(x)}{t^p}=0,$$

$$\left[\left(1+\arctan\left(\sin_3 t\right)\right)^{-\frac{p}{q}}\Phi\left(x'\right)\right]'+q^{-p}\left(1+\frac{\pi q\left|\sin t\right|}{4\log^2 t}\right)\frac{\Phi(x)}{t^p}=0$$

are non-oscillatory.

Theorem 2.3.1 implies new results in many special cases. We obtain a new result even for linear equations with constant and periodic coefficients which is formulated as the corollary below.

Corollary 2.3.1. *Let f, g be periodic and continuous functions such that $M(f), M(g) \ge 0$ and $M(f) + M(g) = 1$. The equation*

$$\left[\left(1+\frac{f(t)}{\log^2 t}\right)^{-1}x'\right]'+\frac{1}{4t^2}\left(1+\frac{g(t)}{\log^2 t}\right)x=0 \tag{2.3.32}$$

is non-oscillatory.

Example 2.3.2. Let $a \in (0, 1)$ and $\varrho, \sigma > 1$ be arbitrary. For the linear equations

$$\left[\frac{x'}{1+(a+\sin_\varrho t)/\log^2 t}\right]'+\frac{1+(1-a+\sin_\sigma t)/\log^2 t}{4t^2}x=0,$$
$$\left[\frac{x'}{1+(a+\sin_\varrho t)/\log^2 t}\right]'+\frac{1+(1-a+\cos_\sigma t)/\log^2 t}{4t^2}x=0,$$
$$\left[\frac{x'}{1+(a+\cos_\varrho t)/\log^2 t}\right]'+\frac{1+(1-a+\sin_\sigma t)/\log^2 t}{4t^2}x=0,$$
$$\left[\frac{x'}{1+(a+\cos_\varrho t)/\log^2 t}\right]'+\frac{1+(1-a+\cos_\sigma t)/\log^2 t}{4t^2}x=0,$$

we can apply Corollary 2.3.1. Thus, the above equations are non-oscillatory.

To proceed further, we mention a direct consequence of a relevant result from the literature.

Theorem 2.3.2. *Let c_1 be a positive α-periodic continuous function, let d_1 be an α-periodic continuous function, and let $c_2, d_2 : \mathbb{R}_a \to \mathbb{R}$ be arbitrary continuous functions for which mean values $M(c_2)$, $M(|c_2|)$, $M(d_2)$, $M(|d_2|)$ exist. Let us consider the equation*

$$\left[\left(c_1(t) + \frac{c_2(t)}{\log^2 t}\right)^{-\frac{p}{q}} \Phi(x')\right]' + \left(d_1(t) + \frac{d_2(t)}{\log^2 t}\right) \frac{\Phi(x)}{t^p} = 0 \tag{2.3.33}$$

and denote

$$\Gamma := 2q^{p-1} M(d_2) \left[M\left(c_1\right)\right]^{\frac{p}{q}} + 2q^{-2} p M(c_2) \left[M(c_1)\right]^{-1}.$$

Let

$$c_1(t) + \frac{c_2(t)}{\log^2 t} > 0, \quad t \geq a, \qquad q^p M(d_1) \left[M(c_1)\right]^{\frac{p}{q}} = 1.$$

The following statements hold.

(i) Eq. (2.3.33) *is oscillatory if* $\Gamma > 1$.

(ii) Eq. (2.3.33) *is non-oscillatory if* $\Gamma < 1$.

Proof. See [22, Theorem 5.1], where it suffices to put $n = 1$. □

Combining Theorems 2.1.4 and 2.3.2, we obtain the following one.

Theorem 2.3.3. *The following statements hold.*

(i) If $[M(r_0)]^{\frac{p}{q}} M(s_0) > q^{-p}$, *then Eq.* (2.3.1) *is oscillatory.*

(ii) If $[M(r_0)]^{\frac{p}{q}} M(s_0) < q^{-p}$, *then Eq.* (2.3.1) *is non-oscillatory.*

(iii) If $[M(r_0)]^{\frac{p}{q}} M(s_0) = q^{-p}$ *and*

$$M(s_1) \left[M(r_0)\right]^{\frac{p}{q}} + \frac{p}{q^{p+1}} M(r_1) \left[M(r_0)\right]^{-1} > \frac{q^{1-p}}{2},$$

then Eq. (2.3.1) *is oscillatory.*

(iv) If $[M(r_0)]^{\frac{p}{q}} M(s_0) = q^{-p}$ *and*

$$M(s_1) \left[M(r_0)\right]^{\frac{p}{q}} + \frac{p}{q^{p+1}} M(r_1) \left[M(r_0)\right]^{-1} < \frac{q^{1-p}}{2},$$

then Eq. (2.3.1) *is non-oscillatory.*

Proof. The theorem follows immediately from Theorem 2.1.4 (parts (i), (ii)) and Theorem 2.3.2 (parts (iii), (iv)). It suffices to consider the identities $p-1 = p/q$, $(1-q)(-p/q) = 1$. □

Applying Theorem 2.3.3, we can improve Theorem 2.3.1 and Corollary 2.3.1 into the following more convenient forms. We give illustrating examples as well.

Theorem 2.3.4. *Eq.* (2.3.4) *is non-oscillatory if and only if*

$$\lim_{t\to\infty}\left[\frac{t}{\alpha^p}\left(\int_a^{a+\alpha} r_0(\tau)\,\mathrm{d}\tau\right)^{\frac{p}{q}}\left(\int_a^{a+\alpha} s_0(\tau)\,\mathrm{d}\tau\right)-q^{-p}t\right.$$
$$\left.+\frac{1}{t^p}\left(\int_a^{a+t}\sum_{i=1}^{n} S_i(\tau)\,\mathrm{d}\tau\right)\left(\int_a^{a+t} r_0(\tau)\,\mathrm{d}\tau\right)^{\frac{p}{q}}+\frac{p\int_a^{a+t}\sum_{i=1}^{m} R_i(\tau)\,\mathrm{d}\tau}{q^{p+1}\int_a^{a+t} r_0(\tau)\,\mathrm{d}\tau}\right]\leq\frac{q^{1-p}}{2}.$$

Proof. It suffices to consider Theorems 2.3.1 and 2.3.3. □

Example 2.3.3. Let $a,b,c,d>0$, $a_1,a_2,a_3,b_1\neq 0$, $p=3/2$. Let us consider the half-linear equation

$$\left[\frac{1}{\sqrt{a+c\left(\cos\left(a_1t\right)\sin\left(a_1t\right)+\cos\left(a_2t\right)+\sin\left(a_3t\right)\right)/\log^2 t}}\cdot\frac{x'}{\sqrt{|x'|}}\right]'$$
$$+\left(b+d\left[\frac{\cos\left(b_1t\right)\sin\left(b_1t\right)}{\log t}\right]^2\right)\frac{x}{\sqrt{t^3|x|}}=0. \tag{2.3.34}$$

Theorem 2.3.4 guarantees the oscillation of Eq. (2.3.34) if $ab^2>1/27$; and its non-oscillation if $ab^2<1/27$. We put $ab^2=1/27$. Since

$$M\left(\cos\left(\alpha t\right)\sin\left(\alpha t\right)\right)=M\left(\cos\left(\alpha t\right)\right)=M\left(\sin\left(\alpha t\right)\right)=0,\quad \alpha\neq 0,$$

and

$$M\left(\left[\cos\left(\alpha t\right)\sin\left(\alpha t\right)\right]^2\right)=\frac{1}{8},\quad \alpha\neq 0,$$

we obtain the oscillation of Eq. (2.3.34) for $ad^2>16/3$ and the non-oscillation in the opposite case $ad^2\leq 16/3$.

Corollary 2.3.2. *Let f,g be periodic and continuous functions such that $M(f),M(g)\geq 0$. Eq.* (2.3.32) *is oscillatory if and only if $M(f)+M(g)>1$.*

Example 2.3.4. Using Corollary 2.3.2 (and Theorem 2.3.4), we can generalize Example 2.3.2. For any $a_1, a_2, b_1, b_2 > 0$ and $\varrho, \sigma > 1$, the linear equations

$$\left[\frac{x'}{a_1 + (b_1 + \sin_\varrho t)/\log^2 t}\right]' + \frac{a_2 + (b_2 + \sin_\sigma t)/\log^2 t}{4t^2}\, x = 0,$$
$$\left[\frac{x'}{a_1 + (b_1 + \sin_\varrho t)/\log^2 t}\right]' + \frac{a_2 + (b_2 + \cos_\sigma t)/\log^2 t}{4t^2}\, x = 0,$$
$$\left[\frac{x'}{a_1 + (b_1 + \cos_\varrho t)/\log^2 t}\right]' + \frac{a_2 + (b_2 + \sin_\sigma t)/\log^2 t}{4t^2}\, x = 0,$$
$$\left[\frac{x'}{a_1 + (b_1 + \cos_\varrho t)/\log^2 t}\right]' + \frac{a_2 + (b_2 + \cos_\sigma t)/\log^2 t}{4t^2}\, x = 0$$

are oscillatory for $a_1 a_2 > 1$ and non-oscillatory for $a_1 a_2 < 1$. In the limiting case $a_1 a_2 = 1$, one can easily rewrite the considered equations in the form of Eq. (2.3.32), where $M(f) = b_1/a_1$ and $M(g) = a_1 b_2$. Therefore, in the case $a_1 a_2 = 1$, the above equations are oscillatory if and only if $b_1 > a_1 (1 - a_1 b_2)$.

If we know that an equation is conditionally oscillatory, then we can use it as a testing equation for many other equations. For example, using the Sturm comparison theorem 1.1.3, we can proceed for perturbed Euler type half-linear equations as follows. Let us consider

$$\left[r_0^{-\frac{p}{q}}(t)\Phi\left(x'\right)\right]' + s_0(t)\,\frac{\Phi(x)}{t^p} + g(t)\Phi(x) = 0 \tag{2.3.35}$$

and

$$\left[\left[r_0(t) + f(t)\right]^{-\frac{p}{q}}\Phi\left(x'\right)\right]' + s_0(t)\,\frac{\Phi(x)}{t^p} = 0, \tag{2.3.36}$$

where f, g are arbitrary continuous functions and r_0, s_0 are α-periodic continuous functions such that r_0, f are positive and $M(r_0) = 1$, $M(s_0) = q^{-p}$.

Eq. (2.3.35) is non-oscillatory if there exist β_i-periodic continuous functions S_i, $i \in \{1, \dots, n\}$, such that

$$M\left(\sum_{i=1}^{n} S_i\right) = 1, \qquad \sum_{i=1}^{n} S_i(t) > 0, \quad t \in \mathbb{R}, \tag{2.3.37}$$

and

$$\limsup_{t\to\infty} \frac{g(t)t^p \log^2 t}{\sum_{i=1}^{n} S_i(t)} < \frac{q^{1-p}}{2}. \tag{2.3.38}$$

Eq. (2.3.35) is oscillatory if the functions S_i satisfy (2.3.37) and

$$\liminf_{t\to\infty} \frac{g(t)t^p \log^2 t}{\sum_{i=1}^{n} S_i(t)} > \frac{q^{1-p}}{2}. \tag{2.3.39}$$

Indeed, from inequality (2.3.38), we obtain $\varepsilon > 0$ with the property that we have

$$g(t) < \left(\frac{q^{1-p}}{2} - \varepsilon\right) \frac{\sum_{i=1}^{n} S_i(t)}{t^p \log^2 t}$$

for all sufficiently large t. Thus, it suffices to use Theorem 2.3.4 and Sturm comparison theorem 1.1.3. Analogously, we get the statement concerning inequality (2.3.39).

Similarly, Eq. (2.3.36) is non-oscillatory if there exist α_i-periodic continuous functions R_i for $i \in \{1, \dots, m\}$ such that

$$M\left(\sum_{i=1}^{m} R_i\right) = 1, \qquad \sum_{i=1}^{m} R_i(t) > 0, \quad t \in \mathbb{R}, \tag{2.3.40}$$

and it holds

$$\limsup_{t\to\infty} \frac{f(t)\log^2 t}{\sum_{i=1}^{m} R_i(t)} < \frac{q^2}{2p}.$$

On the other hand, if the functions R_i satisfy (2.3.40) and

$$\liminf_{t\to\infty} \frac{f(t)\log^2 t}{\sum_{i=1}^{m} R_i(t)} > \frac{q^2}{2p},$$

then Eq. (2.3.36) is oscillatory.

2.4 Modified Euler type equations

In this section, our aim is to extend the family of conditionally oscillatory equations. More precisely, we identify the critical oscillation constant for the Euler type equations in the form

$$\left[r(t)t^{p-1}\Phi(x')\right]' + \frac{s(t)}{t\log^p t}\Phi(x) = 0, \tag{2.4.1}$$

where $r : \mathbb{R}_e \to \mathbb{R}^+$ and $s : \mathbb{R}_e \to \mathbb{R}$ are continuous functions. We introduce another modification of the half-linear Prüfer angle in combination with the Riccati transformation.

The main motivation comes from Theorem 2.1.4 and the following theorem proved in [49].

Theorem 2.4.1. *Let $r : \mathbb{R}_1 \to \mathbb{R}^+$ and $s : \mathbb{R}_1 \to \mathbb{R}$ be continuous functions such that there exist $\delta > 1$ and $\varepsilon \in (0, 1/2)$ for which*

$$\int_t^{t+1} r(\tau)\,\mathrm{d}\tau < \delta t^{\varepsilon}, \quad \int_t^{t+1} |s(\tau)|\,\mathrm{d}\tau < \delta t^{\varepsilon}, \qquad t \in \mathbb{R}_1. \tag{2.4.2}$$

Consider the equation

$$\left[r^{1-p}(t)\,\Phi\left(x'\right)\right]' + \frac{s(t)}{t^p}\Phi(x) = 0. \tag{2.4.3}$$

(i) If there exist $\alpha, R, S \in \mathbb{R}^+$ *satisfying*

$$R^{p-1}S > \left(\frac{p-1}{p}\right)^p$$

and

$$\frac{1}{\alpha}\int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau \geq R, \quad \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau \geq S$$

for all sufficiently large t*, then Eq.* (2.4.3) *is oscillatory.*

(ii) If there exist $\alpha, R, S \in \mathbb{R}^+$ *satisfying*

$$R^{p-1}S < \left(\frac{p-1}{p}\right)^p$$

and

$$\frac{1}{\alpha}\int_t^{t+\alpha} r(\tau)\,\mathrm{d}\tau \leq R, \quad \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau \leq S$$

for all sufficiently large t*, then Eq.* (2.4.3) *is non-oscillatory.*

We remark that the results proven in this section are stronger than the ones known for Eq. (2.4.3) (compare the below given assumptions (2.4.13) and (2.4.14) with (2.4.2)). To the best of our knowledge, the presented results are new even for Eq. (2.4.1) with periodic coefficients r, s.

The section is organized as follows. The notion of the modified Prüfer angle is mentioned in the next paragraph. Auxiliary results are collected in § 2.4.2. The content of § 2.4.1 and § 2.4.2 gives the description of our method which is used in the proofs of the main results in § 2.4.3 which is finished by corollaries and examples.

§ 2.4.1 Equation for Prüfer angle

In this paragraph, we derive the equation for the modified half-linear Prüfer angle which will be fundamental for our investigation. To do this, we have to use different transformations than in the previous sections and combine it with the Riccati equation (1.1.13) corresponding to the general half-linear equation (1.1.1) via (1.1.12).

We introduce the modified Prüfer transformation in the form

$$x(t) = \rho(t)\sin_p \varphi(t), \quad r^{q-1}(t)x'(t) = \frac{\rho(t)}{\log t}\cos_p \varphi(t) \tag{2.4.4}$$

together with the substitution

$$v(t) = (\log t)^{\frac{p}{q}} w(t), \qquad t \in \mathbb{R}_{\mathrm{e}}. \tag{2.4.5}$$

Note that substitutions similar to (2.4.5) can be used also in the Riccati equation (1.1.13). This approach leads to the so-called adapted (or weighted) Riccati equation. Nevertheless, we use this process only partially (see below) and we have to take into consideration the modified Prüfer transformation (2.4.4) as well.

Using (2.4.5), (1.1.12), (2.4.4), and (1.1.4) successively, we obtain

$$\begin{aligned} v(t) &= (\log t)^{\frac{p}{q}} w(t) = (\log t)^{\frac{p}{q}} r(t)\Phi\left(\frac{x'(t)}{x(t)}\right) \\ &= (\log t)^{\frac{p}{q}} r(t)\,\frac{\Phi\left(r^{1-q}(t)\cos_p\varphi(t)\right)}{\Phi\left(\log t\ \sin_p\varphi(t)\right)} = \Phi\left(\frac{\cos_p\varphi(t)}{\sin_p\varphi(t)}\right). \end{aligned} \tag{2.4.6}$$

One can easily verify that

$$\hat{v}(t) := \Phi\left(\frac{\cos_p t}{\sin_p t}\right) \tag{2.4.7}$$

solves the equation

$$\hat{v}'(t) + p - 1 + (p-1)|\hat{v}(t)|^q = 0. \tag{2.4.8}$$

Eq. (2.4.8) is the Riccati equation associated to the equation in (1.1.2). Hence, due to (1.1.5), (1.1.4), (2.4.7), and (2.4.8), we have

$$\begin{aligned} v'(t) &= [\hat{v}(\varphi(t))]' = [-p+1-(p-1)|\hat{v}(\varphi(t))|^q]\,\varphi'(t) \\ &= (1-p)\left[1+\left|\frac{\cos_p\varphi(t)}{\sin_p\varphi(t)}\right|^p\right]\varphi'(t) = \frac{1-p}{|\sin_p\varphi(t)|^p}\,\varphi'(t). \end{aligned} \tag{2.4.9}$$

On the other side, considering (2.4.5) together with (1.1.13), we have

$$\begin{aligned} v'(t) &= \frac{p}{q}\,(\log t)^{\frac{p}{q}-1}\,\frac{w(t)}{t} + (\log t)^{\frac{p}{q}} w'(t) \\ &= \frac{p}{q}\cdot\frac{v(t)}{t\log t} + (\log t)^{\frac{p}{q}}\left[-z(t)-(p-1)r^{1-q}(t)|w(t)|^q\right] \\ &= \frac{p}{q}\cdot\frac{v(t)}{t\log t} - (\log t)^{\frac{p}{q}} z(t) - (p-1)r^{1-q}(t)\,\frac{|v(t)|^q}{\log t}. \end{aligned} \tag{2.4.10}$$

We combine (2.4.9) and (2.4.10). This leads to

$$\frac{1-p}{|\sin_p\varphi(t)|^p}\,\varphi'(t) = \frac{p}{q}\cdot\frac{v(t)}{t\log t} - (\log t)^{\frac{p}{q}} z(t) - (p-1)r^{1-q}(t)\,\frac{|v(t)|^q}{\log t}.$$

Taking into account (2.4.6), we obtain

$$\begin{aligned} (1-p)\,\varphi'(t) = &\ \frac{p}{q}\cdot\frac{1}{t\log t}\,\Phi\left(\cos_p\varphi(t)\right)\sin_p\varphi(t) \\ &- (\log t)^{\frac{p}{q}} z(t)\,|\sin_p\varphi(t)|^p - (p-1)r^{1-q}(t)\,\frac{|\cos_p\varphi(t)|^p}{\log t} \end{aligned}$$

which gives us the desired equation for the Prüfer angle as

$$\begin{aligned}\varphi'(t) = r^{1-q}(t)\,\frac{|\cos_p \varphi(t)|^p}{\log t}\\ - \frac{1}{t\log t}\,\Phi\left(\cos_p \varphi(t)\right)\sin_p \varphi(t) + \frac{(\log t)^{\frac{p}{q}}}{p-1}\,z(t)\,|\sin_p \varphi(t)|^p .\end{aligned} \tag{2.4.11}$$

In this section, we apply Eq. (2.4.11) to the study of Eq. (2.4.1), i.e., we have the equation for the Prüfer angle in the form

$$\begin{aligned}\varphi'(t) = \frac{1}{t\log t}\Bigg[r^{1-q}(t)|\cos_p \varphi(t)|^p - \Phi\left(\cos_p \varphi(t)\right)\sin_p \varphi(t)\\ + s(t)\,\frac{|\sin_p \varphi(t)|^p}{p-1}\Bigg],\end{aligned} \tag{2.4.12}$$

which can be simply verified.

§ 2.4.2 Prüfer angle of average function

In this paragraph, we consider that the coefficients $r : \mathbb{R}_e \to \mathbb{R}^+$ and $s : \mathbb{R}_e \to \mathbb{R}$ in Eq. (2.4.1) are such that

$$\lim_{t\to\infty} \frac{\int_t^{t+\alpha} r^{1-q}(\tau)\,\mathrm{d}\tau}{\sqrt{t\log t}} = 0, \tag{2.4.13}$$

$$\lim_{t\to\infty} \frac{\int_t^{t+\alpha} |s(\tau)|\,\mathrm{d}\tau}{\sqrt{t\log t}} = 0 \tag{2.4.14}$$

hold for some $\alpha \in \mathbb{R}^+$. For this number α, we define the function ψ which determines the average value of an arbitrarily given solution φ of Eq. (2.4.12) over intervals of the length α, i.e., we put

$$\psi(t) := \frac{1}{\alpha}\int_t^{t+\alpha} \varphi(\tau)\;\mathrm{d}\tau, \qquad t \in \mathbb{R}_e,$$

where φ is a solution of Eq. (2.4.12) on $\mathbb{R}_e$. We formulate and prove auxiliary results concerning the function ψ.

Lemma 2.4.1. *It holds*

$$\lim_{t\to\infty} \sqrt{t\log t}\,|\varphi(s) - \psi(t)| = 0 \tag{2.4.15}$$

uniformly with respect to $s \in [t, t+\alpha]$.

Proof. For $s \in [t, t+\alpha]$, it is seen that

$$\begin{aligned}
&\limsup_{t\to\infty} \sqrt{t\log t}\,|\varphi(s)-\psi(t)| \\
&\le \limsup_{t\to\infty} \sqrt{t\log t} \int_t^{t+\alpha} |\varphi'(\tau)| \,\mathrm{d}\tau \\
&= \limsup_{t\to\infty} \sqrt{t\log t} \int_t^{t+\alpha} \left| \frac{1}{\tau\log\tau} \Bigg[r^{1-q}(\tau)|\cos_p\varphi(\tau)|^p \right. \\
&\qquad \left. - \Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau) + s(\tau)\frac{|\sin_p\varphi(\tau)|^p}{p-1} \Bigg] \right| \mathrm{d}\tau \\
&\le \limsup_{t\to\infty} \sqrt{t\log t} \int_t^{t+\alpha} \frac{1}{t\log t} \Bigg[r^{1-q}(\tau)|\cos_p\varphi(\tau)|^p \\
&\qquad + |\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)| + |s(\tau)|\frac{|\sin_p\varphi(\tau)|^p}{p-1} \Bigg] \,\mathrm{d}\tau.
\end{aligned}$$

Since (see directly (1.1.5))

$$|\sin_p x|^p \le 1, \quad |\cos_p x|^p \le 1, \qquad x\in\mathbb{R}, \tag{2.4.16}$$

and, consequently,

$$|\Phi(\cos_p x)\sin_p x| = |\cos_p x|^{p-1}|\sin_p x| \le 1, \qquad x\in\mathbb{R}, \tag{2.4.17}$$

we have

$$\begin{aligned}
0 &\le \limsup_{t\to\infty} \sqrt{t\log t}\,|\varphi(s)-\psi(t)| \\
&\le \limsup_{t\to\infty} \frac{1}{\sqrt{t\log t}} \int_t^{t+\alpha} r^{1-q}(\tau) + 1 + \frac{|s(\tau)|}{p-1} \,\mathrm{d}\tau, \qquad s\in[t,t+\alpha].
\end{aligned}$$

Using (2.4.13) and (2.4.14), we obtain

$$0 \le \liminf_{t\to\infty} \sqrt{t\log t}\,|\varphi(s)-\psi(t)| \le \limsup_{t\to\infty} \sqrt{t\log t}\,|\varphi(s)-\psi(t)| = 0$$

uniformly with respect to $s\in[t,t+\alpha]$. $\square$

Lemma 2.4.2. *It holds*

$$\begin{aligned}
\psi'(t) = \frac{1}{t\log t} \Bigg[&\frac{|\cos_p\psi(t)|^p}{\alpha} \int_t^{t+\alpha} r^{1-q}(\tau)\,\mathrm{d}\tau - \Phi\left(\cos_p\psi(t)\right)\sin_p\psi(t) \\
&+ \frac{|\sin_p\psi(t)|^p}{(p-1)\alpha} \int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau + \Psi(t) \Bigg]
\end{aligned}$$

for all $t > \mathrm{e}$*, where* $\Psi : \mathbb{R}_\mathrm{e} \to \mathbb{R}$ *is a continuous function such that* $\lim_{t\to\infty} \Psi(t) = 0$*.*

Proof. For any $t > \mathrm{e}$, we have

$$\psi'(t) = \frac{1}{\alpha} \int\limits_t^{t+\alpha} \frac{1}{\tau \log \tau} \Bigg[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p \\ - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \Bigg] \, \mathrm{d}\tau.$$

We can replace $\psi'(t)$ by

$$\frac{1}{\alpha} \int\limits_t^{t+\alpha} \frac{1}{t \log t} \Bigg[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p \\ - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \Bigg] \, \mathrm{d}\tau,$$

because we can easily estimate (see (2.4.13), (2.4.14), (2.4.16), (2.4.17))

$$\limsup_{t\to\infty} \frac{t \log t}{\alpha} \Bigg| \int\limits_t^{t+\alpha} \frac{1}{\tau \log \tau} \Bigg[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p \\ - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \Bigg] \, \mathrm{d}\tau \\ - \int\limits_t^{t+\alpha} \frac{1}{t \log t} \Bigg[r^{1-q}(\tau) |\cos_p \varphi(\tau)|^p \\ - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \Bigg] \, \mathrm{d}\tau \Bigg| \\ \leq \limsup_{t\to\infty} \frac{t \log t}{\alpha} \int\limits_t^{t+\alpha} \left[\frac{1}{t \log t} - \frac{1}{\tau \log \tau} \right] \left[r^{1-q}(\tau) + 1 + |s(\tau)| \frac{1}{p-1} \right] \mathrm{d}\tau$$

$$
\begin{aligned}
&\leq \limsup_{t\to\infty} \frac{t\log t}{\alpha} \int\limits_t^{t+\alpha} \left[\frac{1}{t\log t} - \frac{1}{(t+\alpha)\log(t+\alpha)}\right] \times \\
&\qquad\qquad \times \left[r^{1-q}(\tau) + 1 + |s(\tau)|\,\frac{1}{p-1}\right] \mathrm{d}\tau \\
&\leq \limsup_{t\to\infty} \frac{t\log t}{\alpha} \cdot \frac{\alpha}{t\,(t+\alpha)\log t} \int\limits_t^{t+\alpha} \left[r^{1-q}(\tau) + 1 + |s(\tau)|\,\frac{1}{p-1}\right] \mathrm{d}\tau \\
&\leq \limsup_{t\to\infty} \frac{1}{t+\alpha} \int\limits_t^{t+\alpha} \left[r^{1-q}(\tau) + 1 + |s(\tau)|\,\frac{1}{p-1}\right] \mathrm{d}\tau = 0.
\end{aligned}
$$

Applying the uniform continuity of the half-linear trigonometric functions and (2.4.15) in Lemma 2.4.1, we see that

$$
\begin{aligned}
\lim_{t\to\infty} \Bigg(& \Phi\left(\cos_p \psi(t)\right) \sin_p \psi(t) \\
& - \frac{1}{\alpha} \int\limits_t^{t+\alpha} \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau)\, \mathrm{d}\tau \Bigg) = 0.
\end{aligned}
\tag{2.4.18}
$$

In addition, the half-linear trigonometric functions are continuously differentiable and periodic (see, e.g., [21, Section 1.1.2]). Hence, they have the Lipschitz property on $\mathbb{R}$ and, consequently, there exists a constant $L \in \mathbb{R}^+$ for which

$$
\big|\, |\cos_p x|^p - |\cos_p y|^p \,\big| \leq L|x-y|, \qquad x, y \in \mathbb{R}, \tag{2.4.19}
$$

$$
\big|\, |\sin_p x|^p - |\sin_p y|^p \,\big| \leq L|x-y|, \qquad x, y \in \mathbb{R}. \tag{2.4.20}
$$

Therefore, from (2.4.13), Lemma 2.4.1, and (2.4.19), it follows

$$
\begin{aligned}
\limsup_{t\to\infty} \Bigg| & \frac{|\cos_p \psi(t)|^p}{\alpha} \int\limits_t^{t+\alpha} r^{1-q}(\tau)\, \mathrm{d}\tau \\
& - \frac{1}{\alpha} \int\limits_t^{t+\alpha} r^{1-q}(\tau)\, |\cos_p \varphi(\tau)|^p \, \mathrm{d}\tau \Bigg| \\
\leq & \frac{1}{\alpha} \limsup_{t\to\infty} \int\limits_t^{t+\alpha} r^{1-q}(\tau) \big|\, |\cos_p \psi(t)|^p - |\cos_p \varphi(\tau)|^p \,\big|\, \mathrm{d}\tau \\
\leq & \frac{1}{\alpha} \limsup_{t\to\infty} \frac{\sqrt{t\log t}}{\sqrt{t\log t}} \int\limits_t^{t+\alpha} r^{1-q}(\tau) L \,|\psi(t) - \varphi(\tau)|\, \mathrm{d}\tau = 0.
\end{aligned}
\tag{2.4.21}
$$

Analogously, from (2.4.14), Lemma 2.4.1, and (2.4.20), we have

$$
\begin{aligned}
&\limsup_{t\to\infty}\left|\frac{|\sin_p \psi(t)|^p}{(p-1)\alpha}\int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau - \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\,\frac{|\sin_p \varphi(\tau)|^p}{p-1}\,\mathrm{d}\tau\right| \\
&\le \frac{1}{(p-1)\alpha}\limsup_{t\to\infty}\int_t^{t+\alpha} |s(\tau)|\cdot\big|\,|\sin_p \psi(t)|^p - |\sin_p \varphi(\tau)|^p\,\big|\,\mathrm{d}\tau \qquad (2.4.22)\\
&\le \frac{1}{(p-1)\alpha}\limsup_{t\to\infty}\frac{\sqrt{t\log t}}{\sqrt{t\log t}}\int_t^{t+\alpha} |s(\tau)|\,L\,|\psi(t)-\varphi(\tau)|\,\mathrm{d}\tau = 0.
\end{aligned}
$$

Finally, the statement of the lemma directly comes from the combination of (2.4.18), (2.4.21), and (2.4.22). The continuity of Ψ is obvious. □

§ 2.4.3 Oscillation constant

At first, we recall known results concerning the studied equations with constant coefficients.

Theorem 2.4.2. *If $A, B \in \mathbb{R}^+$ satisfy $B/A > q^{-p}$, then the equation*

$$\left[At^{p-1}\Phi(x')\right]' + \frac{B}{t\log^p t}\Phi(x) = 0$$

is oscillatory.

Proof. See, e.g., [21, Theorem 1.4.4] (or directly [27] and [26]). □

Theorem 2.4.3. *If $C, D \in \mathbb{R}^+$ satisfy $D/C < q^{-p}$, then the equation*

$$\left[Ct^{p-1}\Phi(x')\right]' + \frac{D}{t\log^p t}\Phi(x) = 0$$

is non-oscillatory.

Proof. See again [21, Theorem 1.4.4] (or [27, 26]). □

Now, we can prove the announced results which identify the critical oscillation constant for the analyzed equations with very general coefficients.

Theorem 2.4.4. *Let $\alpha, R, S \in \mathbb{R}^+$ be such that* (2.4.14) *is valid and $R^{p-1}S > q^{-p}$. If there exists $T > \mathrm{e}$ with the property that*

$$\frac{1}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\,\mathrm{d}\tau \ge R, \quad \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau \ge S, \qquad t \in \mathbb{R}_T,$$

then Eq. (2.4.1) *is oscillatory.*

Proof. We will deal with the equation for the Prüfer angle corresponding to the average function ψ. It is well-known that the non-oscillation of solutions of Eq. (2.4.1) is equivalent to the boundedness from above of the Prüfer angle φ given by Eq. (2.4.12). We refer, e.g., to [17, 19, 39, 71]. It also suffices to consider directly (2.4.4) and Eq. (2.4.12) when $\sin_p \varphi(t) = 0$. Based on Lemma 2.4.1, we know that the boundedness (from above) of φ is equivalent to the boundedness (from above) of ψ. Hence, we will show that ψ is unbounded from above. At first, we assume that (2.4.13) is true.

Taking into account Lemma 2.4.2, we have

$$\begin{aligned}\psi'(t) &= \frac{1}{t\log t}\left[\frac{|\cos_p \psi(t)|^p}{\alpha}\int\limits_t^{t+\alpha} r^{1-q}(\tau)\,\mathrm{d}\tau - \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t)\right.\\ &\qquad\left. + \frac{|\sin_p \psi(t)|^p}{(p-1)\alpha}\int\limits_t^{t+\alpha} s(\tau)\,\mathrm{d}\tau + \Psi(t)\right]\\ &\geq \frac{1}{t\log t}\left[R\,|\cos_p \psi(t)|^p - \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t)\right.\\ &\qquad\left. + S\,\frac{|\sin_p \psi(t)|^p}{p-1} + \Psi(t)\right]\end{aligned}$$

for all $t \in \mathbb{R}_T$ and for some continuous function $\Psi : \mathbb{R}_T \to \mathbb{R}$ satisfying

$$\lim_{t\to\infty} \Psi(t) = 0. \tag{2.4.23}$$

Let $\varepsilon > 0$ be arbitrary. From (1.1.5) and (2.4.23), we have

$$\varepsilon\left(|\cos_p x|^p + \frac{|\sin_p x|^p}{p-1}\right) > |\Psi(t)| \tag{2.4.24}$$

for all $x \in \mathbb{R}$ and for all large $t \in \mathbb{R}_T$. Thus,

$$\begin{aligned}\psi'(t) > \frac{1}{t\log t}&\left[(R-\varepsilon)\,|\cos_p \psi(t)|^p - \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t)\right.\\ &\qquad\left. + (S-\varepsilon)\,\frac{|\sin_p \psi(t)|^p}{p-1}\right]\end{aligned} \tag{2.4.25}$$

for all large $t \in \mathbb{R}_T$. Let ε be so small that $(R-\varepsilon)^{p-1}(S-\varepsilon) > q^{-p}$ and $R-\varepsilon > 0$. Using Theorem 2.4.2 for $A = (R-\varepsilon)^{\frac{1}{1-q}}$ and $B = S-\varepsilon$, we know that the equation

$$\left[(R-\varepsilon)^{\frac{1}{1-q}}\,t^{p-1}\Phi(x')\right]' + \frac{S-\varepsilon}{t\log^p t}\Phi(x) = 0$$

is oscillatory, i.e., any solution $\hat{\varphi} : \mathbb{R}_T \to \mathbb{R}$ of the equation

$$\hat{\varphi}'(t) = \frac{1}{t \log t} \Bigg[(R - \varepsilon)\, |\cos_p \hat{\varphi}(t)|^p - \Phi(\cos_p \hat{\varphi}(t)) \sin_p \hat{\varphi}(t) + (S - \varepsilon)\, \frac{|\sin_p \hat{\varphi}(t)|^p}{p-1} \Bigg] \tag{2.4.26}$$

has the property that $\limsup_{t\to\infty} \hat{\varphi}(t) = \infty$. Indeed, one can simply compute that

$$B/A = (R - \varepsilon)^{p-1}\, (S - \varepsilon) > q^{-p}.$$

Considering (2.4.25) with (2.4.26) and the $2\pi_p$-periodicity of the functions $\sin_p$ and $\cos_p$, we see that $\limsup_{t\to\infty} \hat{\varphi}(t) = \infty$ implies $\limsup_{t\to\infty} \psi(t) = \infty$.

To finish the proof, we have to consider the case when (2.4.13) is not valid. Evidently, there exists a continuous function $\tilde{r} : \mathbb{R}_{\mathrm{e}} \to \mathbb{R}^+$ with the properties

$$r^{1-q}(t) \geq \tilde{r}^{1-q}(t), \quad \frac{1}{\alpha} \int_t^{t+\alpha} \tilde{r}^{1-q}(\tau)\, \mathrm{d}\tau \geq R, \qquad t \in \mathbb{R}_T,$$

and

$$\lim_{t\to\infty} \frac{\int_t^{t+\alpha} \tilde{r}^{1-q}(\tau)\, \mathrm{d}\tau}{\sqrt{t \log t}} = 0. \tag{2.4.27}$$

We actually know that the equation

$$\left[\tilde{r}(t) t^{p-1} \Phi(x')\right]' + \frac{s(t)}{t \log^p t} \Phi(x) = 0$$

is oscillatory (cf. (2.4.13) and (2.4.27)). Since $r(t) \leq \tilde{r}(t)$ for all $t \in \mathbb{R}_T$, the Sturm half-linear comparison theorem 1.1.3 gives the oscillation of Eq. (2.4.1). $\square$

Theorem 2.4.5. *Let $\alpha, R, S \in \mathbb{R}^+$ be such that* (2.4.14) *is valid and $R^{p-1}S < q^{-p}$. If there exists $T > \mathrm{e}$ with the property that*

$$\frac{1}{\alpha} \int_t^{t+\alpha} r^{1-q}(\tau)\, \mathrm{d}\tau \leq R, \quad \frac{1}{\alpha} \int_t^{t+\alpha} s(\tau)\, \mathrm{d}\tau \leq S, \qquad t \in \mathbb{R}_T, \tag{2.4.28}$$

then Eq. (2.4.1) *is non-oscillatory.*

Proof. We will proceed analogously as in the proof of Theorem 2.4.4. In this proof, we will show that ψ is bounded from above. Note that (2.4.13) is valid (see the first inequality in (2.4.28)).

From Lemma 2.4.2, we obtain

$$\psi'(t) \leq \frac{1}{t\log t}\Bigg[R\left|\cos_p \psi(t)\right|^p - \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t) \\ + S\,\frac{\left|\sin_p \psi(t)\right|^p}{p-1} + \Psi(t)\Bigg]$$

for any $t \in \mathbb{R}_T$ and for a continuous function $\Psi : \mathbb{R}_T \to \mathbb{R}$ satisfying (2.4.23). In addition, using (2.4.24), we obtain

$$\psi'(t) < \frac{1}{t\log t}\Bigg[(R+\varepsilon)\left|\cos_p \psi(t)\right|^p - \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t) \\ + (S+\varepsilon)\,\frac{\left|\sin_p \psi(t)\right|^p}{p-1}\Bigg] \tag{2.4.29}$$

for any $\varepsilon > 0$ and all sufficiently large $t \in \mathbb{R}_T$. We choose ε in such a way that

$$(R+\varepsilon)^{p-1}(S+\varepsilon) < q^{-p}.$$

We put $C = (R+\varepsilon)^{\frac{1}{1-q}}$ and $D = S+\varepsilon$ in Theorem 2.4.3. Since

$$D/C = (R+\varepsilon)^{p-1}(S+\varepsilon) < q^{-p},$$

we know that the equation

$$\left[(R+\varepsilon)^{\frac{1}{1-q}}\,t^{p-1}\Phi(x')\right]' + \frac{S+\varepsilon}{t\log^p t}\Phi(x) = 0$$

is non-oscillatory. This fact means that any solution $\check{\varphi} : \mathbb{R}_T \to \mathbb{R}$ of the equation

$$\check{\varphi}'(t) = \frac{1}{t\log t}\Bigg[(R+\varepsilon)\left|\cos_p \check{\varphi}(t)\right|^p - \Phi(\cos_p \check{\varphi}(t))\sin_p \check{\varphi}(t) \\ + (S+\varepsilon)\,\frac{\left|\sin_p \check{\varphi}(t)\right|^p}{p-1}\Bigg] \tag{2.4.30}$$

has the property that $\limsup_{t\to\infty} \check{\varphi}(t) < \infty$. Finally, considering (2.4.29) together with (2.4.30) and considering the $2\pi_p$-periodicity of the generalized trigonometric functions, we have the inequality

$$\limsup_{t\to\infty} \psi(t) < \infty.$$

Therefore, the statement of the theorem is proven. □

Now, we mention definitions which enable us to formulate the below given Corollaries 2.4.1 and 2.4.2 (and which we use later as well).

Definition 2.4.1. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called *almost periodic* if, for all $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ of the real line contains at least one point s for which

$$|f(t+s) - f(t)| < \varepsilon, \qquad t \in \mathbb{R}.$$

It is well-known that there exist different (equivalent) ways to define almost periodic functions. The above given definition is the so-called Bohr definition. Another way is given by the Bochner definition which follows.

Definition 2.4.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We say that f is *almost periodic* if, from any sequence of the form $\{f(t+s_n)\}_{n\in\mathbb{N}}$, where s_n are real numbers, one can extract a subsequence which converges uniformly with respect to $t \in \mathbb{R}$.

We remark that the equivalence of Definitions 2.4.1 and 2.4.2 is shown, e.g., in [29, Theorem 1.14]. As a direct generalization of the almost periodicity, we consider the notion of the so-called asymptotic almost periodicity.

Definition 2.4.3. We say that a continuous function $f : \mathbb{R}_e \to \mathbb{R}$ is *asymptotically almost periodic* if f can be expressed in the form $f(t) = f_1(t) + f_2(t)$, $t \in \mathbb{R}_e$, where f_1 is almost periodic and f_2 has the property that $\lim_{t\to\infty} f_2(t) = 0$.

Concerning coefficients with mean values, we obtain a new result which reads as follows.

Corollary 2.4.1. *Let continuous functions $r : \mathbb{R}_e \to \mathbb{R}^+$ and $s : \mathbb{R}_e \to \mathbb{R}$ be such that the mean values $M(r^{1-q}) \in \mathbb{R}^+$, $M(s) \in \mathbb{R}$ exist and let* (2.4.14) *be valid for some $\alpha \in \mathbb{R}^+$. Let $[M(r^{1-q})]^{p-1} M(s) \neq q^{-p}$. Eq.* (2.4.1) *is oscillatory if and only if $[M(r^{1-q})]^{p-1} M(s) > q^{-p}$.*

Proof. The corollary follows from Theorems 2.4.4 and 2.4.5. Let $\varepsilon > 0$ be arbitrary. The existence of $M(r^{1-q})$ and $M(s)$ implies the existence of $n \in \mathbb{N}$ such that

$$\left| \frac{1}{n\alpha} \int_t^{t+n\alpha} r^{1-q}(\tau)\,\mathrm{d}\tau - M(r^{1-q}) \right| < \varepsilon, \quad \left| \frac{1}{n\alpha} \int_t^{t+n\alpha} s(\tau)\,\mathrm{d}\tau - M(s) \right| < \varepsilon$$

for all $t \in \mathbb{R}_e$. If $[M(r^{1-q})]^{p-1} M(s) > q^{-p}$, then it suffices to choose ε so that

$$M(r^{1-q}) - \varepsilon > 0, \qquad \left[M(r^{1-q}) - \varepsilon\right]^{p-1} [M(s) - \varepsilon] > q^{-p},$$

to put $R = M(r^{1-q}) - \varepsilon$ and $S = M(s) - \varepsilon$, and to replace α by $n\alpha$ in Theorem 2.4.4. Obviously, (2.4.14) is true also for $n\alpha$. If $[M(r^{1-q})]^{p-1} M(s) < q^{-p}$, then we choose ε so that

$$M(s) + \varepsilon > 0, \qquad \left[M(r^{1-q}) + \varepsilon\right]^{p-1} [M(s) + \varepsilon] < q^{-p},$$

we consider $R = M(r^{1-q}) + \varepsilon$ and $S = M(s) + \varepsilon$, and we replace α by $n\alpha$ in Theorem 2.4.5. $\square$

Remark 2.4.1. We point out that the requirement about the validity of (2.4.14) for some $\alpha \in \mathbb{R}^+$ cannot be omitted in the statement of Corollary 2.4.1. Indeed, the existence of $M(s)$ does not imply (2.4.14). We remark that, from the same reason, Theorem 2.1.4 does not follow from Theorem 2.4.1.

For asymptotically almost periodic coefficients, we get a new result as well. Again, we formulate it explicitly.

Corollary 2.4.2. *Let functions* $r^{1-q} : \mathbb{R}_e \to \mathbb{R}^+$ *and* $s : \mathbb{R}_e \to \mathbb{R}$ *be asymptotically almost periodic and let* $[M(r^{1-q})]^{p-1} M(s) \in \mathbb{R}^+ \setminus \{q^{-p}\}$. *Then, Eq.* (2.4.1) *is oscillatory if and only if* $[M(r^{1-q})]^{p-1} M(s) > q^{-p}$.

Proof. Since any asymptotically almost periodic function has mean value and it is bounded (see, e.g., [12, 29]), this corollary is a consequence of Corollary 2.4.1. □

Remark 2.4.2. Let us pay our attention to Eq. (2.1.65). We repeat that the result about Eq. (2.1.65), which corresponds to Corollary 2.4.1, is proven in Section 2.1 and that the one, which corresponds to Corollary 2.4.2, is proven in [38].

Remark 2.4.3. In Corollaries 2.4.1 and 2.4.2, the case $[M(r^{1-q})]^{p-1} M(s) = q^{-p}$ cannot be solved as oscillatory or non-oscillatory for general coefficients (which have mean values or which are asymptotically almost periodic). We conjecture that this case is not possible to solve even for general almost periodic coefficients. Our conjecture is based on constructions of almost periodic functions mentioned in [76] (see also [75, 77]).

At the end, we give some examples to illustrate the proven results.

Example 2.4.1. Let us consider constants $u > 0$, $v \in \mathbb{R}$, and $w \neq 0$ and the function $h : \mathbb{R}_1 \to \mathbb{R}$ given by the formula

$$h(t) := \begin{cases} v + nw2^n\,(t-n)\,, & t \in \left[n, n + \dfrac{1}{2^n}\right), \ n \in \mathbb{N}; \\ v + nw2^n \left(n + \dfrac{2}{2^n} - t\right), & t \in \left[n + \dfrac{1}{2^n}, n + \dfrac{2}{2^n}\right), \ n \in \mathbb{N}; \\ v, & t \in \left[n + \dfrac{2}{2^n}, n + 1\right), \ n \in \mathbb{N}. \end{cases}$$

We analyze the equation

$$\left[\frac{(tx')^3}{u}\right]' + \frac{h(t)}{t\log^4 t}\,x^3 = 0. \tag{2.4.31}$$

Hence, we deal with Eq. (2.4.1), where $p = 4$ and

$$r(t) = \frac{1}{u}, \quad s(t) = h(t), \qquad t \in \mathbb{R}_e.$$

One can verify that

$$M(h) = v, \qquad M\left(r^{1-q}\right) = M\left(r^{-\frac{1}{3}}\right) = M\left(u^{\frac{1}{3}}\right) = u^{\frac{1}{3}} > 0.$$

Therefore, applying Corollary 2.4.1 (condition (2.4.14) is trivially valid for all $\alpha \in \mathbb{R}^+$), we know that Eq. (2.4.31) is oscillatory when $4^4uv > 3^4$ and non-oscillatory when $4^4uv < 3^4$. Note that the second coefficient h has mean value, but it is not asymptotically almost periodic (it suffices to consider that $\limsup_{t\to\infty} |h(t)| = \infty$).

Example 2.4.2. For $a, b, c > 0$, we consider the equation

$$\begin{aligned}\left[\left(\frac{t^2x'}{1+t\,[1+\sin t\cos t]}\right)^5\right]' \\ + \frac{|\sin(at)| + \arctan[\sin(bt)+\cos(bt)]}{c\left(t+\sqrt{t}\right)\log^6 t}\,x^5 = 0\end{aligned} \tag{2.4.32}$$

which is in the form of Eq. (2.4.1) with

$$r(t) = \left(\frac{t}{1+t\,[1+\sin t\cos t]}\right)^5, \qquad t \in \mathbb{R}_e,$$
$$s(t) = \frac{t\,(|\sin(at)| + \arctan[\sin(bt)+\cos(bt)])}{c\left(t+\sqrt{t}\right)}, \qquad t \in \mathbb{R}_e,$$

and $p = 6$. Since r^{1-q} and s are asymptotically almost periodic functions (see, e.g., [12, 29]), we can use directly Corollary 2.4.2. We have

$$M\left(r^{1-q}\right) = M\left(r^{-\frac{1}{5}}\right) = M\left(\frac{1}{t} + 1 + \sin t\cos t\right) = 1,$$

$$cM(s) = M\left(|\sin(at)| + \arctan[\sin(bt)+\cos(bt)]\right) = M\left(|\sin(at)|\right) = \frac{2}{\pi}.$$

Therefore, Eq. (2.4.32) is oscillatory for

$$c < \overline{\Gamma} := \left(\frac{6}{5}\right)^6 \frac{2}{\pi}$$

and non-oscillatory for $c > \overline{\Gamma}$.

Example 2.4.3. Let $p = 3/2$ and let $f, g : [-1, 1] \to \mathbb{R}^+$ be continuous functions. We find the oscillation constant for the equation

$$\left[\sqrt{tf\,(\sin t)}\,\Phi(x')\right]' + \frac{g\,(\sin t)}{t\log^p t}\,\Phi(x) = 0. \tag{2.4.33}$$

Evidently, the functions $r(t) = \sqrt{f\,(\sin t)}$ and $s(t) = g\,(\sin t)$ are periodic and

$$M\left(r^{-2}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}t}{f\,(\sin t)}, \quad M(s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g\,(\sin t)\,\mathrm{d}t.$$

We can apply, e.g., Corollary 2.4.2. If

$$\widehat{\Gamma} := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\sin t)\,\mathrm{d}t \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}t}{f(\sin t)}} > 3^{-3/2},$$

then Eq. (2.4.33) is oscillatory. If $\widehat{\Gamma} < 3^{-3/2}$, then Eq. (2.4.33) is non-oscillatory.

2.5 Modified Euler type equations in critical case

In this section, we study the oscillation behavior of the equation

$$\left[r^{-\frac{p}{q}}(t)\, t^{p-1} \Phi(x')\right]' + \frac{s(t)}{t \log^p t}\, \Phi(x) = 0, \tag{2.5.1}$$

where $r > 0$ and s are continuous functions. The motivation comes from Section 2.4, where the equation

$$\left[r(t)\, t^{p-1} \Phi(x')\right]' + \frac{s(t)}{t \log^p t}\, \Phi(x) = 0 \tag{2.5.2}$$

is proved to be conditionally oscillatory. Note that, in Section 2.4, Eq. (2.5.2) is considered without the power $-p/q$ in the first term. Nevertheless, since function r is positive, it does not have any impact. We consider Eq. (2.5.2) in the presented form only due to technical reasons, i.e., the technical parts of our processes are more transparent.

Since the critical case when the coefficients indicate exactly the critical value is open, the aim of this section is to fill this gap. We will consider Eq. (2.5.1) with periodic continuous coefficients. We will not require any common period for coefficients r and s.

§ 2.5.1 Preliminaries

In this paragraph, we describe the equation for the modified half-linear Prüfer angle given by the studied type of equations and we prove the auxiliary lemmata.

Now, let us turn our attention to the half-linear equation

$$\left[r^{-\frac{p}{q}}(t)\, t^{p-1} \Phi(x')\right]' + \frac{s(t)}{t \log^p t}\, \Phi(x) = 0 \tag{2.5.3}$$

and the corresponding equation for the Prüfer angle

$$\begin{aligned} \varphi'(t) = \frac{1}{t \log t} \Bigg[& r(t) |\cos_p \varphi(t)|^p - \Phi(\cos_p \varphi(t)) \sin_p \varphi(t) \\ & + s(t)\, \frac{|\sin_p \varphi(t)|^p}{p-1} \Bigg], \end{aligned} \tag{2.5.4}$$

where $r : \mathbb{R} \to \mathbb{R}$ is a continuous, positive, and α-periodic function and $s : \mathbb{R} \to \mathbb{R}$ is a continuous and β-periodic function.

Since Eq. (2.5.4) for the Prüfer angle of Eq. (2.5.3) was obtained analogously as in § 2.4.1, we mention only main steps. We used the Riccati type transformation

$$w(t) = r^{-\frac{p}{q}}(t)\, t^{p-1} \Phi\left(\frac{x'(t)}{x(t)}\right)$$

to Eq. (2.5.3). This lead to the equation

$$w'(t) + \frac{s(t)}{t \log^p t} + (p-1)\left[r^{-\frac{p}{q}}(t)\, t^{p-1}\right]^{\frac{1}{1-p}} |w(t)|^{\frac{p}{p-1}} = 0. \tag{2.5.5}$$

Then, using the substitution

$$v(t) = (\log t)^{\frac{p}{q}} w(t), \quad t \in (\mathrm{e}, \infty),$$

in Eq. (2.5.5) and taking into account the modified Prüfer transformation

$$x(t) = \rho(t) \sin_p \varphi(t), \quad \left[r^{-\frac{p}{q}}(t)\, t^{p-1}\right]^{q-1} x'(t) = \frac{\rho(t)}{\log t} \cos_p \varphi(t),$$

we obtained Eq. (2.5.4).

Further, let us mention the definition of the mean value of an arbitrary periodic function which is essential for our results. (See also the more general Definition 2.1.1.)

Definition 2.5.1. The *mean value* $M(f)$ of a periodic function $f : \mathbb{R} \to \mathbb{R}$ with period $P > 0$ is defined as

$$M(f) := \frac{1}{P} \int_0^P f(\tau)\, \mathrm{d}\tau.$$

Finally, for the upcoming use, we put

$$\tilde{r} := \sup\{r(t);\, t > \mathrm{e}\}, \quad \tilde{s} := \sup\{|s(t)|;\, t > \mathrm{e}\}, \tag{2.5.6}$$

we denote $2\varrho := \min\{p-1, 1\}$, and we mention that the half-linear trigonometric functions satisfy

$$|\cos_p a| \le 1, \quad |\sin_p a| \le 1, \quad a \in \mathbb{R}. \tag{2.5.7}$$

Further, for $\vartheta > 0$ be arbitrary, we define

$$\psi(t) := \frac{1}{\sqrt{t}} \int_t^{t+\sqrt{t}} \varphi(\tau)\ \mathrm{d}\tau, \quad t \ge \mathrm{e} + \vartheta, \tag{2.5.8}$$

where φ is a solution of Eq. (2.5.4) on $[\mathrm{e} + \vartheta, \infty)$. Now, we formulate and prove auxiliary results concerning this function ψ.

Lemma 2.5.1. *If φ is a solution of Eq.* (2.5.4) *on* $[\mathrm{e}+\vartheta, \infty)$, *then the function* $\psi : [\mathrm{e}+\vartheta, \infty) \to \mathbb{R}$ *defined by* (2.5.8) *satisfies*

$$|\varphi(\tau) - \psi(t)| \leq \frac{C}{\sqrt{t}\log t}, \quad t \geq \mathrm{e}+\vartheta,\, \tau \in \left[t, t+\sqrt{t}\right], \tag{2.5.9}$$

for some constant $C > 0$.

Proof. The continuity of function φ implies that, for any $t \geq \mathrm{e}+\vartheta$, there exists $\tilde{t} \in \left[t, t+\sqrt{t}\right]$ such that $\psi(t) = \varphi\left(\tilde{t}\right)$. Hence, for all $t \geq \mathrm{e}+\vartheta$, $\tau \in \left[t, t+\sqrt{t}\right]$, we get

$$\begin{aligned}
|\varphi(\tau) - \psi(t)| = \left|\varphi(\tau) - \varphi\left(\tilde{t}\right)\right| &\leq \int_t^{t+\sqrt{t}} |\varphi'(\tau)| \,\mathrm{d}\tau \\
&\leq \frac{1}{t\log t}\left[\int_t^{t+\sqrt{t}} r(\tau)|\cos_p \varphi(\tau)|^p + |\Phi(\cos_p \varphi(\tau))\sin_p \varphi(\tau)| \,\mathrm{d}\tau \right.\\
&\qquad \left. + \int_t^{t+\sqrt{t}} \frac{|\sin_p \varphi(\tau)|^p}{p-1} |s(\tau)| \,\mathrm{d}\tau\right],
\end{aligned}$$

i.e., we obtain (see (2.5.7), (2.5.6))

$$|\varphi(\tau) - \psi(t)| \leq \frac{1}{t\log t}\int_t^{t+\sqrt{t}} \left(\tilde{r} + 1 + \frac{\tilde{s}}{p-1}\right) \mathrm{d}\tau \leq \frac{C}{\sqrt{t}\log t},$$

where

$$C := \tilde{r} + 1 + \frac{\tilde{s}}{p-1}. \tag{2.5.10}$$

□

Lemma 2.5.2. *The inequality*

$$\left|\psi'(t) - \frac{1}{t\log t}\left[\frac{|\cos_p \psi(t)|^p}{\sqrt{t}}\int_t^{t+\sqrt{t}} r(\tau)\,\mathrm{d}\tau - \Phi\left(\cos_p \psi(t)\right)\sin_p \psi(t) \right.\right.$$
$$\left.\left. + \frac{|\sin_p \psi(t)|^p}{(p-1)\sqrt{t}}\int_t^{t+\sqrt{t}} s(\tau)\,\mathrm{d}\tau\right]\right| < \frac{D}{t^{1+\varrho}\log t}$$

holds for some $D > 0$ and for all $t > \mathrm{e}+\vartheta$.

Proof. For all $t > \mathrm{e} + \vartheta$, we have

$$\begin{aligned}
\psi'(t) &= \left(1 + \frac{1}{2\sqrt{t}}\right) \frac{\varphi\left(t + \sqrt{t}\right)}{\sqrt{t}} - \frac{\varphi(t)}{\sqrt{t}} - \frac{1}{2\sqrt{t^3}} \int\limits_t^{t+\sqrt{t}} \varphi(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} \varphi'(\tau)\,\mathrm{d}\tau + \frac{1}{2t}\varphi\left(t + \sqrt{t}\right) - \frac{1}{2\sqrt{t^3}} \int\limits_t^{t+\sqrt{t}} \varphi(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} \frac{1}{\tau \log \tau} \Big[r(\tau) |\cos_p \varphi(\tau)|^p - \Phi\left(\cos_p \varphi(\tau)\right) \sin_p \varphi(\tau) \\
&\qquad + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \Big] \,\mathrm{d}\tau + \frac{1}{2\sqrt{t^3}} \int\limits_t^{t+\sqrt{t}} \Big[\varphi\left(t + \sqrt{t}\right) - \varphi(\tau) \Big] \,\mathrm{d}\tau.
\end{aligned}$$

Since (see also (2.5.7), (2.5.6), and (2.5.10))

$$\begin{aligned}
&\left| \frac{1}{2\sqrt{t^3}} \int\limits_t^{t+\sqrt{t}} \Big[\varphi\left(t + \sqrt{t}\right) - \varphi(\tau) \Big] \,\mathrm{d}\tau \right| \le \frac{1}{2\sqrt{t^3}} \int\limits_t^{t+\sqrt{t}} \int\limits_\tau^{t+\sqrt{t}} \left|\varphi'(\sigma)\right| \mathrm{d}\sigma\,\mathrm{d}\tau \\
&\le \frac{1}{2\sqrt{t^3}} \int\limits_t^{t+\sqrt{t}} \int\limits_\tau^{t+\sqrt{t}} \frac{1}{\sigma \log \sigma} \Big| r(\sigma) |\cos_p \varphi(\sigma)|^p - \Phi\left(\cos_p \varphi(\sigma)\right) \sin_p \varphi(\sigma) \\
&\qquad + s(\sigma) \frac{|\sin_p \varphi(\sigma)|^p}{p-1} \Big| \,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&\le \frac{1}{2\sqrt{t^5}\log t} \int\limits_t^{t+\sqrt{t}} \int\limits_t^{t+\sqrt{t}} \left[\tilde{r} + 1 + \frac{\tilde{s}}{p-1} \right] \mathrm{d}\sigma\,\mathrm{d}\tau \le \frac{C}{2\sqrt{t^3}\log t},
\end{aligned}$$

it suffices to consider

$$\begin{aligned}
&\frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} \frac{1}{\tau \log \tau} \Big[r(\tau) |\cos_p \varphi(\tau)|^p - \Phi\left(\cos_p \varphi(\tau)\right) \sin_p \varphi(\tau) \\
&\qquad + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \Big] \,\mathrm{d}\tau.
\end{aligned}$$

In fact, we will consider

$$\begin{aligned}
&\frac{1}{\sqrt{t^3}\log t} \int\limits_t^{t+\sqrt{t}} \Big[r(\tau) |\cos_p \varphi(\tau)|^p - \Phi\left(\cos_p \varphi(\tau)\right) \sin_p \varphi(\tau) \\
&\qquad + s(\tau) \frac{|\sin_p \varphi(\tau)|^p}{p-1} \Big] \,\mathrm{d}\tau,
\end{aligned} \tag{2.5.11}$$

because

$$\Bigg| \int\limits_t^{t+\sqrt{t}} \frac{1}{\tau \log \tau} \left[r(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) \right] \mathrm{d}\tau$$

$$+ \int\limits_t^{t+\sqrt{t}} \frac{1}{\tau \log \tau} \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau)\, \mathrm{d}\tau$$

$$- \int\limits_t^{t+\sqrt{t}} \frac{1}{t \log t} \left[r(\tau) |\cos_p \varphi(\tau)|^p - \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau) \right] \mathrm{d}\tau$$

$$- \int\limits_t^{t+\sqrt{t}} \frac{1}{t \log t} \frac{|\sin_p \varphi(\tau)|^p}{p-1} s(\tau)\, \mathrm{d}\tau \Bigg|$$

$$\leq \int\limits_t^{t+\sqrt{t}} \left[\tilde{r} + 1 + \frac{\tilde{s}}{p-1} \right] \left[\frac{1}{t \log t} - \frac{1}{\tau \log \tau} \right] \mathrm{d}\tau$$

$$\leq C\sqrt{t}\, \frac{\left(t+\sqrt{t}\right) \log\left(t+\sqrt{t}\right) - t \log t}{t \left(t+\sqrt{t}\right) \log\left(t+\sqrt{t}\right) \log t} \leq \frac{KC}{t \log t}$$

for all $t \geq \mathrm{e} + \vartheta$, where $K > 0$ is such a constant that

$$\frac{\left(t+\sqrt{t}\right) \log\left(t+\sqrt{t}\right) - t \log t}{\log\left(t+\sqrt{t}\right)} \leq K\sqrt{t}, \quad t \geq \mathrm{e} + \vartheta.$$

Considering the form of (2.5.11), to finish the proof, it suffices to prove the following inequalities

$$\Bigg| \frac{|\cos_p \psi(t)|^p}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} r(\tau)\, \mathrm{d}\tau - \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} r(\tau) |\cos_p \varphi(\tau)|^p\, \mathrm{d}\tau \Bigg| \leq \frac{E_1}{\sqrt{t} \log t}, \tag{2.5.12}$$

$$\Bigg| \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} \Phi(\cos_p \psi(t)) \sin_p \psi(t)\, \mathrm{d}\tau - \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} \Phi(\cos_p \varphi(\tau)) \sin_p \varphi(\tau)\, \mathrm{d}\tau \Bigg| \leq \frac{E_2}{t^{\varrho} \log^{2\varrho} t}, \tag{2.5.13}$$

$$\left| \frac{|\sin_p \psi(t)|^p}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} s(\tau)\,\mathrm{d}\tau - \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} s(\tau)|\sin_p \varphi(\tau)|^p\,\mathrm{d}\tau \right| \le \frac{E_3}{\sqrt{t}\log t} \tag{2.5.14}$$

for some constants $E_1, E_2, E_3 > 0$ and for all $t \ge \mathrm{e} + \vartheta$.

The half-linear trigonometric functions $\sin_p$ and $\cos_p$ are continuously differentiable with period $2\pi_p$. Thus, there exists $A > 0$ for which

$$\big| |\cos_p a|^p - |\cos_p b|^p \big| \le A|a-b|, \quad a, b \in \mathbb{R}, \tag{2.5.15}$$

$$\big| \cos_p a - \cos_p b \big| \le A|a-b|, \quad a, b \in \mathbb{R}, \tag{2.5.16}$$

$$\big| |\sin_p a|^p - |\sin_p b|^p \big| \le A|a-b|, \quad a, b \in \mathbb{R}, \tag{2.5.17}$$

$$\big| \sin_p a - \sin_p b \big| \le A|a-b|, \quad a, b \in \mathbb{R}. \tag{2.5.18}$$

In addition, directly from the definition of Φ, it follows the existence of $B > 0$ such that

$$|\Phi(a) - \Phi(b)| \le [B|a-b|]^{\min\{1,p-1\}}, \quad a, b \in [-1, 1]. \tag{2.5.19}$$

At first, we consider inequality (2.5.12) which comes from (see also (2.5.6), (2.5.9), and (2.5.15))

$$\left| \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} r(\tau)\left(|\cos_p \psi(t)|^p - |\cos_p \varphi(\tau)|^p \right) \mathrm{d}\tau \right|$$

$$\le \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} r(\tau) A |\psi(t) - \varphi(\tau)|\,\mathrm{d}\tau \le \frac{\tilde{r}AC}{\sqrt{t}\log t}, \quad t \ge \mathrm{e} + \vartheta.$$

Similarly, we can obtain (2.5.14) from (see (2.5.6), (2.5.9), and (2.5.17))

$$\left| \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} s(\tau)\left(|\sin_p \psi(t)|^p - |\sin_p \varphi(\tau)|^p \right) \mathrm{d}\tau \right|$$

$$\le \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} |s(\tau)|\, A\, |\psi(t) - \varphi(\tau)|\,\mathrm{d}\tau \le \frac{\tilde{s}AC}{\sqrt{t}\log t}, \quad t \ge \mathrm{e} + \vartheta.$$

It remains to show (2.5.13). We have (see (2.5.7))

$$\left|\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\left[\Phi(\cos_p\psi(t))\sin_p\psi(t)-\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)\right]\mathrm{d}\tau\right|$$
$$\le\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\left|\Phi(\cos_p\psi(t))\sin_p\psi(t)-\Phi(\cos_p\psi(t))\sin_p\varphi(\tau)\right|\mathrm{d}\tau$$
$$+\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\left|\Phi(\cos_p\psi(t))\sin_p\varphi(\tau)-\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)\right|\mathrm{d}\tau$$

$$\le\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\left|\sin_p\psi(t)-\sin_p\varphi(\tau)\right|\mathrm{d}\tau$$
$$+\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\left|\Phi(\cos_p\psi(t))-\Phi(\cos_p\varphi(\tau))\right|\mathrm{d}\tau$$

for all $t\ge \mathrm{e}+\vartheta$ and, using (2.5.9), (2.5.16), (2.5.18), and (2.5.19), we obtain

$$\left|\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\left[\Phi(\cos_p\psi(t))\sin_p\psi(t)-\Phi(\cos_p\varphi(\tau))\sin_p\varphi(\tau)\right]\mathrm{d}\tau\right|$$
$$\le\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}A\left|\psi(t)-\varphi(\tau)\right|\mathrm{d}\tau$$
$$+\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\left[B\left|\cos_p\psi(t)-\cos_p\varphi(\tau)\right|\right]^{\min\{1,p-1\}}\mathrm{d}\tau$$
$$\le\frac{1}{\sqrt{t}}\int_t^{t+\sqrt{t}}\frac{AC}{\sqrt{t}\log t}+\left[AB\left|\psi(t)-\varphi(\tau)\right|\right]^{\min\{1,p-1\}}\mathrm{d}\tau$$
$$\le\frac{AC}{\sqrt{t}\log t}+\left(\frac{ABC}{\sqrt{t}\log t}\right)^{\min\{1,p-1\}}$$

for all $t\ge \mathrm{e}+\vartheta$, i.e., (2.5.13) is valid for

$$E_2:=AC+[ABC]^{\min\{1,p-1\}}.$$

The proof is complete. □

Now, we recall one known result and we provide its direct consequence which we will use in the proof of Theorem 2.5.2 in the next paragraph.

Theorem 2.5.1. *If $M, N > 0$ are such that $M^{p-1}N = q^{-p}$, then the equation*

$$\left[\left(M + \frac{1}{t}\right)^{-\frac{p}{q}} \Phi(x')\right]' + \frac{1}{t^p}\left(N + \frac{1}{t}\right)\Phi(x) = 0 \tag{2.5.20}$$

is non-oscillatory.

Proof. See, e.g., [17]. □

Corollary 2.5.1. *If $M, N > 0$ are such that $M^{p-1}N = q^{-p}$, then the equation*

$$\left[\left(M + \frac{1}{\log t}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]' + \frac{N + \frac{1}{\log t}}{t \log^p t}\Phi(x) = 0 \tag{2.5.21}$$

is non-oscillatory.

Proof. Let us consider Eq. (2.5.21), where $x = x(t)$ and $(\cdot)' = \frac{\mathrm{d}}{\mathrm{d}t}$. Using the transformation of the independent variable $s = \log t$ when $x(t) = y(s)$, we have

$$\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(M + \frac{1}{s}\right)^{-\frac{p}{q}} t^{p-1}\Phi\left(\frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}s}\right)\right]' + \frac{1}{ts^p}\left(N + \frac{1}{s}\right)\Phi(y) = 0.$$

This equation can be easily simplified into the form

$$\left[\left(M + \frac{1}{s}\right)^{-\frac{p}{q}} \Phi(y')\right]' + \frac{1}{s^p}\left(N + \frac{1}{s}\right)\Phi(y) = 0. \tag{2.5.22}$$

Hence (cf. Eq. (2.5.20) and Eq. (2.5.22)), it suffices to apply Theorem 2.5.1. □

§ 2.5.2 Results and examples

Applying Lemma 2.5.2 and Corollary 2.5.1, we prove the following theorem.

Theorem 2.5.2. *Let $\alpha, \beta > 0$. If $r : \mathbb{R} \to \mathbb{R}$ is α-periodic and $s : \mathbb{R} \to \mathbb{R}$ is β-periodic such that*

$$\left[\frac{1}{\alpha}\int_0^\alpha r(\tau)\,\mathrm{d}\tau\right]^{p-1} \frac{1}{\beta}\int_0^\beta s(\tau)\,\mathrm{d}\tau = [M(r)]^{p-1} M(s) = q^{-p}, \tag{2.5.23}$$

then Eq. (2.5.3) is non-oscillatory.

Proof. In this proof, we consider the equation for the Prüfer angle φ and the corresponding equation for ψ. The used method is based on the fact that the non-oscillation of solutions of Eq. (2.5.3) is equivalent to the boundedness from above of a solution φ of Eq. (2.5.4). We can refer to [42] or also to each one of papers [17, 19, 54, 71, 39]. In addition, Lemma 2.5.1 implies that a solution $\varphi : [\mathrm{e} + \vartheta, \infty) \to \mathbb{R}$ of Eq. (2.5.4) is bounded from above if and only if ψ given by (2.5.8) is bounded from above.

From Lemma 2.5.2, we have

$$\psi'(t) < \frac{1}{t \log t} \left[\frac{|\cos_p \psi(t)|^p}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} r(\tau)\,\mathrm{d}\tau - \Phi\left(\cos_p \psi(t)\right) \sin_p \psi(t) \right.$$
$$\left. + \frac{|\sin_p \psi(t)|^p}{(p-1)\sqrt{t}} \int\limits_t^{t+\sqrt{t}} s(\tau)\,\mathrm{d}\tau + \frac{D}{t^{\varrho}} \right]$$

for all $t > \mathrm{e} + \vartheta$ and for some D. Especially,

$$\begin{aligned} \psi'(t) < \frac{1}{t \log t} \Bigg[& \frac{|\cos_p \psi(t)|^p}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} r(\tau)\,\mathrm{d}\tau - \Phi\left(\cos_p \psi(t)\right) \sin_p \psi(t) \\ & + \frac{|\sin_p \psi(t)|^p}{(p-1)\sqrt{t}} \int\limits_t^{t+\sqrt{t}} s(\tau)\,\mathrm{d}\tau + \frac{D}{\log^2 t} \Bigg] \end{aligned} \tag{2.5.24}$$

for all $t > \mathrm{e} + \vartheta$. Then, using the periodicity of coefficients r, s, we obtain (see (2.5.6) and (2.5.24))

$$\begin{aligned} \psi'(t) < \frac{1}{t \log t} \Bigg[& |\cos_p \psi(t)|^p \left(M(r) + \frac{\tilde{r}\alpha}{\sqrt{t}} \right) \\ & - \Phi\left(\cos_p \psi(t)\right) \sin_p \psi(t) \\ & + \frac{|\sin_p \psi(t)|^p}{p-1} \left(M(s) + \frac{\tilde{s}\beta}{\sqrt{t}} \right) + \frac{D}{\log^2 t} \Bigg] \end{aligned} \tag{2.5.25}$$

for all $t > \mathrm{e} + \vartheta$. Indeed, for any periodic continuous function f with period $P > 0$ and positive mean value $M(f)$, we have

$$\begin{aligned} \frac{1}{\sqrt{t}} \int\limits_t^{t+\sqrt{t}} f(\tau)\,\mathrm{d}\tau &= \frac{1}{\sqrt{t}} \left(\int\limits_t^{t+Pn} f(\tau)\,\mathrm{d}\tau + \int\limits_{t+Pn}^{t+\sqrt{t}} f(\tau)\,\mathrm{d}\tau \right) \\ &\leq \frac{1}{Pn} \int\limits_t^{t+Pn} f(\tau)\,\mathrm{d}\tau + \frac{1}{\sqrt{t}} \int\limits_{t+Pn}^{t+P(n+1)} |f(\tau)|\,\mathrm{d}\tau \leq M(f) + \frac{\tilde{f}P}{\sqrt{t}}, \end{aligned}$$

where $\tilde{f} := \max\{|f(t)|;\, t \in [0,P]\}$ and $n \in \mathbb{N} \cup \{0\}$ is such that $Pn \le \sqrt{t}$ and that $P(n+1) > \sqrt{t}$.

For $R := \max\{1, p-1\}$, the Pythagorean identity (1.1.5) gives

$$R\left(|\cos_p a|^p + \frac{|\sin_p a|^p}{p-1}\right) \ge 1, \quad a \in \mathbb{R}. \tag{2.5.26}$$

Considering (2.5.25) and (2.5.26), we have

$$\begin{aligned}\psi'(t) < \frac{1}{t\log t}\Bigg[& |\cos_p \psi(t)|^p \left(M(r) + \frac{\tilde{r}\alpha}{\sqrt{t}} + \frac{RD}{\log^2 t}\right) \\ & - \Phi\left(\cos_p \psi(t)\right) \sin_p \psi(t) \\ & + \frac{|\sin_p \psi(t)|^p}{p-1}\left(M(s) + \frac{\tilde{s}\beta}{\sqrt{t}} + \frac{RD}{\log^2 t}\right)\Bigg]\end{aligned}$$

for all $t > \mathrm{e} + \vartheta$ and, consequently, we have

$$\begin{aligned}\psi'(t) < \frac{1}{t\log t}\Bigg[& |\cos_p \psi(t)|^p \left(M(r) + \frac{1}{\log t}\right) \\ & - \Phi\left(\cos_p \psi(t)\right) \sin_p \psi(t) \\ & + \frac{|\sin_p \psi(t)|^p}{p-1}\left(M(s) + \frac{1}{\log t}\right)\Bigg]\end{aligned} \tag{2.5.27}$$

for all large t.

The equation

$$\begin{aligned}\varphi'(t) = \frac{1}{t\log t}\Bigg[& |\cos_p \varphi(t)|^p \left(M(r) + \frac{1}{\log t}\right) \\ & - \Phi\left(\cos_p \varphi(t)\right) \sin_p \varphi(t) \\ & + \frac{|\sin_p \varphi(t)|^p}{p-1}\left(M(s) + \frac{1}{\log t}\right)\Bigg]\end{aligned} \tag{2.5.28}$$

has the form of the equation for the Prüfer angle φ which corresponds to Eq. (2.5.21), where $M = M(r)$ and $N = M(s)$. Therefore (see (2.5.23)), Corollary 2.5.1 guarantees that any solution $\varphi : [\mathrm{e} + \vartheta, \infty) \to \mathbb{R}$ of Eq. (2.5.28) is bounded from above. Comparing (2.5.27) with Eq. (2.5.28) and considering the $2\pi_p$-periodicity of the half-linear trigonometric functions, we know that the considered function ψ is bounded from above. This means that any non-zero solution of Eq. (2.5.3) is non-oscillatory. □

Now, we explicitly mention a corollary of the main results of the previous section that we will combine with Theorem 2.5.2.

Theorem 2.5.3. *Let $r, s : \mathbb{R} \to \mathbb{R}$ be periodic.*

(i) If $[M(r)]^{p-1} M(s) > q^{-p}$, then Eq. (2.5.3) *is oscillatory.*

(ii) If $[M(r)]^{p-1} M(s) < q^{-p}$, then Eq. (2.5.3) *is non-oscillatory.*

Proof. The statements of the theorem can be obtained immediately from Theorem 2.4.4 and 2.4.5. □

Using Theorem 2.5.3, we can generalize Theorem 2.5.2 as follows.

Theorem 2.5.4. *Let $r, s : \mathbb{R} \to \mathbb{R}$ be periodic. Eq.* (2.5.3) *is oscillatory if and only if*

$$[M(r)]^{p-1} M(s) > q^{-p}.$$

We get a new result even for linear equations. Thus, we formulate the corollary below.

Corollary 2.5.2. *Let $r : \mathbb{R} \to \mathbb{R}$ be continuous, positive, and periodic function and let $s : \mathbb{R} \to \mathbb{R}$ be continuous and periodic function. The equation*

$$\left[\frac{t}{r(t)}\, x'\right]' + \frac{s(t)}{t \log^2 t}\, x = 0 \tag{2.5.29}$$

is oscillatory if and only if $4M(r)M(s) > 1$.

To illustrate the presented results, we give some examples of equations whose oscillation properties do not follow from previously known oscillation criteria. At first, we mention an example to illustrate Theorem 2.5.2.

Example 2.5.1. For any $p > 1$, the equation

$$\left[\left(\frac{2 + \sin\left(\sqrt{q}t\right)}{2q}\right)^{-\frac{p}{q}} t^{p-1} \Phi\left(x'\right)\right]' + \frac{p - 1 + \cos\left(pt\right)}{pt \log^p t} \Phi(x) = 0 \tag{2.5.30}$$

is in the critical case because

$$M(r) = M\left(\frac{2 + \sin\left(\sqrt{q}t\right)}{2q}\right) = \frac{1}{q} = M\left(\frac{p - 1 + \cos\left(pt\right)}{p}\right) = M(s).$$

Hence, $[M(r)]^{p-1} M(s) = q^{-p}$ and Eq. (2.5.30) is non-oscillatory due to Theorem 2.5.2.

Of course, the oscillation behavior of Eq. (2.5.30) is solvable in many slightly modified situations as well. For example, its coefficients may involve parameters. Thus, we can apply Theorem 2.5.4 as follows.

Example 2.5.2. Let $a > 1$ and $b, c, d \neq 0$ be real parameters. We consider the equation

$$\left[\left(\frac{a + \sin(ct)}{q}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]' + \frac{p - 1 + \cos(dt)}{bt \log^p t}\Phi(x) = 0 \tag{2.5.31}$$

with

$$M(r) = M\left(\frac{a + \sin(ct)}{q}\right) = \frac{a}{q}$$

and

$$M(s) = M\left(\frac{p - 1 + \cos(dt)}{b}\right) = \frac{p-1}{b}.$$

Therefore, by Theorem 2.5.4, Eq. (2.5.31) is oscillatory for $a^{p-1}p/b > 1$ and non-oscillatory otherwise.

Finally, we mention the following simple example of linear equations whose oscillation properties are solvable by Corollary 2.5.2.

Example 2.5.3. Consider the equation

$$\begin{aligned}\left[\frac{t}{a_1 + b_1\sin(c_1t) + d_1\cos(c_1t)}x'\right]' \\ + \frac{a_2 + b_2\sin(c_2t)\cos(c_2t) + d_2\arcsin[\cos(c_2t)]}{t\log^2 t}x = 0,\end{aligned} \tag{2.5.32}$$

where $a_i, b_i, c_i, d_i \in \mathbb{R}$, $c_i \neq 0$, $i \in \{1, 2\}$, $a_1 > |b_1| + |d_1|$. It is seen that $M(r) = a_1$ and $M(s) = a_2$ (cf. Eq. (2.5.29) and Eq. (2.5.32)). Hence, Eq. (2.5.32) is oscillatory for $a_1a_2 > 1/4$ and non-oscillatory for $a_1a_2 \leq 1/4$. We emphasize that this conclusion remains valid even for, e.g., $c_1 = 1$ and $c_2 = \pi$ or $c_2 = \sqrt{2}$, when r and s do not possess any common period.

2.6 Riemann–Weber type equations

This section is devoted to the study of the half-linear differential equations

$$\left[r(t)t^{p-1}\Phi(x')\right]' + \frac{s(t)}{t\log^p t}\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1}\operatorname{sgn} x \tag{2.6.1}$$

and

$$\begin{aligned}\left[\left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]' \\ + \frac{1}{t\log^p t}\left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2}\right)\Phi(x) = 0\end{aligned} \tag{2.6.2}$$

with continuous coefficients $r > 0, s, r_1 > 0, r_2, s_1, s_2$.

In this section, we fully solve the critical case of Eq. (2.6.1) with periodic coefficients r and s, i.e., we solve the oscillation behavior of this equation in full. At the same time, we turn our attention to the perturbed equation (2.6.2), where the coefficients r_1, s_1 are periodic and the coefficients r_2, s_2 in the perturbations are very general and they can change their signs.

Now, we describe in detail related results and the main motivation for the research presented in this section. At first, we mention paper [19]. Its main result deals with the Euler type equation

$$[r(t)\Phi(x')]' + \frac{\gamma c(t)}{t^p}\Phi(x) = 0 \tag{2.6.3}$$

and with the Riemann–Weber type equation

$$[r(t)\Phi(x')]' + \frac{1}{t^p}\left[\gamma c(t) + \frac{\mu d(t)}{\log^2 t}\right]\Phi(x) = 0, \tag{2.6.4}$$

where r, c, and d are periodic positive functions with the same period. Since [19] is one of the main motivations for our research, we reformulate its main result in full. We should recall that the mean value of a periodic function f over its period, say $T > 0$, is the number

$$M(f) = \frac{1}{T}\int\limits_a^{a+T} f(x)\,\mathrm{d}x,$$

where $a \in \mathbb{R}$ is arbitrary. We can also refer to Definition 2.1.1. Further, we will use the notation

$$\gamma_p = \left(\frac{p-1}{p}\right)^p, \qquad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

Theorem 2.6.1 ([19])**.** *Eq.* (2.6.3) *is non-oscillatory if and only if*

$$\gamma \le \gamma_{rc} := \gamma_p \left[M\left(r^{1-q}\right)\right]^{1-p}[M(c)]^{-1}.$$

In the limiting case $\gamma = \gamma_{rc}$, Eq. (2.6.4) *is non-oscillatory if*

$$\mu < \mu_{rd} := \mu_p \left[M\left(r^{1-q}\right)\right][M(d)]^{-1},$$

and it is oscillatory if $\mu > \mu_{rd}$.

The next motivation comes from papers [14, 17, 16, 18]. At this place, we state a result concerning the equation

$$\left[\left(\alpha_1 + \frac{\alpha_2}{\log^2 t}\right)^{-\frac{p}{q}}\Phi(x')\right]' + \frac{1}{t^p}\left(\beta_1 + \frac{\beta_2}{\log^2 t}\right)\Phi(x) = 0, \tag{2.6.5}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants and $\alpha_1 > 0$. Note that, due to the exponent in the first term of Eq. (2.6.5), the formulations of results are technically easier and the exponent does not mean any restriction and can be removed. The following theorem can be obtained, e.g., as a direct corollary of the main result of [17] (or deduced from [14, 16, 18]). We will also use this theorem in the proof of Lemma 2.6.2 below which is essential to prove our main result.

Theorem 2.6.2 ([17])**.** *The following statements hold.*

(i) Eq. (2.6.5) *is oscillatory if* $\beta_1\alpha_1^{p-1} > \gamma_p$*, and non-oscillatory if* $\beta_1\alpha_1^{p-1} < \gamma_p$*.*

(ii) Let $\beta_1\alpha_1^{p-1} = \gamma_p$*. Eq.* (2.6.5) *is oscillatory if*

$$\beta_2\alpha_1^{p-1} + (p-1)\gamma_p\alpha_2\alpha_1^{-1} > \mu_p,$$

and non-oscillatory if

$$\beta_2\alpha_1^{p-1} + (p-1)\gamma_p\alpha_2\alpha_1^{-1} < \mu_p.$$

As the third result which is strongly connected to the presented one, we mention Theorem 2.5.3 which comes from the results of Section § 2.4.3.

In this section, we generalize Theorem 2.5.3 into a very general situation. Especially, we solve the critical case $[M(r)]^{p-1}\, M(s) = \gamma_p$. Our aim is to obtain a result similar to Theorem 2.6.1 which covers also non-periodic coefficients. To make this, we apply Theorem 2.6.2 and the method based on the combination of the modified half-linear Prüfer angle and the Riccati equation.

This section is organized as follows. In the following paragraph, we derive the equation for the modified Prüfer angle, which will be an important tool in the rest of this chapter. Then, we study the behavior of the Prüfer angle. This leads to the proof of the main result in § 2.6.2. The section is finished by corollaries and examples in § 2.6.3.

§ 2.6.1 Modified Prüfer angle and average function

At this place, we provide some background calculations which lead to auxiliary equations that are necessary for our approach. Throughout this section, we will use the notation $\mathbb{R}^+_a := (a, \infty)$ for $a \in \mathbb{R}$. In our main result (see Theorem 2.6.3 below), we will consider the equation

$$\begin{aligned}\left[\left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]'\\ + \frac{1}{t\log^p t}\left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2}\right)\Phi(x) = 0,\end{aligned} \tag{2.6.6}$$

where $r_1 : \mathbb{R} \to \mathbb{R}^+$ and $s_1 : \mathbb{R} \to \mathbb{R}$ are α-periodic continuous functions for some $\alpha \in \mathbb{R}^+$ and where $r_2, s_2 : \mathbb{R}^+_{\mathrm{e}} \to \mathbb{R}$ are continuous functions such that

$$r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} > 0, \qquad t \in \mathbb{R}^+_{\mathrm{e}}, \tag{2.6.7}$$

$$\lim_{t\to\infty} \frac{1}{\sqrt{t\log t}} \int_t^{t+\alpha} |r_2(u)|\,\mathrm{d}u = 0, \tag{2.6.8}$$

and

$$\lim_{t\to\infty} \frac{1}{\sqrt{t\log t}} \int_t^{t+\alpha} |s_2(u)|\,\mathrm{d}u = 0. \tag{2.6.9}$$

For future use, we put

$$r_1^+ := \max_{t\in[0,\alpha]} r_1(t), \qquad s_1^+ := \max_{t\in[0,\alpha]} |s_1(t)|. \tag{2.6.10}$$

For our investigation of Eq. (2.6.6), we need to express the half-linear Prüfer angle in a very special form. Let us briefly describe its derivation. At first, we apply the Riccati transformation

$$w(t) = \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi\left(\frac{x'(t)}{x(t)}\right), \tag{2.6.11}$$

where x is a nontrivial solution of Eq. (2.6.6). The obtained function w satisfies the Riccati equation

$$\begin{aligned} w'(t) + \frac{1}{t\log^p t} \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2} \right) \\ + \frac{p-1}{t} \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right) |w(t)|^q = 0 \end{aligned} \tag{2.6.12}$$

associated to Eq. (2.6.6) whenever $x(t) \neq 0$. For details about the Riccati transformation and equation, see Section 1.1 (we also refer to [21, Section 1.1.4]).

Now, we use the transformation

$$v(t) = (\log t)^{\frac{p}{q}} w(t), \qquad t \in \mathbb{R}_{\mathrm{e}}^+, \tag{2.6.13}$$

in Eq. (2.6.12) which brings us to the adapted (or weighted) Riccati type equation

$$\begin{aligned} v'(t) &= \frac{p}{q} (\log t)^{\frac{p}{q}-1} \frac{w(t)}{t} + (\log t)^{\frac{p}{q}} w'(t) \\ &= \frac{p}{q} \frac{v(t)}{t\log t} - \frac{1}{t\log t} \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2} \right) \\ &\qquad - \frac{p-1}{t} \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right) \frac{|v(t)|^q}{\log t}. \end{aligned} \tag{2.6.14}$$

Thus, in one hand, we keep the adapted Riccati equation (2.6.14). In the other hand, we have the modified half-linear Prüfer transformation

$$x(t) = \rho(t) \sin_p \varphi(t), \qquad \left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2} \right)^{-1} tx'(t) = \frac{\rho(t)}{\log t} \cos_p \varphi(t), \tag{2.6.15}$$

where $\sin_p$ and $\cos_p$ stands for the half-linear sine and cosine function, respectively. For fundamental properties of the half-linear trigonometric functions $\sin_p$ and $\cos_p$, see Section 1.1 (or [21, Section 1.1.2]). In this section, we have to mention only that the half-linear sine and cosine functions are periodic and continuously differentiable and that they satisfy the half-linear Pythagorean identity (1.1.5). Especially,

$$|\sin_p x| \le 1, \qquad |\cos_p x| \le 1, \qquad |\Phi(\cos_p x)| \le 1, \qquad x \in \mathbb{R}. \tag{2.6.16}$$

We combine the adapted Riccati equation (2.6.14) with the Prüfer transformation (2.6.15). We begin with the observations that the function

$$y(t) = \Phi\left(\frac{\cos_p t}{\sin_p t}\right)$$

solves the equation

$$y'(t) + p - 1 + (p-1)|y(t)|^q = 0$$

and that (see (2.6.11), (2.6.13), and (2.6.15))

$$v(t) = (\log t)^{\frac{p}{q}}\left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi\left(\frac{x'(t)}{x(t)}\right) = \Phi\left(\frac{\cos_p \varphi(t)}{\sin_p \varphi(t)}\right). \tag{2.6.17}$$

Using (1.1.5), these two observations lead to the second expression (the first one is the adapted Riccati equation (2.6.14) itself)

$$\begin{aligned} v'(t) = [y(\varphi(t))]' &= [1 - p + (1-p)|y(\varphi(t))|^q]\,\varphi'(t) \\ &= (1-p)\left[1 + \left|\Phi\left(\frac{\cos_p \varphi(t)}{\sin_p \varphi(t)}\right)\right|^q\right]\varphi'(t) \\ &= (1-p)\left[1 + \left|\frac{\cos_p \varphi(t)}{\sin_p \varphi(t)}\right|^p\right]\varphi'(t) = \frac{1-p}{|\sin_p \varphi(t)|^p}\,\varphi'(t). \end{aligned} \tag{2.6.18}$$

Finally, we compare both of the expressions for $v'(t)$, namely (2.6.14) and (2.6.18). Hence, we have

$$\begin{aligned} &\frac{1-p}{|\sin_p \varphi(t)|^p}\,\varphi'(t) \\ &\quad = \frac{p}{q}\,\frac{v(t)}{t\log t} - \frac{1}{t\log t}\left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2}\right) - \frac{p-1}{t}\left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2}\right)\frac{|v(t)|^q}{\log t}, \end{aligned}$$

from where we immediately express the derivative of the modified Prüfer angle (we are aware of (2.6.17))

$$\begin{aligned} \varphi'(t) = \frac{1}{t\log t}\Bigg[&\left(r_1(t) + \frac{r_2(t)}{[\log(\log t)]^2}\right)|\cos_p \varphi(t)|^p - \Phi(\cos_p \varphi(t))\sin_p \varphi(t) \\ &+ \left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2}\right)\frac{|\sin_p \varphi(t)|^p}{p-1}\Bigg]. \end{aligned} \tag{2.6.19}$$

We will use Eq. (2.6.19) to the study of oscillatory properties of Eq. (2.6.6).

For the period α of the functions r_1, s_1, we define the function ψ which determines the average value of an arbitrarily given solution $\varphi : \mathbb{R}_e^+ \to \mathbb{R}$ of Eq. (2.6.19) over intervals of the length α, i.e., we put

$$\psi(t) := \frac{1}{\alpha} \int_t^{t+\alpha} \varphi(u)\,\mathrm{d}u, \qquad t \in \mathbb{R}_e^+, \tag{2.6.20}$$

where φ is a solution of Eq. (2.6.19) on $\mathbb{R}_e^+$.

We prove an auxiliary result concerning the function ψ.

Lemma 2.6.1. *The limit*

$$\lim_{t\to\infty} \sqrt{t\log t}\,|\varphi(s) - \psi(t)| = 0 \tag{2.6.21}$$

exists uniformly with respect to $s \in [t, t+\alpha]$.

Proof. For $s \in [t, t+\alpha]$, we have

$$\begin{aligned}
0 &\le \liminf_{t\to\infty} \sqrt{t\log t}\,|\varphi(s) - \psi(t)| \le \limsup_{t\to\infty} \sqrt{t\log t}\,|\varphi(s) - \psi(t)| \\
&= \limsup_{t\to\infty} \sqrt{t\log t}\left|\varphi(s) - \frac{1}{\alpha}\int_t^{t+\alpha} \varphi(u)\,\mathrm{d}u\right| = \limsup_{t\to\infty} \sqrt{t\log t}\left|\frac{1}{\alpha}\int_t^{t+\alpha} \varphi(s) - \varphi(u)\,\mathrm{d}u\right| \\
&\le \limsup_{t\to\infty} \sqrt{t\log t} \max_{s_1,s_2\in[t,t+\alpha]} |\varphi(s_1) - \varphi(s_2)| = \limsup_{t\to\infty} \sqrt{t\log t} \max_{s_1,s_2\in[t,t+\alpha]} \left|\int_{s_1}^{s_2} \varphi'(u)\,\mathrm{d}u\right| \\
&= \limsup_{t\to\infty} \sqrt{t\log t} \max_{s_1,s_2\in[t,t+\alpha]} \left| \int_{s_1}^{s_2} \frac{1}{u\log u}\left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2}\right)|\cos_p \varphi(u)|^p \right.\right. \\
&\qquad \left.\left. - \Phi(\cos_p \varphi(u))\sin_p \varphi(u) + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2}\right)\frac{|\sin_p \varphi(u)|^p}{p-1}\right]\mathrm{d}u\right|
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{t\to\infty} \sqrt{t\log t} \left\{ \max_{s_1,s_2\in[t,t+\alpha]} \left| \frac{1}{s_1\log s_1} \int_{s_1}^{s_3} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \right.\right.\right. \\
&\qquad \left. - \Phi(\cos_p\varphi(u))\sin_p\varphi(u) + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p\varphi(u)|^p}{p-1} \right] \mathrm{d}u \\
&\qquad\qquad + \frac{1}{s_2\log s_2} \int_{s_3}^{s_2} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p\varphi(u)|^p \right. \\
&\qquad\qquad\qquad \left.\left.\left. - \Phi(\cos_p\varphi(u))\sin_p\varphi(u) + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p\varphi(u)|^p}{p-1} \right] \mathrm{d}u \right| \right\} \\
&\le \limsup_{t\to\infty} \sqrt{t\log t} \left\{ \max_{s_1\in[t,t+\alpha]} \left| \frac{1}{s_1\log s_1} \right| \cdot \left| \int_{s_1}^{s_3} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p\varphi(u)|^p \right.\right.\right. \\
&\qquad \left.\left. - \Phi(\cos_p\varphi(u))\sin_p\varphi(u) + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p\varphi(u)|^p}{p-1} \right] \mathrm{d}u \right| \\
&\qquad\qquad + \max_{s_2\in[t,t+\alpha]} \left| \frac{1}{s_2\log s_2} \right| \cdot \left| \int_{s_3}^{s_2} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p\varphi(u)|^p \right.\right. \\
&\qquad\qquad\qquad \left.\left.\left. - \Phi(\cos_p\varphi(u))\sin_p\varphi(u) + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p\varphi(u)|^p}{p-1} \right] \mathrm{d}u \right| \right\} \\
&\le \limsup_{t\to\infty} \sqrt{t\log t} \left\{ \frac{1}{t\log t} \max_{s_1,s_2\in[t,t+\alpha]} \left| \int_{s_1}^{s_2} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p\varphi(u)|^p \right.\right.\right. \\
&\qquad \left.\left. - \Phi(\cos_p\varphi(u))\sin_p\varphi(u) + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p\varphi(u)|^p}{p-1} \right] \mathrm{d}u \right| \\
&\qquad\qquad + \frac{1}{t\log t} \max_{s_1,s_2\in[t,t+\alpha]} \left| \int_{s_1}^{s_2} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p\varphi(u)|^p \right.\right. \\
&\qquad\qquad\qquad \left.\left.\left. - \Phi(\cos_p\varphi(u))\sin_p\varphi(u) + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p\varphi(u)|^p}{p-1} \right] \mathrm{d}u \right| \right\} \\
&\le \limsup_{t\to\infty} \frac{2}{\sqrt{t\log t}} \max_{s_1,s_2\in[t,t+\alpha]} \int_{s_1}^{s_2} \left| \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p\varphi(u)|^p \right. \\
&\qquad\qquad \left. - \Phi(\cos_p\varphi(u))\sin_p\varphi(u) + \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) \frac{|\sin_p\varphi(u)|^p}{p-1} \right| \mathrm{d}u
\end{aligned}
$$

$$\le 2 \limsup_{t\to\infty} \frac{1}{\sqrt{t\log t}} \max_{s_1,s_2\in[t,t+\alpha]} \int_{s_1}^{s_2} \Bigg[r_1^+ + \frac{|r_2(u)|}{[\log(\log u)]^2} + 1 + \frac{1}{p-1}\left(s_1^+ + \frac{|s_2(u)|}{[\log(\log u)]^2} \right) \Bigg] \mathrm{d}u$$

$$\le 2 \limsup_{t\to\infty} \frac{1}{\sqrt{t\log t}} \int_{t}^{t+\alpha} \left[r_1^+ + \frac{|r_2(u)|}{[\log(\log t)]^2} + 1 + \frac{1}{p-1}\left(s_1^+ + \frac{|s_2(u)|}{[\log(\log t)]^2} \right) \right] \mathrm{d}u = 0,$$

where (2.6.8), (2.6.9), (2.6.10), and (2.6.16) are used. □

§ 2.6.2 Preliminaries and results

At first, we discuss the oscillation behavior of the equation

$$\left[\left(\alpha_1 + \frac{\alpha_2}{[\log(\log t)]^2} \right)^{-\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{1}{t\log^p t}\left(\beta_1 + \frac{\beta_2}{[\log(\log t)]^2} \right) \Phi(x) = 0 \quad (2.6.22)$$

with constant coefficients $\alpha_1 \in \mathbb{R}^+$, $\alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Applying a simple transformation, one can get the following lemma.

Lemma 2.6.2. *Eq.* (2.6.22) *is oscillatory for* $\alpha_1^{p-1}\beta_1 > q^{-p}$ *and non-oscillatory for* $\alpha_1^{p-1}\beta_1 < q^{-p}$. *In the limiting case* $\alpha_1^{p-1}\beta_1 = q^{-p}$, *Eq.* (2.6.22) *is oscillatory if*

$$\beta_2\alpha_1^{p-1} + \frac{p-1}{q^p}\frac{\alpha_2}{\alpha_1} > \frac{q^{1-p}}{2}, \quad (2.6.23)$$

and non-oscillatory if

$$\beta_2\alpha_1^{p-1} + \frac{p-1}{q^p}\frac{\alpha_2}{\alpha_1} < \frac{q^{1-p}}{2}. \quad (2.6.24)$$

Proof. In Eq. (2.6.22), we have $x = x(t)$ and $(\cdot)' = \mathrm{d}/\mathrm{d}t$. Using the transformation of the independent variable

$$s = \log t, \qquad \text{i.e.,} \qquad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}s},$$

we obtain (we put $x(t) = y(s)$)

$$\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(\alpha_1 + \frac{\alpha_2}{\log^2 s} \right)^{-\frac{p}{q}} t^{p-1} \Phi\left(\frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}s} \right) \right] + \frac{1}{ts^p}\left(\beta_1 + \frac{\beta_2}{\log^2 s} \right)\Phi(y) = 0.$$

This leads to the equation

$$\left[\left(\alpha_1 + \frac{\alpha_2}{\log^2 s} \right)^{-\frac{p}{q}} \Phi(y') \right]' + \frac{1}{s^p}\left(\beta_1 + \frac{\beta_2}{\log^2 s} \right)\Phi(y) = 0,$$

where $y = y(s)$ and $(\cdot)' = \mathrm{d}/\mathrm{d}s$. Now, it suffices to use Theorem 2.6.2. □

From Lemma 2.6.2, we get the lemma below which closes the preliminary results.

Lemma 2.6.3. *Let $M(r_1), M(s_1) \in \mathbb{R}^+$ be such that $[M(r_1)]^{p-1}M(s_1) = q^{-p}$.*

(i) If $X, Y \in \mathbb{R}$ satisfy

$$[M(r_1)]^{p-1}Y + \frac{p-1}{q^p}\frac{X}{M(r_1)} > \frac{q^{1-p}}{2}, \tag{2.6.25}$$

then any solution $\theta : \mathbb{R}^+_{\mathrm{e}} \to \mathbb{R}$ of the equation

$$\begin{aligned}\theta'(t) = \frac{1}{t\log t}\Bigg[&\left(M(r_1) + \frac{X}{[\log(\log t)]^2}\right)|\cos_p\theta(t)|^p \\ &-\Phi\left(\cos_p\theta(t)\right)\sin_p\theta(t) + \left(M(s_1) + \frac{Y}{[\log(\log t)]^2}\right)\frac{|\sin_p\theta(t)|^p}{p-1}\Bigg]\end{aligned} \tag{2.6.26}$$

is unbounded from above.

(ii) If $V, W \in \mathbb{R}$ satisfy

$$[M(r_1)]^{p-1}W + \frac{p-1}{q^p}\frac{V}{M(r_1)} < \frac{q^{1-p}}{2}, \tag{2.6.27}$$

then any solution $\xi : \mathbb{R}^+_{\mathrm{e}} \to \mathbb{R}$ of the equation

$$\begin{aligned}\xi'(t) = \frac{1}{t\log t}\Bigg[&\left(M(r_1) + \frac{V}{[\log(\log t)]^2}\right)|\cos_p\xi(t)|^p \\ &-\Phi\left(\cos_p\xi(t)\right)\sin_p\xi(t) + \left(M(s_1) + \frac{W}{[\log(\log t)]^2}\right)\frac{|\sin_p\xi(t)|^p}{p-1}\Bigg]\end{aligned} \tag{2.6.28}$$

is bounded from above.

Proof. Comparing Eq. (2.6.26) and Eq. (2.6.28) with Eq. (2.6.19), one can see that Eq. (2.6.26) and Eq. (2.6.28) is the equation of the Prüfer angle for

$$\begin{aligned}&\left[\left(M(r_1) + \frac{X}{[\log(\log t)]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]' \\ &\qquad + \frac{1}{t\log^p t}\left(M(s_1) + \frac{Y}{[\log(\log t)]^2}\right)\Phi(x) = 0\end{aligned} \tag{2.6.29}$$

and

$$\begin{aligned}&\left[\left(M(r_1) + \frac{V}{[\log(\log t)]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]' \\ &\qquad + \frac{1}{t\log^p t}\left(M(s_1) + \frac{W}{[\log(\log t)]^2}\right)\Phi(x) = 0,\end{aligned} \tag{2.6.30}$$

respectively.

Let us focus on the first case. The assumption $[M(r_1)]^{p-1}M(s_1) = q^{-p}$ and (2.6.25) give that Eq. (2.6.29) is oscillatory (see (2.6.23) in Lemma 2.6.2). Now, it suffices to consider directly the Prüfer transformation (2.6.15) and take into account the form of Eq. (2.6.26), where $\sin_p \theta(t) = 0$ implies $\theta'(t) > 0$ for all large t. Therefore, Eq. (2.6.29) is oscillatory if and only if its Prüfer angle θ is unbounded from above. Part (*i*) is proved.

Considering (2.6.24) and (2.6.27), the case (*ii*) is analogous (Eq. (2.6.30) is non-oscillatory if and only if the Prüfer angle ξ is bounded from above). $\square$

Now, we are ready to formulate and to prove the main result of this section.

Theorem 2.6.3. *Let* $[M(r_1)]^{p-1}M(s_1) = q^{-p}$.

(*i*) *If there exist* $R, S \in \mathbb{R}$ *such that*

$$\frac{1}{\alpha}\int_t^{t+\alpha} r_2(u)\,\mathrm{d}u \ge R, \qquad \frac{1}{\alpha}\int_t^{t+\alpha} s_2(u)\,\mathrm{d}u \ge S \tag{2.6.31}$$

for all sufficiently large t *and that*

$$[M(r_1)]^{p-1}S + \frac{p-1}{q^p}\,\frac{R}{M(r_1)} > \frac{q^{1-p}}{2}, \tag{2.6.32}$$

then Eq. (2.6.6) *is oscillatory.*

(*ii*) *If there exist* $R, S \in \mathbb{R}$ *such that*

$$\frac{1}{\alpha}\int_t^{t+\alpha} r_2(u)\,\mathrm{d}u \le R, \qquad \frac{1}{\alpha}\int_t^{t+\alpha} s_2(u)\,\mathrm{d}u \le S \tag{2.6.33}$$

for all sufficiently large t *and that*

$$[M(r_1)]^{p-1}S + \frac{p-1}{q^p}\,\frac{R}{M(r_1)} < \frac{q^{1-p}}{2}, \tag{2.6.34}$$

then Eq. (2.6.6) *is non-oscillatory.*

Proof. Let us consider the function ψ given by (2.6.20), where φ is an arbitrary solution of Eq. (2.6.19) on $\mathbb{R}_{\mathrm{e}}^{+}$. It holds

$$\begin{aligned}
\psi'(t) &= \frac{1}{\alpha}\left[\varphi(t+\alpha)-\varphi(t)\right] = \frac{1}{\alpha}\int\limits_t^{t+\alpha}\varphi'(u)\,\mathrm{d}u \\
&= \frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{1}{u\log u}\left[\left(r_1(u)+\frac{r_2(u)}{[\log(\log u)]^2}\right)|\cos_p\varphi(u)|^p\right. \\
&\quad \left. -\,\Phi\left(\cos_p\varphi(u)\right)\sin_p\varphi(u) + \left(s_1(u)+\frac{s_2(u)}{[\log(\log u)]^2}\right)\frac{|\sin_p\varphi(u)|^p}{p-1}\right]\mathrm{d}u \\
&= \frac{1}{\alpha}\left[\int\limits_t^{t+\alpha}\frac{1}{u\log u}\left(r_1(u)+\frac{r_2(u)}{[\log(\log u)]^2}\right)|\cos_p\varphi(u)|^p\,\mathrm{d}\right. \\
&\qquad -\int\limits_t^{t+\alpha}\frac{1}{u\log u}\Phi\left(\cos_p\varphi(u)\right)\sin_p\varphi(u)\,\mathrm{d}u \\
&\qquad \left. +\int\limits_t^{t+\alpha}\frac{1}{u\log u}\left(s_1(u)+\frac{s_2(u)}{[\log(\log u)]^2}\right)\frac{|\sin_p\varphi(u)|^p}{p-1}\,\mathrm{d}u\right]
\end{aligned} \tag{2.6.35}$$

for any $t\in\mathbb{R}_{\mathrm{e}}^{+}$. Let $\varepsilon\in\mathbb{R}^{+}$ be arbitrarily given.

We have

$$\left|\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{1}{u\log u}\left(r_1(u)+\frac{r_2(u)}{[\log(\log u)]^2}\right)|\cos_p\varphi(u)|^p\,\mathrm{d}u\right.$$
$$\left. -\frac{1}{\alpha\,t\log t}\int\limits_t^{t+\alpha}\left(r_1(u)+\frac{r_2(u)}{[\log(\log t)]^2}\right)\mathrm{d}u\,|\cos_p\psi(t)|^p\right|$$

$$
\begin{aligned}
&\leq \left| \frac{1}{\alpha} \int\limits_{t}^{t+\alpha} \frac{1}{u \log u} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \,\mathrm{d}u \right. \\
&\qquad\qquad \left. - \frac{1}{\alpha} \int\limits_{t}^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \,\mathrm{d}u \right| \\
&+ \left| \frac{1}{\alpha} \int\limits_{t}^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \,\mathrm{d}u \right. \\
&\qquad\qquad \left. - \frac{1}{\alpha} \int\limits_{t}^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \psi(t)|^p \,\mathrm{d}u \right| \\
&+ \left| \frac{1}{\alpha} \int\limits_{t}^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \psi(t)|^p \,\mathrm{d}u \right. \\
&\qquad\qquad \left. - \frac{1}{\alpha} \int\limits_{t}^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) |\cos_p \psi(t)|^p \,\mathrm{d}u \right|
\end{aligned}
$$

for all $t \in \mathbb{R}_{\mathrm{e}}^{+}$. Since

$$
\lim_{t\to\infty} t(t+\alpha)\log t \left(\frac{1}{t \log t} - \frac{1}{(t+\alpha)\log(t+\alpha)} \right) = \alpha, \tag{2.6.36}
$$

we obtain (see (2.6.8), (2.6.10), and (2.6.16))

$$
\begin{aligned}
&\left| \frac{1}{\alpha} \int\limits_{t}^{t+\alpha} \left[\frac{1}{u \log u} - \frac{1}{t \log t} \right] \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \,\mathrm{d}u \right| \\
&\quad \leq \frac{1}{\alpha} \int\limits_{t}^{t+\alpha} \left[\frac{1}{t \log t} - \frac{1}{(t+\alpha)\log(t+\alpha)} \right] \left(r_1(u) + \frac{|r_2(u)|}{[\log(\log t)]^2} \right) |\cos_p \varphi(u)|^p \,\mathrm{d}u \\
&\quad \leq 2 \int\limits_{t}^{t+\alpha} \frac{1}{t^2 \log t} \left(r_1^+ + \frac{|r_2(u)|}{[\log(\log t)]^2} \right) \mathrm{d}u < \frac{1}{t^{\frac{3}{2}}} \int\limits_{t}^{t+\alpha} \frac{1}{\sqrt{t \log t}} \left(r_1^+ + |r_2(u)| \right) \mathrm{d}u < \frac{1}{t^{\frac{3}{2}}}
\end{aligned}
$$

for all large t. Especially, we can assume that

$$
\begin{aligned}
&\left| \frac{1}{\alpha} \int\limits_{t}^{t+\alpha} \left[\frac{1}{u \log u} - \frac{1}{t \log t} \right] \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \,\mathrm{d}u \right| \\
&\qquad\qquad < \frac{\varepsilon}{t \log t \,[\log(\log t)]^2}.
\end{aligned} \tag{2.6.37}
$$

We recall that the half-linear trigonometric functions $\sin_p$ and $\cos_p$ are periodic and continuously differentiable. In particular, these facts imply the existence of a positive number L such that

$$||\sin_p x| - |\sin_p y|| \le L\,|x-y|\,, \qquad ||\cos_p x| - |\cos_p y|| \le L\,|x-y|\,, \tag{2.6.38}$$

and

$$||\sin_p x|^p - |\sin_p y|^p| \le L\,|x-y|\,, \qquad ||\cos_p x|^p - |\cos_p y|^p| \le L\,|x-y|\,, \tag{2.6.39}$$

for any $x, y \in \mathbb{R}$. Applying the second inequality in (2.6.39), we have (see (2.6.21) in Lemma 2.6.1 and again (2.6.8) and (2.6.10))

$$\begin{aligned}
&\left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{1}{t\log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) [|\cos_p \varphi(u)|^p - |\cos_p \psi(t)|^p] \,\mathrm{d}u \right| \\
&\qquad \le \frac{L}{\alpha} \int_t^{t+\alpha} \frac{1}{t\log t} \left(r_1^+ + \frac{|r_2(u)|}{[\log(\log t)]^2} \right) |\varphi(u) - \psi(t)| \,\mathrm{d}u \\
&\qquad\qquad \le \frac{L}{\alpha} \int_t^{t+\alpha} \frac{1}{t\log t} \left(r_1^+ + \frac{|r_2(u)|}{[\log(\log t)]^2} \right) \frac{1}{\sqrt{t\log t}} \,\mathrm{d}u \\
&\qquad\qquad\qquad < \frac{\varepsilon}{t\log t\,[\log(\log t)]^2}
\end{aligned} \tag{2.6.40}$$

for sufficiently large t.

Using

$$\lim_{t\to\infty} t\log t \left(\frac{1}{[\log(\log t)]^2} - \frac{1}{[\log(\log[t+\alpha])]^2} \right) = 0$$

and (2.6.8), we obtain the estimation

$$\begin{aligned}
&\left| \frac{1}{\alpha} \int_t^{t+\alpha} \frac{r_2(u)}{[\log(\log t)]^2} \,\mathrm{d}u - \frac{1}{\alpha} \int_t^{t+\alpha} \frac{r_2(u)}{[\log(\log u)]^2} \,\mathrm{d}u \right| \\
&\qquad \le \frac{1}{\alpha} \int_t^{t+\alpha} |r_2(u)| \left(\frac{1}{[\log(\log t)]^2} - \frac{1}{[\log(\log[t+\alpha])]^2} \right) \mathrm{d}u \\
&\qquad\qquad \le \frac{1}{t\log t} \int_t^{t+\alpha} |r_2(u)| \,\mathrm{d}u \le \frac{1}{\sqrt{t\log t}}
\end{aligned}$$

for every large t, which gives (consider also (2.6.16))

$$\begin{aligned}
&\Bigg| \frac{1}{\alpha} \int\limits_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \psi(t)|^p \,\mathrm{d}u \\
&\qquad - \frac{1}{\alpha} \int\limits_t^{t+\alpha} \frac{1}{t \log t} \left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) |\cos_p \psi(t)|^p \,\mathrm{d}u \Bigg| \\
&\le \frac{1}{\alpha t \log t} \int\limits_t^{t+\alpha} \left| \frac{r_2(u)}{[\log(\log u)]^2} - \frac{r_2(u)}{[\log(\log t)]^2} \right| |\cos_p \psi(t)|^p \,\mathrm{d}u \\
&\qquad \le \left(\frac{1}{t \log t} \right)^{\frac{3}{2}} < \frac{\varepsilon}{t \log t \,[\log(\log t)]^2}
\end{aligned} \tag{2.6.41}$$

for all large t.

Thus (see (2.6.37), (2.6.40), and (2.6.41)), we have

$$\begin{aligned}
&\Bigg| \frac{1}{\alpha} \int\limits_t^{t+\alpha} \frac{1}{u \log u} \left(r_1(u) + \frac{r_2(u)}{[\log(\log u)]^2} \right) |\cos_p \varphi(u)|^p \,\mathrm{d}u \\
&\qquad - \frac{1}{\alpha\, t \log t} \int\limits_t^{t+\alpha} \left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) \mathrm{d}u \, |\cos_p \psi(t)|^p \Bigg| \\
&\qquad < \frac{3\varepsilon}{t \log t \,[\log(\log t)]^2}
\end{aligned} \tag{2.6.42}$$

for all large t.

Analogously (cf. (2.6.8) and (2.6.9)), one can show that

$$\begin{aligned}
&\Bigg| \frac{1}{\alpha(p-1)} \int\limits_t^{t+\alpha} \frac{1}{u \log u} \left(s_1(u) + \frac{s_2(u)}{[\log(\log u)]^2} \right) |\sin_p \varphi(u)|^p \,\mathrm{d}u \\
&\qquad - \frac{1}{\alpha(p-1)\, t \log t} \int\limits_t^{t+\alpha} \left(s_1(u) + \frac{s_2(u)}{[\log(\log t)]^2} \right) \mathrm{d}u \, |\sin_p \psi(t)|^p \Bigg| \\
&\qquad < \frac{3\varepsilon}{t \log t \,[\log(\log t)]^2}
\end{aligned} \tag{2.6.43}$$

for all large t.

For large t, we have (see (2.6.16) and (2.6.36))

$$\begin{aligned}
&\left|\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{\Phi(\cos_p\varphi(u))\sin_p\varphi(u)}{t\log t}\,\mathrm{d}u-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{\Phi(\cos_p\varphi(u))\sin_p\varphi(u)}{u\log u}\,\mathrm{d}u\right|\\
&\qquad\le\frac{1}{\alpha}\int\limits_t^{t+\alpha}\left|\frac{1}{t\log t}-\frac{1}{u\log u}\right|\left|\Phi(\cos_p\varphi(u))\sin_p\varphi(u)\right|\mathrm{d}u\\
&\qquad\qquad\le\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{1}{t\log t}-\frac{1}{(t+\alpha)\log(t+\alpha)}\,\mathrm{d}u\\
&\qquad\qquad\qquad=\frac{1}{t\log t}-\frac{1}{(t+\alpha)\log(t+\alpha)}\le\frac{2\alpha}{t^2\log t}
\end{aligned}\tag{2.6.44}$$

and (see (2.6.16), (2.6.21) in Lemma 2.6.1, and (2.6.38))

$$\begin{aligned}
&\left|\Phi\left(\cos_p\psi(t)\right)\sin_p\psi(t)-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\Phi(\cos_p\varphi(u))\sin_p\varphi(u)\,\mathrm{d}u\right|\\
&\le\left|\Phi\left(\cos_p\psi(t)\right)\sin_p\psi(t)-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\Phi(\cos_p\psi(t))\sin_p\varphi(u)\,\mathrm{d}u\right|\\
&\qquad+\left|\frac{1}{\alpha}\int\limits_t^{t+\alpha}\Phi(\cos_p\psi(t))\sin_p\varphi(u)\,\mathrm{d}u-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\Phi(\cos_p\varphi(u))\sin_p\varphi(u)\,\mathrm{d}u\right|\\
&\le\frac{1}{\alpha}\int\limits_t^{t+\alpha}\left|\sin_p\psi(t)-\sin_p\varphi(u)\right|\mathrm{d}u+\frac{1}{\alpha}\int\limits_t^{t+\alpha}\left|\Phi(\cos_p\psi(t))-\Phi(\cos_p\varphi(u))\right|\mathrm{d}u\\
&\qquad\le\frac{L}{\alpha}\int\limits_t^{t+\alpha}\left|\psi(t)-\varphi(u)\right|\mathrm{d}u+\frac{K}{\alpha}\int\limits_t^{t+\alpha}\left|\psi(t)-\varphi(u)\right|^{\min\{p-1,1\}}\mathrm{d}u\\
&\qquad\qquad\le\left(\frac{1}{\sqrt{t}}\right)^{\min\{p-1,1\}}
\end{aligned}\tag{2.6.45}$$

for some $K\in\mathbb{R}^+$.

Hence (see (2.6.44) and (2.6.45)), it holds

$$
\begin{aligned}
&\left|\frac{1}{t\log t}\,\Phi(\cos_p\psi(t))\sin_p\psi(t)-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{1}{u\log u}\,\Phi(\cos_p\varphi(u))\sin_p\varphi(u)\,\mathrm{d}u\right|\\
&\leq\frac{1}{t\log t}\left|\Phi(\cos_p\psi(t))\sin_p\psi(t)-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\Phi(\cos_p\varphi(u))\sin_p\varphi(u)\,\mathrm{d}u\right|\\
&\qquad+\left|\frac{1}{\alpha t\log t}\int\limits_t^{t+\alpha}\Phi(\cos_p\varphi(u))\sin_p\varphi(u)\,\mathrm{d}u\right.\\
&\qquad\qquad\left.-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{1}{u\log u}\,\Phi(\cos_p\varphi(u))\sin_p\varphi(u)\,\mathrm{d}u\right|\\
&\qquad\qquad\leq\frac{1}{t\log t}\left(\frac{1}{\sqrt{t}}\right)^{\min\{p-1,1\}}+\frac{2\alpha}{t^2\log t}<\frac{\varepsilon}{t\log t\,[\log(\log t)]^2}
\end{aligned}
\tag{2.6.46}
$$

for all large t.

Finally (see (2.6.35), (2.6.42), (2.6.43), and (2.6.46)), we have

$$
\begin{aligned}
&\left|\psi'(t)-\frac{1}{\alpha}\int\limits_t^{t+\alpha}\frac{1}{t\log t}\left[\left(r_1(u)+\frac{r_2(u)}{[\log(\log t)]^2}\right)|\cos_p\psi(t)|^p\right.\right.\\
&\left.\left.-\Phi\left(\cos_p\psi(t)\right)\sin_p\psi(t)+\left(s_1(u)+\frac{s_2(u)}{[\log(\log t)]^2}\right)\frac{|\sin_p\psi(t)|^p}{p-1}\right]\mathrm{d}u\right|\\
&\qquad<\frac{7\varepsilon}{t\log t\,[\log(\log t)]^2}
\end{aligned}
\tag{2.6.47}
$$

for any sufficiently large t.

Part (i). Let $\vartheta\in\mathbb{R}^+$ be such that (see (2.6.32))

$$
[M(r_1)]^{p-1}(S-\vartheta)+\frac{p-1}{q^p}\,\frac{R-\vartheta}{M(r_1)}>\frac{q^{1-p}}{2}.
\tag{2.6.48}
$$

We consider $\varepsilon\in\mathbb{R}^+$ such that

$$
7\varepsilon<\vartheta,\qquad 7\varepsilon\,(p-1)<\vartheta.
\tag{2.6.49}
$$

For large t, we have (see (1.1.5), (2.6.31), (2.6.47), and (2.6.49))

$$\psi'(t) > \frac{1}{\alpha t \log t} \int\limits_t^{t+\alpha} \left[\left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) |\cos_p \psi(t)|^p \right.$$
$$- \Phi(\cos_p \psi(t)) \sin_p \psi(t) + \left(s_1(u) + \frac{s_2(u)}{[\log(\log t)]^2} \right) \times$$
$$\left. \times \frac{|\sin_p \psi(t)|^p}{p-1} \right] du - \frac{7\varepsilon}{t \log t \, [\log(\log t)]^2}$$
$$= \frac{1}{\alpha t \log t} \left[\int\limits_t^{t+\alpha} \left(r_1(u) + \frac{r_2(u) - 7\varepsilon}{[\log(\log t)]^2} \right) |\cos_p \psi(t)|^p \right.$$
$$- \Phi(\cos_p \psi(t)) \sin_p \psi(t)$$
$$\left. + \left(s_1(u) + \frac{s_2(u) - 7\varepsilon(p-1)}{[\log(\log t)]^2} \right) \frac{|\sin_p \psi(t)|^p}{p-1} \, du \right] \tag{2.6.50}$$
$$= \frac{1}{t \log t} \left[\left(M(r_1) + \frac{\frac{1}{\alpha} \int_t^{t+\alpha} r_2(u) \, du - 7\varepsilon}{[\log(\log t)]^2} \right) |\cos_p \psi(t)|^p \right.$$
$$- \Phi(\cos_p \psi(t)) \sin_p \psi(t)$$
$$\left. + \left(M(s_1) + \frac{\frac{1}{\alpha} \int_t^{t+\alpha} s_2(u) \, du - 7\varepsilon(p-1)}{[\log(\log t)]^2} \right) \frac{|\sin_p \psi(t)|^p}{p-1} \right]$$
$$> \frac{1}{t \log t} \left[\left(M(r_1) + \frac{R - \vartheta}{[\log(\log t)]^2} \right) |\cos_p \psi(t)|^p \right.$$
$$- \Phi(\cos_p \psi(t)) \sin_p \psi(t)$$
$$\left. + \left(M(s_1) + \frac{S - \vartheta}{[\log(\log t)]^2} \right) \frac{|\sin_p \psi(t)|^p}{p-1} \right].$$

It suffices to use Lemma 2.6.3, (i) (compare (2.6.25) with (2.6.48) and Eq. (2.6.26) with the form of (2.6.50) for $X = R - \vartheta$, $Y = S - \vartheta$). Since the Prüfer angle φ is unbounded from above (consider (2.6.21) in Lemma 2.6.1), Eq. (2.6.6) is oscillatory. Therefore, the first part of the theorem is proved.

Part (ii). We consider $\vartheta \in \mathbb{R}^+$ such that (see (2.6.34))

$$[M(r_1)]^{p-1}(S + \vartheta) + \frac{p-1}{q^p} \frac{R + \vartheta}{M(r_1)} < \frac{q^{1-p}}{2} \tag{2.6.51}$$

and $\varepsilon \in \mathbb{R}^+$ satisfying (2.6.49). We can proceed analogously as in the first case.

For large t, we have (see (1.1.5), (2.6.33), (2.6.47), and (2.6.49))

$$\begin{aligned}
\psi'(t) < \frac{1}{\alpha t \log t} \int_t^{t+\alpha} \Bigg[& \left(r_1(u) + \frac{r_2(u)}{[\log(\log t)]^2} \right) |\cos_p \psi(t)|^p \\
& - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \\
& + \left(s_1(u) + \frac{s_2(u)}{[\log(\log t)]^2} \right) \frac{|\sin_p \psi(t)|^p}{p-1} \Bigg] \mathrm{d}u \\
& + \frac{7\varepsilon}{t \log t \, [\log(\log t)]^2} \\
< \frac{1}{t \log t} \Bigg[& \left(M(r_1) + \frac{R + \vartheta}{[\log(\log t)]^2} \right) |\cos_p \psi(t)|^p \\
& - \Phi(\cos_p \psi(t)) \sin_p \psi(t) \\
& + \left(M(s_1) + \frac{S + \vartheta}{[\log(\log t)]^2} \right) \frac{|\sin_p \psi(t)|^p}{p-1} \Bigg].
\end{aligned} \tag{2.6.52}$$

Using Lemma 2.6.1 and Lemma 2.6.3, *(ii)* (cf. (2.6.27), (2.6.28) and (2.6.51), (2.6.52) for $V = R + \vartheta$, $W = S + \vartheta$), we know that the Prüfer angle is bounded from above, which implies the non-oscillation of Eq. (2.6.6). The proof is complete. □

§ 2.6.3 Corollaries and examples

In this paragraph, we illustrate the novelty of Theorem 2.6.3 on corollaries and examples which are not covered by any previously known criteria. As a corollary of Theorem 2.6.3, we obtain the following new result which solves the oscillation behavior of the non-perturbed equation in the critical case (cf. Theorem 2.5.3).

Corollary 2.6.1. *If $r : \mathbb{R} \to \mathbb{R}^+$ and $s : \mathbb{R} \to \mathbb{R}$ are continuous α-periodic functions such that*

$$[M(r)]^{p-1} M(s) = q^{-p},$$

then the equation

$$\left[r^{-\frac{p}{q}}(t) t^{p-1} \Phi(x') \right]' + \frac{s(t)}{t \log^p t} \Phi(x) = 0 \tag{2.6.53}$$

is non-oscillatory.

Proof. It suffices to consider $r_1(t) = r(t)$, $r_2(t) \equiv 0$, $s_1(t) = s(t)$, and $s_2(t) \equiv 0$ in Eq (2.6.6) and to put $R := 0$ and $S := 0$ in Theorem 2.6.3, *(ii)*. □

Example 2.6.1. We can apply Corollary 2.6.1, e.g., to the equation

$$\left[\left(\frac{2}{1 + 2\sin^2 t} \right)^{\frac{p}{q}} t^{p-1} \Phi(x') \right]' + \frac{q^{-p} + p \sin t - q \cos t}{t \log^p t} \Phi(x) = 0 \tag{2.6.54}$$

which is in the form of Eq. (2.6.53), where

$$M(r) = M\left(\frac{1+2\sin^2 t}{2}\right) = 1,$$
$$M(s) = M\left(q^{-p} + p\sin t - q\cos t\right) = q^{-p}.$$

Since $[M(r)]^{p-1} M(s) = q^{-p}$, Eq. (2.6.54) is in the critical case which means that it is non-oscillatory.

Now, we formulate a direct consequence of Theorems 2.5.3 and 2.6.3 and Corollary 2.6.1 for linear equations.

Corollary 2.6.2. *Consider the equations*

$$\left[\frac{tx'}{r_1(t)}\right]' + \frac{s_1(t)x}{t\log^2 t} = 0, \tag{2.6.55}$$

$$\left(\frac{[\log(\log t)]^2 tx'}{r_1(t)[\log(\log t)]^2 + r_2(t)}\right)' + \frac{1}{t\log^2 t}\left(s_1(t) + \frac{s_2(t)}{[\log(\log t)]^2}\right)x = 0 \tag{2.6.56}$$

with continuous α-periodic coefficients $r_1 : \mathbb{R} \to \mathbb{R}^+$, $s_1 : \mathbb{R} \to \mathbb{R}$ and with continuous coefficients $r_2, s_2 : \mathbb{R}_e^+ \to \mathbb{R}$ satisfying (2.6.7), (2.6.8), *and* (2.6.9).

(i) If $4M(r_1)M(s_1) > 1$, then Eq. (2.6.55) *is oscillatory.*

(ii) If $4M(r_1)M(s_1) \leq 1$, then Eq. (2.6.55) *is non-oscillatory.*

(iii) If $4M(r_1)M(s_1) = 1$ and if there exist $R, S \in \mathbb{R}$ satisfying

$$\frac{S}{M(s_1)} + \frac{R}{M(r_1)} > 1 \quad \text{and} \quad \frac{1}{\alpha}\int_t^{t+\alpha} r_2(u)\,\mathrm{d}u \geq R, \quad \frac{1}{\alpha}\int_t^{t+\alpha} s_2(u)\,\mathrm{d}u \geq S, \quad t \in \mathbb{R}_e^+,$$

then Eq. (2.6.56) *is oscillatory.*

(iv) If $4M(r_1)M(s_1) = 1$ and if there exist $R, S \in \mathbb{R}$ satisfying

$$\frac{S}{M(s_1)} + \frac{R}{M(r_1)} < 1 \quad \text{and} \quad \frac{1}{\alpha}\int_t^{t+\alpha} r_2(u)\,\mathrm{d}u \leq R, \quad \frac{1}{\alpha}\int_t^{t+\alpha} s_2(u)\,\mathrm{d}u \leq S, \quad t \in \mathbb{R}_e^+,$$

then Eq. (2.6.56) *is non-oscillatory.*

Example 2.6.2. Let $a \in \mathbb{R}_1^+$ and $b, c, d \in \mathbb{R}^+$. From Corollary 2.6.2, we know that the equation

$$\left[\frac{tx'}{a+\sin(cx)}\right]' + \frac{b+\cos(cx)}{t\log^2 t}x = 0$$

is oscillatory if and only if $4ab > 1$. Note that the case $4ab = 1$ is covered by Corollary 2.6.1 and the case $4ab \neq 1$ by Theorem 2.5.3. In addition, applying Corollary 2.6.2, we know that the equation

$$\left(\frac{tx'}{a+\sin(cx)}\right)' + \frac{1}{t\log^2 t}\left(\frac{1}{4a} + \cos(cx) + \frac{d+\sin(cx)\cos(cx)}{\left[\log\left(\log t\right)\right]^2}\right) x = 0$$

is oscillatory for $4ad > 1$ and non-oscillatory for $4ad < 1$.

To formulate the next corollary, we need the definitions of almost periodicity 2.4.1 and 2.4.2 and the definition of asymptotic almost periodicity 2.4.3. For more detail, we refer to books [12, 29].

From Definition 2.4.3, it is seen that (2.6.8) and (2.6.9) hold for all asymptotically almost periodic functions r_2, s_2. At the same time, (2.6.7) is valid for all large t if r_2 is asymptotically almost periodic. Therefore, we can use Theorem 2.6.3 for any equation of the form (2.6.6) with α-periodic coefficients r_1, s_1 and asymptotically almost periodic coefficients r_2, s_2. To be as clear as possible, we use in Corollary 2.6.3 and Example 2.6.3 below the fact, that any asymptotically almost periodic function has its mean value in the sense of the following definition. Note that Theorem 2.6.3 can be applied also for equations with coefficients which have mean values (see Definition 2.1.1) and which are not asymptotically almost periodic.

Corollary 2.6.3. *Let $R_1 : \mathbb{R} \to \mathbb{R}^+$, $S_1 : \mathbb{R} \to \mathbb{R}$ be continuous α-periodic functions such that*

$$\left[M\left(R_1\right)\right]^{p-1} M(S_1) = q^{-p}$$

and let $R_2, S_2 : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ be asymptotically almost periodic functions.

(i) If

$$[M(R_1)]^{p-1} M(S_2) + \frac{p-1}{q^p}\,\frac{M(R_2)}{M(R_1)} > \frac{q^{1-p}}{2},$$

then the equation

$$\begin{aligned}\left[\left(R_1(t) + \frac{R_2(t)}{[\log(\log t)]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]'& \\ + \frac{1}{t\log^p t}\left(S_1(t) + \frac{S_2(t)}{[\log(\log t)]^2}\right)\Phi(x) = 0&\end{aligned} \tag{2.6.57}$$

is oscillatory.

(ii) If

$$[M(R_1)]^{p-1} M(S_2) + \frac{p-1}{q^p}\,\frac{M(R_2)}{M(R_1)} < \frac{q^{1-p}}{2},$$

then Eq. (2.6.57) is non-oscillatory.

Proof. The corollary follows from Theorem 2.6.3 as well. It suffices to replace α by $n\alpha$ for a sufficiently large number $n \in \mathbb{N}$ and to use the definition of the mean value given in (2.1.6) and the existence of $\delta \in \mathbb{R}^+$ with the property that

$$[M(R_1)]^{p-1}\delta + \frac{p-1}{q^p}\frac{\delta}{M(R_1)} < \left|[M(R_1)]^{p-1}M(S_2) + \frac{p-1}{q^p}\frac{M(R_2)}{M(R_1)} - \frac{q^{1-p}}{2}\right|$$

if

$$[M(R_1)]^{p-1}M(S_2) + \frac{p-1}{q^p}\frac{M(R_2)}{M(R_1)} \neq \frac{q^{1-p}}{2}.$$

□

Example 2.6.3. Let $a, b, c \in \mathbb{R}$ and $u, v \in \mathbb{R} \setminus \{0\}$ determine the coefficients of the equation

$$\begin{aligned}\left[\left(\frac{3+\Phi(\sin t)}{3} + \frac{a+\sin(bt)+\sin(ct)}{[\log(\log[t+1])]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]' \\ + \frac{1}{t\log^p t}\left(\frac{2\sin^2 t}{q^p} + \left[\frac{\sin(ut)\cos(ut)+v+t^{-2}}{\log(\log t)}\right]^2\right)\Phi(x) = 0\end{aligned} \tag{2.6.58}$$

which has the form of Eq. (2.6.57) for

$$R_1(t) = \frac{3+\Phi(\sin t)}{3}, \qquad S_1(t) = \frac{2\sin^2 t}{q^p},$$

$$R_2(t) = [a+\sin(bt)+\sin(ct)]\left[\frac{\log(\log t)}{\log(\log[t+1])}\right]^2, \quad S_2(t) = \left[\sin(ut)\cos(ut)+v+\frac{1}{t^2}\right]^2.$$

One can verify that R_2, S_2 are asymptotically almost periodic functions and that

$$M(R_1) = 1, \qquad M(S_1) = q^{-p}, \qquad M(R_2) = a, \qquad M(S_2) = \frac{8v^2+1}{8}.$$

Especially, $[M(R_1)]^{p-1}M(S_1) = q^{-p}$. Hence, we can apply Corollary 2.6.3 which gives the oscillation of Eq. (2.6.58) for

$$\frac{8v^2+1}{8} + \frac{a(p-1)}{q^p} > \frac{q^{1-p}}{2}$$

and its non-oscillation for

$$\frac{8v^2+1}{8} + \frac{a(p-1)}{q^p} < \frac{q^{1-p}}{2}.$$

In the final corollary and example, we consider Eq. (2.6.6) with constant coefficients r_1, s_1 and periodic coefficients r_2, s_2, which do not need to have any common period. We point out that we get a new result even in the case when the periods of r_2, s_2 are same.

Corollary 2.6.4. *Let $a, b \in \mathbb{R}^+$ satisfy $a^{p-1}b = q^{-p}$. Let $R, S : \mathbb{R} \to \mathbb{R}$ be periodic continuous functions.*

(i) If

$$a^{p-1}M(S) + \frac{p-1}{aq^p}M(R) > \frac{q^{1-p}}{2},$$

then the equation

$$\left[\left(a + \frac{R(t)}{[\log(\log t)]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]' + \frac{1}{t\log^p t}\left(b + \frac{S(t)}{[\log(\log t)]^2}\right)\Phi(x) = 0 \tag{2.6.59}$$

is oscillatory.

(ii) If

$$a^{p-1}M(S) + \frac{p-1}{aq^p}M(R) < \frac{q^{1-p}}{2},$$

then Eq. (2.6.59) *is non-oscillatory.*

Proof. The corollary is a special case of Corollary 2.6.3. □

Example 2.6.4. We illustrate Corollary 2.6.4 by the equation

$$\left[\left(1 + \frac{c + d\sin t}{[\log(\log t)]^2}\right)^{-\frac{p}{q}} t^{p-1}\Phi(x')\right]' + \frac{1}{t\log^p t}\left(q^{-p} + \frac{C + D\cos\left(\sqrt{2}\,t\right)}{[\log(\log t)]^2}\right)\Phi(x) = 0, \tag{2.6.60}$$

where $c, d, C, D \in \mathbb{R}$ are arbitrary constants. For $a := 1$, $b := q^{-p}$, $R(t) := c + d\sin t$, and $S(t) := C + D\cos\left(\sqrt{2}\,t\right)$, we have $a^{p-1}b = q^{-p}$ and $M(R) = c$, $M(S) = C$. Hence, Eq. (2.6.60) is oscillatory for

$$C + \frac{p-1}{q^p}c > \frac{q^{1-p}}{2}, \qquad \text{i.e.,} \qquad Cq^p + (p-1)\,c > \frac{q}{2},$$

and non-oscillatory for

$$C + \frac{p-1}{q^p}c < \frac{q^{1-p}}{2}, \qquad \text{i.e.,} \qquad Cq^p + (p-1)\,c < \frac{q}{2}.$$

Chapter

3 Difference equations

3.1 Half-linear difference equations with coefficients having mean values

This chapter is devoted to oscillatory properties of the half-linear difference equations

$$\Delta[r_k\Phi(\Delta x_k)] + c_k\Phi(x_{k+1}) = 0, \tag{3.1.1}$$

where r_k is positive for all considered k. Throughout the whole chapter, we consider integers $k \geq a$ for a sufficiently large number $a \in \mathbb{N}$. For reader's convenience, we use the notation $\mathbb{N}_a := \{n \in \mathbb{N} : n \geq a\}$ for $a \in \mathbb{N}$. Our aim is to find explicit oscillation constants for all equations (3.1.1) from a large class of equations with

$$\{c_k\}_{k\in\mathbb{N}_a} \equiv \left\{\frac{\gamma s_k}{(k+1)^{(p)}}\right\}_{k\in\mathbb{N}_a},$$

i.e., in the form

$$\Delta[r_k\Phi(\Delta x_k)] + \frac{\gamma s_k}{(k+1)^{(p)}}\,\Phi(x_{k+1}) = 0, \tag{3.1.2}$$

where $\gamma \in \mathbb{R}$ and $k^{(p)}$ stands for the generalized power function (also called the falling factorial power) given by

$$k^{(p)} = \frac{\Gamma(k+1)}{\Gamma(k+1-p)}, \quad \Gamma(x) = \int_0^\infty \mathrm{e}^{-s}\, s^{x-1}\, \mathrm{d}s,\ x > 0.$$

For details about $k^{(p)}$, see, e.g., [51, Chapter 2]. The basic terminology is analogous to the one from the continuous case. Eq. (3.1.2) is said to be *conditionally oscillatory* if there exists a (positive) constant Γ such that Eq. (3.1.2) is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$. The constant Γ (which is dependent on coefficients) is called the *critical oscillation constant* of Eq. (3.1.2).

Concerning the conditional oscillation of the studied difference equations, the first result comes from [62], where the equation

$$\Delta^2 x_k + \frac{\gamma}{(k+1)k}\, x_{k+1} = 0$$

was proved to be conditionally oscillatory with the oscillation constant $\Gamma = 1/4$. Equations with non-constant coefficients were analyzed in [58]. In [36], the conditional oscillation of the linear equation

$$\Delta(r_k \Delta x_k) + \frac{\gamma s_k}{(k+1)k}\, x_{k+1} = 0 \tag{3.1.3}$$

with almost periodic coefficients was obtained. In [37], this result was generalized for half-linear equations of the form (3.1.2) with positive asymptotically almost periodic sequences $\{r_k\}$, $\{s_k\}$. From other results from the oscillation theory of difference equations, we mention at least papers [6, 13, 50, 63, 80].

Since the main result of [37] is one of the basic motivations for the research presented here, we reformulate it as follows. We remark that the symbol $M(\cdot)$ stands for the mean values of indicated sequences clarified in the below given Definition 3.1.2 and that the definition of asymptotic almost periodicity is mentioned in Definition 3.1.4 below.

Theorem 3.1.1. *Let $\gamma \in \mathbb{R}$ be given and let $\{r_k\}_{k\in\mathbb{N}_a}$ and $\{s_k\}_{k\in\mathbb{N}_a}$ be arbitrary positive asymptotically almost periodic sequences such that*

$$\inf\{r_k;\, k \in \mathbb{N}_a\} > 0, \quad \limsup_{k\to\infty} s_k > 0.$$

Let

$$\Gamma := \left(\frac{p-1}{p}\right)^p \left[M\left(\left\{r_k^{\frac{1}{1-p}}\right\}\right)\right]^{1-p} \left[M\left(\{s_k\}\right)\right]^{-1}.$$

If $\gamma > \Gamma$, then Eq. (3.1.2) *is oscillatory. If $\gamma < \Gamma$, then Eq.* (3.1.2) *is non-oscillatory.*

Our second basic motivation comes from the continuous case which is described in chapters 1 and 2. The most relevant result for the research contained in this chapter is Theorem 2.1.3 which can be reformulated as follows.

Theorem 3.1.2. *Let $r : \mathbb{R}_a \to \mathbb{R}$ be a continuous function, for which mean value $M\left(r^{\frac{1}{1-p}}\right)$ exists and for which it holds*

$$0 < \inf_{t\in\mathbb{R}_a} r(t) \le \sup_{t\in\mathbb{R}_a} r(t) < \infty,$$

and let $s : \mathbb{R}_a \to \mathbb{R}$ be a continuous function having mean value $M(s)$. Let

$$\Gamma := \left(\frac{p-1}{p}\right)^p \left[M\left(r^{\frac{1}{1-p}}\right)\right]^{1-p} = \left(\frac{p-1}{p}\right)^p \left[\lim_{t\to\infty} \frac{1}{t}\int_a^{a+t} r^{\frac{1}{1-p}}(\tau)\,\mathrm{d}\tau\right]^{1-p}.$$

Consider the equation

$$[r(t)\Phi(x')]' + \frac{s(t)}{t^p}\,\Phi(x) = 0. \tag{3.1.4}$$

Eq. (3.1.4) *is oscillatory if $M(s) > \Gamma$, and non-oscillatory if $M(s) < \Gamma$.*

In this chapter, we intend to generalize Theorem 3.1.1 into the case when the coefficients have mean values and the second coefficient can change sign. It means that our aim is to prove the discrete counterpart of Theorem 3.1.2. For this purpose, we improve the method from [37]. Since we study equations with coefficients from more general classes, we have to prove some new auxiliary results and inequalities (especially, we need Lemmata 3.1.1 and 3.1.2 below). Note that we partially apply the processes used in [37] (see the proof of Theorem 3.1.5 below, where it is explicitly mentioned).

The chapter is organized as follows. In the next paragraph, we state the necessary background and we recall the discrete Riccati technique, which is essential for our investigation. In § 3.1.2, the reader can find preparatory lemmata, results, and corollaries. In § 3.1.3, we collect illustrative examples.

§ 3.1.1 Preliminaries

In this paragraph, we mention the needed background concerning the oscillation theory of half-linear difference equations. For more details, we refer to [1, Chapter 3] and [21, Chapter 8] with references cited therein. In addition, we recall the concept of mean values which is necessary to find general oscillation constants. We also state the concept of the (adapted) discrete half-linear Riccati equation which is the main tool in our investigation.

At first, we recall the basic notions from the oscillation theory of the half-linear equation

$$\Delta[r_k\Phi(\Delta x_k)] + c_k\Phi(x_{k+1}) = 0, \tag{3.1.5}$$

where $r_k > 0$ for all considered $k \in \mathbb{N}_a$. An interval $(l, l+1]$, $l \in \mathbb{N}_a$, contains the *generalized zero* of a solution $\{x_k\}$ of Eq. (3.1.5) if $x_l \neq 0$ and $x_l x_{l+1} \leq 0$. We say that Eq. (3.1.5) is *disconjugate* on a set $\{l, l+1, \dots, l+n\}$ if any solution of Eq. (3.1.5) has at most one generalized zero on $(l, l+n+1]$ and a solution $\{\tilde{x}_k\}$ given by the initial value $\tilde{x}_l = 0$ has no generalized zero on $(l, l+n+1]$. Otherwise, Eq. (3.1.5) is called *conjugate* on $\{l, l+1, \dots, l+n\}$. Now, we can formulate the following definition.

Definition 3.1.1. Eq. (3.1.5) is called *non-oscillatory* if there exists $l \in \mathbb{N}$ with the property that Eq. (3.1.5) is disconjugate on $\{l, l+1, \dots, l+n\}$ for all $n \in \mathbb{N}$. In the opposite case, Eq. (3.1.5) is called *oscillatory.*

The Sturm type separation theorem (see, e.g., [1, Theorem 3.3.6]) enables us to give Definition 3.1.1, because the oscillation of an arbitrary non-zero solution of Eq. (3.1.5) implies the oscillation of all solutions of Eq. (3.1.5). We will also use a consequence of the Sturm type comparison theorem. We mention only the form that is prepared for our purpose.

Theorem 3.1.3. *Let $\{y_k\}_{k\in\mathbb{N}_a}, \{Y_k\}_{k\in\mathbb{N}_a}, \{z_k\}_{k\in\mathbb{N}_a}, \{Z_k\}_{k\in\mathbb{N}_a}$ be sequences satisfying the inequalities $y_k \geq Y_k > 0$, $Z_k \geq z_k$ for all sufficiently large k. Let us consider the equations*

$$\Delta[y_k\Phi(\Delta x_k)] + z_k\Phi(x_{k+1}) = 0, \tag{3.1.6}$$

$$\Delta[Y_k\Phi(\Delta x_k)] + Z_k\Phi(x_{k+1}) = 0. \tag{3.1.7}$$

If Eq. (3.1.7) *is non-oscillatory, then Eq.* (3.1.6) *is non-oscillatory as well.*

Proof. The theorem follows, e.g., from [1, Theorem 3.3.5]. □

To obtain explicit oscillation constants, we need the definition of the mean value of a sequence.

Definition 3.1.2. Let a sequence $\{f_k\}_{k\in\mathbb{N}_a} \subset \mathbb{R}$ be given and let the limit

$$M(\{f_k\}) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=i}^{n+i-1} f_k \tag{3.1.8}$$

be finite and exist uniformly with respect to $i \in \mathbb{N}_a$. The number $M(\{f_k\})$ introduced in (3.1.8) is called the *mean value* of $\{f_k\}$.

An important class of sequences having mean values is formed by asymptotically almost periodic sequences (see also [37]). Hence, we formulate the next definitions.

Definition 3.1.3. A sequence $\{f_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ is called *almost periodic* if, for any $\varepsilon > 0$, there exists $P(\varepsilon) \in \mathbb{N}$ such that any set of the form $\{i, i+1, \dots, i+P(\varepsilon)-1\} \subset \mathbb{Z}$ contains an integer l for which $|f_k - f_{k+l}| < \varepsilon$, $k \in \mathbb{Z}$.

Definition 3.1.4. We say that a sequence $\{f_k\}_{k\in\mathbb{N}_a} \subset \mathbb{R}$ is *asymptotically almost periodic* if there exists a pair of sequences $\{f_k^1\}_{k\in\mathbb{Z}}, \{f_k^2\}_{k\in\mathbb{N}_a} \subset \mathbb{R}$ such that $\{f_k^1\}$ is almost periodic, $\{f_k^2\}$ satisfies $\lim_{k\to\infty} f_k^2 = 0$, and $\{f_k\}_{k\in\mathbb{N}_a} \equiv \{f_k^1 + f_k^2\}_{k\in\mathbb{N}_a}$.

Finally, we describe the half-linear Riccati equation and its adapted version. Using the Riccati substitution

$$w_k = r_k \Phi\left(\frac{\Delta x_k}{x_k}\right)$$

to Eq. (3.1.5), we obtain the associated Riccati equation

$$\Delta w_k + c_k + w_k\left(1 - \frac{r_k}{\Phi[\Phi^{-1}(r_k) + \Phi^{-1}(w_k)]}\right) = 0, \tag{3.1.9}$$

where Φ^{-1} denotes the inverse function of Φ, i.e., $\Phi^{-1}(x) = |x|^{q-1}\operatorname{sgn} x$. Under the condition $w_k + r_k > 0$, we can express (see [1, Lemma 3.2.6, (I_8)])

$$w_k\left(1 - \frac{r_k}{\Phi[\Phi^{-1}(r_k) + \Phi^{-1}(w_k)]}\right) = \frac{(p-1)\,|w_k|^q\,|\beta_k|^{p-2}}{\Phi[\Phi^{-1}(r_k) + \Phi^{-1}(w_k)]},$$

where β_k is between $\Phi^{-1}(r_k)$ and $\Phi^{-1}(r_k) + \Phi^{-1}(w_k)$, i.e., for $w_k + r_k > 0$, we have the Riccati equation (3.1.9) associated to Eq. (3.1.5) in the form

$$\Delta w_k + c_k + \frac{(p-1)\,|w_k|^q\,|\beta_k|^{p-2}}{\Phi[\Phi^{-1}(r_k) + \Phi^{-1}(w_k)]} = 0. \tag{3.1.10}$$

The following theorem is typically known as the discrete Riccati method. It shows the way in which the non-oscillation of Eq. (3.1.5) is connected to the solvability of Eq. (3.1.10).

Theorem 3.1.4. *Eq.* (3.1.5) *is non-oscillatory if and only if there exist an integer b and a sequence of w_k which solves Eq.* (3.1.10) *and satisfies $w_k + r_k > 0$ for $k \in \mathbb{N}_b$.*

Proof. The theorem is a consequence of the well-known discrete Reid roundabout theorem (see, e.g., [1, Theorem 3.3.4] or directly [21, Theorem 8.2.5]). □

Taking into account the second substitution

$$\zeta_k = -k^{(p-1)} w_k \tag{3.1.11}$$

together with the Riccati equation (3.1.10), we obtain the adapted Riccati equation associated to Eq. (3.1.5) as

$$\begin{aligned}\Delta\zeta_k = \frac{1}{k-p+2}\Bigg[&(p-1)\zeta_k + (k+1)^{(p)} c_k \\ &+ \frac{(k+1)(p-1)\,|\beta_k|^{p-2}\,|\zeta_k|^q}{\left[k^{(p-1)}\right]^{q-1}\Phi\left[\Phi^{-1}(r_k) + \Phi^{-1}\left(-\frac{\zeta_k}{k^{(p-1)}}\right)\right]}\Bigg],\end{aligned} \tag{3.1.12}$$

where β_k is between $\Phi^{-1}(r_k)$ and $\Phi^{-1}(r_k) + \Phi^{-1}\left(-\frac{\zeta_k}{k^{(p-1)}}\right)$.

In fact, we will consider Eq. (3.1.5) in the form

$$\Delta\left[r_k \Phi(\Delta x_k)\right] + \frac{s_k}{(k+1)^{(p)}}\,\Phi(x_{k+1}) = 0, \tag{3.1.13}$$

where sequence $\{r_k\}_{k\in\mathbb{N}_a}$ has mean value $M(\{r_k^{1-q}\}) = 1$ and

$$0 < r^- := \inf_{k\in\mathbb{N}_a} r_k \le \sup_{k\in\mathbb{N}_a} r_k =: r^+ < \infty \tag{3.1.14}$$

and where sequence $\{s_k\}_{k\in\mathbb{N}_a}$ has a positive mean value, i.e., $M(\{s_k\}) > 0$. Therefore, we will deal with the Riccati equation associated to Eq. (3.1.13) in the form (see Eq. (3.1.10))

$$\Delta w_k + \frac{s_k}{(k+1)^{(p)}} + \frac{(p-1)\,|w_k|^q\,|\beta_k|^{p-2}}{\Phi\left[\Phi^{-1}(r_k) + \Phi^{-1}(w_k)\right]} = 0 \tag{3.1.15}$$

and with the adapted Riccati equation (see Eq. (3.1.12))

$$\Delta\zeta_k = \frac{1}{k-p+2}\left[(p-1)\zeta_k + s_k + \frac{(k+1)(p-1)\,|\beta_k|^{p-2}\,|\zeta_k|^q}{\left[k^{(p-1)}\right]^{q-1}\Phi\left[\Phi^{-1}(r_k) + \Phi^{-1}\left(-\frac{\zeta_k}{k^{(p-1)}}\right)\right]}\right]. \tag{3.1.16}$$

More precisely, we will study Eq. (3.1.13) using Eq. (3.1.16).

§ 3.1.2 Results

To prove the main results, we need the following lemmata.

Lemma 3.1.1. *Let a sequence* $\{f_k\}_{k\in\mathbb{N}_a} \subset \mathbb{R}$ *have mean value* $M(\{f_k\})$*. There exists a positive number* $K(\{f_k\})$ *for which* $|f_k| < K(\{f_k\})$*,* $k \in \mathbb{N}_a$*.*

Proof. The existence of $M(\{f_k\})$ gives $m \in \mathbb{N}$ such that

$$\left| M(\{f_k\}) - \frac{1}{m+l} \sum_{k=i}^{i+m+l-1} f_k \right| < 1, \quad i \in \mathbb{N}_a,\, l \in \mathbb{N} \cup \{0\}. \tag{3.1.17}$$

From (3.1.17) it follows

$$\left| \frac{1}{m} \sum_{k=i+1}^{i+m} f_k - \frac{1}{m} \sum_{k=i}^{i+m-1} f_k \right| < 2, \quad i \in \mathbb{N}_a,$$

$$\left| \frac{1}{m+1} \sum_{k=i+1}^{i+m+1} f_k - \frac{1}{m+1} \sum_{k=i}^{i+m} f_k \right| < 2, \quad i \in \mathbb{N}_a.$$

Especially,

$$|\, f_i - f_{m+i} \,| < 2m, \quad i \in \mathbb{N}_a,$$

$$|\, f_i - f_{m+i+1} \,| < 2(m+1), \quad i \in \mathbb{N}_a.$$

Thus, we have

$$|\, f_{m+i} - f_{m+i+1} \,| < 4m+2, \quad i \in \mathbb{N}_a.$$

Finally, it holds

$$|\, f_i - f_{i+1} \,| \le L(\{f_k\}) := \max\{4m+2, |f_a - f_{a+1}|, \dots, |f_{m+a-1} - f_{m+a}|\}, \quad i \in \mathbb{N}_a.$$

On contrary, let us suppose that $\limsup_{k\to\infty} |f_k| = \infty$. If

$$f_i \ge M(\{f_k\}) + 1 + (m-1)L(\{f_k\})$$

for some $i \in \mathbb{N}_a$, then

$$f_{i+j} \ge M(\{f_k\}) + 1, \quad j \in \{0, 1, \dots, m-1\}. \tag{3.1.18}$$

Analogously, if

$$f_i \le M(\{f_k\}) - 1 - (m-1)L(\{f_k\})$$

for some $i \in \mathbb{N}_a$, then

$$f_{i+j} \le M(\{f_k\}) - 1, \quad j \in \{0, 1, \dots, m-1\}. \tag{3.1.19}$$

Of course, each one of inequalities (3.1.18), (3.1.19) gives a contradiction with (3.1.17) for $l = 0$. It means that it suffices to put

$$K(\{f_k\}) := |M(\{f_k\})| + 1 + (m-1)L(\{f_k\})$$

for m from (3.1.17). □

Henceforth, let $m \in \mathbb{N}$ be such that

$$\frac{3M(\{s_k\})}{2} > \frac{1}{m+l} \sum_{k=i}^{i+m+l-1} s_k > \frac{M(\{s_k\})}{2} > 0, \quad i \in \mathbb{N}_a,\ l \in \mathbb{N} \cup \{0\}. \tag{3.1.20}$$

We also put (cf. (3.1.14))

$$s^+ := \sup_{k \in \mathbb{N}_a} |s_k| < \infty, \tag{3.1.21}$$

where we use Lemma 3.1.1.

Lemma 3.1.2. *If Eq.* (3.1.13) *is non-oscillatory, then there exist* $L \in \mathbb{N}$ *and a negative solution* $\{\zeta_k\}_{k \in \mathbb{N}_L}$ *of Eq.* (3.1.16) *such that*

$$\lim_{k \to \infty} \frac{\zeta_k}{k^{(p-1)}} = 0. \tag{3.1.22}$$

Proof. Considering Theorem 3.1.4, the non-oscillation of Eq. (3.1.13) implies that there exist $L \in \mathbb{N}$ and a solution $\{w_k\}_{k \in \mathbb{N}_L}$ of Eq. (3.1.15) such that $w_k + r_k > 0$ for $k \geq L$. Considering (3.1.11), it gives the solution $\{\zeta_k\}_{k \in \mathbb{N}_L} \equiv \{-w_k k^{(p-1)}\}_{k \in \mathbb{N}_L}$ of Eq. (3.1.16). We show that this solution $\{\zeta_k\}$ is negative and satisfies (3.1.22).

In fact, we show that the sequences $\{w_{mk}\}$, $\{w_{mk+1}\}$, $\ldots$, $\{w_{mk+m-1}\}$ are decreasing for sufficiently large k and tend to zero. Let $j \in \{0, 1, \ldots, m-1\}$ be arbitrarily given. From Eq. (3.1.15), we have

$$w_{(k+1)m+j} - w_{mk+j} = -\sum_{i=mk+j}^{(k+1)m+j-1} \frac{s_i}{(i+1)^{(p)}} - \sum_{i=mk+j}^{(k+1)m+j-1} \frac{(p-1)\,|w_i|^q\,|\beta_i|^{p-2}}{\Phi\left[\Phi^{-1}(r_i) + \Phi^{-1}(w_i)\right]} \tag{3.1.23}$$

for all considered k. Since $\{s_k\}$ is bounded (consider Lemma 3.1.1) and

$$\lim_{l \to \infty} \frac{(l+m)^{(p)}}{(l+1)^{(p)}} = 1,$$

using (3.1.20), we obtain

$$-\sum_{i=mk+j}^{(k+1)m+j-1} \frac{s_i}{(i+1)^{(p)}} < 0 \tag{3.1.24}$$

for all large k. Considering $w_k + r_k > 0$ for $k \geq L$, it is seen that

$$\Phi\left[\Phi^{-1}(r_k) + \Phi^{-1}(w_k)\right] > 0, \quad k \geq L. \tag{3.1.25}$$

From (3.1.23), (3.1.24), (3.1.25), we get that $w_{(k+1)m+j} < w_{mk+j}$ for all large k. Thus, there exist limits (as real numbers or $-\infty$)

$$L_0 := \lim_{k \to \infty} w_{mk},\ L_1 := \lim_{k \to \infty} w_{mk+1},\ \ldots,\ L_{m-1} := \lim_{k \to \infty} w_{mk+m-1}.$$

Our aim is to prove that $L_j = 0$ for each $j \in \{0, 1, \dots, m-1\}$. On contrary, let us consider that $L_j \neq 0$ for some j. Denote $W := \max_{k \in \mathbb{N}_L} w_k$.

Let $L_j > \varepsilon > 0$. We know that

$$\Phi^{-1}(W) + \Phi^{-1}(r^+) \geq \beta_{km+j} \geq \Phi^{-1}(r_{km+j}) \geq \Phi^{-1}(r^-) > 0 \tag{3.1.26}$$

and that

$$\Phi\left[\Phi^{-1}\left(r_{km+j}\right) + \Phi^{-1}(w_{mk+j})\right] < \Phi\left[\Phi^{-1}\left(r^+\right) + \Phi^{-1}(W)\right] \tag{3.1.27}$$

for all k. Hence (see (3.1.26)), there exists $B_j > 0$ with the property that

$$|\beta_{mk+j}|^{p-2} > B_j, \quad k \geq L. \tag{3.1.28}$$

In addition, for large k, we have (see (3.1.23), (3.1.24), (3.1.25), (3.1.27), and (3.1.28))

$$\begin{aligned} w_{(m+1)k+j} - w_{mk+j} &< -\sum_{i=mk+j}^{(k+1)m+j-1} \frac{(p-1)\,|w_i|^q\,|\beta_i|^{p-2}}{\Phi\left[\Phi^{-1}(r_i) + \Phi^{-1}(w_i)\right]} \\ &< -\frac{(p-1)\,|w_{mk+j}|^q\,|\beta_{mk+j}|^{p-2}}{\Phi\left[\Phi^{-1}(r_{mk+j}) + \Phi^{-1}(w_{mk+j})\right]} < -\frac{(p-1)\varepsilon^q B_j}{\Phi\left[\Phi^{-1}(r^+) + \Phi^{-1}(W)\right]}. \end{aligned}$$

But, we obtain the contradiction $L_j = -\infty$, because the last term is a negative constant.

Let $L_j < -\varepsilon < 0$, i.e., $w_{mk+j} < -\varepsilon$ for large k. In this case, for large k, we have

$$\begin{aligned} w_{(m+1)k+j} - w_{mk+j} &< -\sum_{i=mk+j}^{(k+1)m+j-1} \frac{(p-1)\,|w_i|^q\,|\beta_i|^{p-2}}{\Phi\left[\Phi^{-1}(r_i) + \Phi^{-1}(w_i)\right]} \\ &< -\frac{(p-1)\,|w_{mk+j}|^q\,|\beta_{mk+j}|^{p-2}}{\Phi\left[\Phi^{-1}(r_{mk+j}) + \Phi^{-1}(w_{mk+j})\right]} \\ &< -\frac{(p-1)\,|\varepsilon|^q\left[\Phi^{-1}(r_{mk+j}) + \Phi^{-1}(w_{mk+j})\right]^{p-2}}{\Phi\left[\Phi^{-1}(r_{mk+j}) + \Phi^{-1}(w_{mk+j})\right]} \\ &= -\frac{(p-1)\,|\varepsilon|^q}{\Phi^{-1}(r_{mk+j}) + \Phi^{-1}(w_{mk+j})} < -\frac{(p-1)\varepsilon^q}{\Phi^{-1}(r^+)} \end{aligned}$$

if $p \geq 2$; and

$$\begin{aligned} w_{(m+1)k+j} - w_{mk+j} &< -\frac{(p-1)\,|w_{mk+j}|^q\,|\beta_{mk+j}|^{p-2}}{\Phi\left[\Phi^{-1}(r_{mk+j}) + \Phi^{-1}(w_{mk+j})\right]} \\ &< -\frac{(p-1)\,|\varepsilon|^q\left[\Phi^{-1}(r_{mk+j})\right]^{p-2}}{\Phi\left[\Phi^{-1}(r_{mk+j})\right]} < -\frac{(p-1)\varepsilon^q}{\Phi^{-1}(r^+)} \end{aligned}$$

if $p \in (1, 2)$. Again, for any $p > 1$, we get $L_j = -\infty$ which cannot be true, because $w_k + r_k > 0$ for all k and $\{r_k\}$ is bounded.

Altogether, we know that $\{w_k\}$ is positive and

$$\lim_{k \to \infty} w_k = 0. \tag{3.1.29}$$

Thus, $\{\zeta_k\}$ is negative and (3.1.22) follows from (3.1.29). □

We remark that, in the case when the sequence of s_k is positive, the statement of Lemma 3.1.2 follows from [64, Lemma 1, (v) and Theorem 1] combined with [1, Lemma 3.5.9] or with [21, Lemma 8.2.2].

Lemma 3.1.3. *If there exists a negative solution* $\{\zeta_k\}_{k\in\mathbb{N}_L}$ *of Eq.* (3.1.16)*, then Eq.* (3.1.13) *is non-oscillatory.*

Proof. A negative solution $\{\zeta_k\}_{k\in\mathbb{N}_L}$ of Eq. (3.1.16) gives $\{w_k\}_{k\in\mathbb{N}_L} \equiv \{-\zeta_k/k^{(p-1)}\}_{k\in\mathbb{N}_L}$ which is a positive solution of Eq. (3.1.15). Thus, the lemma follows from Theorem 3.1.4. □

Applying the above lemmata, we can obtain the announced result. For readers' convenience, we recall the assumptions on coefficients.

Theorem 3.1.5. *Let sequence* $\{r_k\}_{k\in\mathbb{N}_a}$ *have mean value* $M(\{r_k^{1-q}\}) = 1$ *and satisfy* (3.1.14) *and let sequence* $\{s_k\}_{k\in\mathbb{N}_a}$ *have mean value* $M(\{s_k\}) > 0$*. Then, Eq.* (3.1.13) *is oscillatory for* $M(\{s_k\}) > q^{-p}$ *and non-oscillatory for* $M(\{s_k\}) < q^{-p}$*.*

Proof. At first, let us approach the oscillatory part of the theorem. By contradiction, we suppose that $M(\{s_k\}) > q^{-p}$ and that Eq. (3.1.13) is non-oscillatory. Using Lemma 3.1.2, we obtain the existence of a negative solution $\{\zeta_k\}_{k\in\mathbb{N}_L}$ of Eq. (3.1.16), i.e., it holds

$$\begin{aligned}\Delta\zeta_k = \frac{1}{k-p+2}\Bigg[&(p-1)\zeta_k + s_k \\ &+ \frac{(k+1)(p-1)\beta_k^{p-2}|\zeta_k|^q}{\left[k^{(p-1)}\right]^{q-1}\Phi\left[\Phi^{-1}(r_k)+\Phi^{-1}\left(-\frac{\zeta_k}{k^{(p-1)}}\right)\right]}\Bigg], \quad k \geq L,\end{aligned} \tag{3.1.30}$$

where

$$0 < \Phi^{-1}(r_k) \leq \beta_k \leq \Phi^{-1}(r_k) + \Phi^{-1}\left(-\frac{\zeta_k}{k^{(p-1)}}\right), \quad k \geq L. \tag{3.1.31}$$

From Lemma 3.1.2 (see (3.1.14), (3.1.22), and (3.1.31)), we also obtain

$$1 \leq \liminf_{k\to\infty} \frac{\beta_k}{\Phi^{-1}(r_k)} \leq \limsup_{k\to\infty} \frac{\beta_k}{\Phi^{-1}(r_k)} \leq \lim_{k\to\infty} \frac{\Phi^{-1}(r_k)+\Phi^{-1}\left(-\frac{\zeta_k}{k^{(p-1)}}\right)}{\Phi^{-1}(r_k)} = 1. \tag{3.1.32}$$

From (3.1.32) it follows

$$\lim_{k\to\infty} \frac{\beta_k^{p-2} r_k^{q-1}}{\Phi\left[\Phi^{-1}(r_k)+\Phi^{-1}\left(-\frac{\zeta_k}{k^{(p-1)}}\right)\right]} = \lim_{k\to\infty} \frac{\left[r_k^{q-1}\right]^{p-2} r_k^{q-1}}{r_k} = \lim_{k\to\infty} \frac{r_k^{2-q} r_k^{q-1}}{r_k} = 1. \tag{3.1.33}$$

It is well-known that

$$\lim_{k\to\infty} \frac{k+1}{\left[k^{(p-1)}\right]^{q-1}} = \lim_{k\to\infty} \frac{k}{\left[k^{(p-1)}\right]^{q-1}} = \lim_{k\to\infty} \frac{k}{\left[k^{p-1}\right]^{q-1}} = 1. \tag{3.1.34}$$

Thus (see (3.1.14), (3.1.33), and (3.1.34)), we can assume that $L > p-2$ is so large that

$$\begin{aligned} \frac{1}{\left(\sqrt{2}r^+\right)^{q-1}} \le \frac{1}{\left(\sqrt{2}r_k\right)^{q-1}} &\le \frac{\beta_k^{p-2}}{\Phi\left[\Phi^{-1}(r_k) + \Phi^{-1}\left(-\frac{\zeta_k}{k^{(p-1)}}\right)\right]} \\ &\le \left(\frac{\sqrt{2}}{r_k}\right)^{q-1} \le \left(\frac{\sqrt{2}}{r^-}\right)^{q-1}, \quad k \ge L, \end{aligned} \tag{3.1.35}$$

and that

$$\left(\frac{1}{\sqrt{2}}\right)^{q-1} \le \frac{k+1}{\left[k^{(p-1)}\right]^{q-1}} \le 2^{\frac{q-1}{2}}, \quad k \ge L. \tag{3.1.36}$$

Combining (3.1.35) and (3.1.36), we obtain

$$\frac{(p-1)|\zeta_k|^q}{[2r^+]^{q-1}} \le \frac{(k+1)(p-1)\beta_k^{p-2}|\zeta_k|^q}{\left[k^{(p-1)}\right]^{q-1}\Phi\left[\Phi^{-1}(r_k) + \Phi^{-1}\left(-\frac{\zeta_k}{k^{(p-1)}}\right)\right]} \le \frac{2^{q-1}(p-1)|\zeta_k|^q}{[r^-]^{q-1}} \tag{3.1.37}$$

for all $k \ge L$.

Considering (3.1.30) and (3.1.37), we have (see also (3.1.21))

$$\Delta\zeta_k \ge \frac{1}{k-p+2}\left[(p-1)\zeta_k - s^+ + \frac{(p-1)|\zeta_k|^q}{[2r^+]^{q-1}}\right], \quad k \ge L, \tag{3.1.38}$$

and

$$\Delta\zeta_k \le \frac{1}{k-p+2}\left[s^+ + \frac{2^{q-1}(p-1)|\zeta_k|^q}{[r^-]^{q-1}}\right], \quad k \ge L. \tag{3.1.39}$$

If

$$\zeta_i < X_1 := -2r^+\left[2 + \frac{s^+}{p-1}\right]^{\frac{1}{q-1}} - 1 \tag{3.1.40}$$

for some $i \ge L$, then

$$\begin{aligned} \zeta_i - \frac{s^+}{p-1} + \frac{|\zeta_i|^q}{[2r^+]^{q-1}} &> |\zeta_i|\left(-1 + \left[\frac{|\zeta_i|}{2r^+}\right]^{q-1} - \frac{s^+}{p-1}\right) \\ &> |\zeta_i|\left(-1+2\right) > -X_1 > 0. \end{aligned} \tag{3.1.41}$$

Thus, in this case, it holds $\zeta_{i+1} > \zeta_i$. Indeed, (3.1.38) and (3.1.41) give

$$\Delta\zeta_i \ge \frac{p-1}{i-p+2}\left[\zeta_i - \frac{s^+}{p-1} + \frac{|\zeta_i|^q}{[2r^+]^{q-1}}\right] > \frac{p-1}{i-p+2}\,|X_1| > 0 \tag{3.1.42}$$

if (3.1.40) is valid. Let us consider the smallest integer $l_0 \ge L$ such that $\zeta_{l_0} \ge X_1$. Note that such a number l_0 has to exist because its existence follows from (3.1.42) and from

$$\sum_{i=L}^{\infty} \frac{p-1}{i-p+2}\,|X_1| \ge (p-1)\sum_{j=L-p+2}^{\infty}\frac{1}{j} = \infty.$$

Using (3.1.38), we have

$$\zeta_{l_0+1} \geq X_2 := X_1 + \frac{1}{l_0 - p + 2}\left[(p-1)X_1 - s^+\right].$$

Analogously, one can get that $\zeta_{j+1} \geq X_2$ if $\zeta_j \geq X_1$ for some $j \geq l_0$. Hence, there exists $N > 0$ satisfying

$$\zeta_k \in (-N, 0), \quad k \geq L. \tag{3.1.43}$$

In fact, it suffices to put

$$N := \min\{\zeta_L, \zeta_{L+1}, \dots, \zeta_{l_0}, X_2\}.$$

Trivially, from (3.1.38) and (3.1.39) (or directly from (3.1.30)), it is seen that

$$|\Delta\zeta_k| \leq \frac{1}{k-p+2}\left[(p-1)N + s^+ + \frac{2^{q-1}(p-1)N^q}{[r^-]^{q-1}}\right], \quad k \geq L. \tag{3.1.44}$$

Therefore, there exists $P > 0$ for which

$$|\Delta\zeta_k| < \frac{P}{k}, \quad k \geq L. \tag{3.1.45}$$

Especially, (3.1.45) gives $Q > 0$ such that

$$|\zeta_{k+i} - \zeta_{k+j}| < \frac{Q}{k}, \quad i, j \in \{0, \dots, m-1\}, \, k \geq L. \tag{3.1.46}$$

Indeed (consider (3.1.44)), inequalities (3.1.45) and (3.1.46) are valid for

$$\begin{aligned} P &:= \sup_{k \in \mathbb{N}_L} \frac{k}{k-p+2}\left[(p-1)N + s^+ + \frac{2^{q-1}(p-1)N^q}{[r^-]^{q-1}}\right] \\ &= \max\left\{1, \frac{L}{L-p+2}\right\}\left[(p-1)N + s^+ + \frac{2^{q-1}(p-1)N^q}{[r^-]^{q-1}}\right] \end{aligned}$$

and $Q := (m-1)\,P$.

In addition (see Definition 3.1.2), we can assume that $m \in \mathbb{N}$ from (3.1.20) is so large that

$$\frac{1}{m+j}\sum_{k=i}^{i+m+j-1} s_k > q^{-p}\left(\frac{1}{m+l}\sum_{k=i}^{i+m+l-1} r_k^{1-q}\right)^{1-p}, \quad i \in \mathbb{N}_a, \, j, l \in \mathbb{N} \cup \{0\}. \tag{3.1.47}$$

To obtain the oscillatory part of the theorem, one can proceed as in the proof of [37, Theorem 4.1], where only (3.1.20), (3.1.33), (3.1.34), (3.1.43), (3.1.46), and (3.1.47) are used to get a contradiction with the existence of the negative solution $\{\zeta_k\}_{k \in \mathbb{N}_L}$ (in fact, these inequalities are used only in special forms therein).

In the non-oscillatory part of the proof, we consider that $m \in \mathbb{N}$ satisfies

$$\frac{1}{m+j}\sum_{k=i}^{i+m+j-1} s_k < q^{-p}\left(\frac{1}{m+l}\sum_{k=i}^{i+m+l-1} r_k^{1-q}\right)^{1-p}, \quad i \in \mathbb{N}_a,\ j,l \in \mathbb{N}\cup\{0\}. \tag{3.1.48}$$

Let $\{\zeta_k\}_{k\in\mathbb{N}_L}$ be the solution of the Cauchy problem given by Eq. (3.1.16) and

$$\zeta_L = -\left(\frac{q}{m}\sum_{i=L}^{L+m-1} r_i^{1-q}\right)^{1-p},$$

where $L \in \mathbb{N}$ is sufficiently large. As in the first part of the proof, we can obtain (3.1.38) and (3.1.39) if ζ_k is negative. Thus, we can assume that L is so large that

$$2\zeta_L < \zeta_{L+i} < 0, \quad i \in \{1,\dots,m\}. \tag{3.1.49}$$

In addition (compare (3.1.49) with (3.1.43)), as in the first part, one can estimate

$$|\Delta\zeta_{L+i}| < \frac{\widetilde{P}}{L}, \quad i \in \{0,1,\dots,m-1\}, \tag{3.1.50}$$

$$|\zeta_{L+i} - \zeta_{L+j}| < \frac{\widetilde{Q}}{L}, \quad i,j \in \{0,1,\dots,m\}, \tag{3.1.51}$$

for some $\widetilde{P},\widetilde{Q} > 0$. Now, the process from the proof of [37, Theorem 4.1] gives that ζ_k is negative for all $k \geq L$, because only (3.1.20), (3.1.33), (3.1.34), (3.1.48), (3.1.50), and (3.1.51) are used therein. Hence, to complete the proof, it suffices to apply Lemma 3.1.3. □

We slightly improve Theorem 3.1.5 into the following form (more common in the literature). In particular, we remove the requirement on sequence $\{s_k\}$ that it has a positive mean value.

Theorem 3.1.6. *Let us consider the equation*

$$\Delta\left[\tilde{r}_k\Phi(\Delta x_k)\right] + \frac{\tilde{s}_k}{(k+1)^{(p)}}\,\Phi(x_{k+1}) = 0, \tag{3.1.52}$$

where the coefficients $\{\tilde{r}_k\}_{k\in\mathbb{N}_a}$, $\{\tilde{s}_k\}_{k\in\mathbb{N}_a}$ be such that the mean values of sequences $\left\{\tilde{r}_k^{1-q}\right\}$, $\{\tilde{s}_k\}$ exist and $\{\tilde{r}_k\}$ is bounded and positive. Let us denote

$$\Gamma := q^{-p}\left[M\left(\{\tilde{r}_k^{1-q}\}\right)\right]^{1-p}. \tag{3.1.53}$$

Eq. (3.1.52) is oscillatory if $M(\{\tilde{s}_k\}) > \Gamma$. Eq. (3.1.52) is non-oscillatory if $M(\{\tilde{s}_k\}) < \Gamma$.

Proof. Considering Lemma 3.1.1 for $\left\{\tilde{r}_k^{1-q}\right\}$ and the boundedness of $\{\tilde{r}_k\}$, we know that

$$0 < \inf_{k\in\mathbb{N}_a} \tilde{r}_k^{1-q} \le \sup_{k\in\mathbb{N}_a} \tilde{r}_k^{1-q} < \infty, \quad \text{i.e.,} \quad \infty > \sup_{k\in\mathbb{N}_a} \tilde{r}_k \ge \inf_{k\in\mathbb{N}_a} \tilde{r}_k > 0. \tag{3.1.54}$$

We use Theorem 3.1.5. Therefore, we assume that $M(\{\tilde{s}_k\}) > 0$.

We divide Eq. (3.1.52) by the constant value $\left[M\left(\left\{\tilde{r}_k^{1-q}\right\}\right)\right]^{1-p} > 0$ (see (3.1.54)). We obtain the equation

$$\Delta\left[\frac{\tilde{r}_k\Phi(\Delta x_k)}{\left[M\left(\left\{\tilde{r}_k^{1-q}\right\}\right)\right]^{1-p}}\right] + \frac{\tilde{s}_k\Phi(x_{k+1})}{(k+1)^{(p)}\left[M\left(\left\{\tilde{r}_k^{1-q}\right\}\right)\right]^{1-p}} = 0 \tag{3.1.55}$$

which has the form of Eq. (3.1.13) with the coefficients

$$r_k = \frac{\tilde{r}_k}{\left[M\left(\left\{\tilde{r}_k^{1-q}\right\}\right)\right]^{1-p}}, \quad s_k = \frac{\tilde{s}_k}{\left[M\left(\left\{\tilde{r}_k^{1-q}\right\}\right)\right]^{1-p}}.$$

Especially, we have

$$\begin{aligned} M\left(\left\{r_k^{1-q}\right\}\right) &= M\left(\left\{\frac{\tilde{r}_k^{1-q}}{\left[M\left(\left\{\tilde{r}_k^{1-q}\right\}\right)\right]^{(1-p)(1-q)}}\right\}\right) \\ &= \left[M\left(\left\{\tilde{r}_k^{1-q}\right\}\right)\right]^{-1} M\left(\left\{\tilde{r}_k^{1-q}\right\}\right) = 1, \end{aligned}$$

where the identity

$$(1-p)(1-q) = 1 \tag{3.1.56}$$

is used.

According to Theorem 3.1.5, Eq. (3.1.55) is oscillatory if

$$M(\{s_k\}) = M\left(\left\{\frac{\tilde{s}_k}{\left[M(\{\tilde{r}_k^{1-q}\})\right]^{1-p}}\right\}\right) = \left[M\left(\left\{\tilde{r}_k^{1-q}\right\}\right)\right]^{p-1} M\left(\{\tilde{s}_k\}\right) > q^{-p},$$

and non-oscillatory if

$$M(\{s_k\}) = \left[M\left(\left\{\tilde{r}_k^{1-q}\right\}\right)\right]^{p-1} M(\{\tilde{s}_k\}) < q^{-p}.$$

This fact implies the statement of Theorem 3.1.6 for any positive mean value of $\{\tilde{s}_k\}$.

Now, let $M(\{\tilde{s}_k\}) \le 0$. Then, there exists a positive constant C such that

$$0 < M(\{\tilde{s}_k\}) + C = M(\{\tilde{s}_k + C\}) < \Gamma.$$

We consider the non-oscillatory equation

$$\Delta\left[\tilde{r}_k\Phi(\Delta x_k)\right] + \frac{\tilde{s}_k + C}{(k+1)^{(p)}}\,\Phi(x_{k+1}) = 0,$$

which is a majorant equation of Eq. (3.1.52). Thus, the proof can be completed by the application of Theorem 3.1.3. $\square$

Since the presented results are new also for linear difference equations (the case when $p = q = 2$), we mention the following direct corollary of Theorem 3.1.6.

Corollary 3.1.1. *Let us consider the equation*

$$\Delta\left[r_k \Delta x_k\right] + \frac{s_k x_{k+1}}{(k+1)k} = 0, \tag{3.1.57}$$

where the sequences $\{r_k\}_{k\in\mathbb{N}_a}$ *and* $\{s_k\}_{k\in\mathbb{N}_a}$ *have the properties that* $M(\{r_k^{-1}\})$ *and* $M(\{s_k\})$ *exist and* $\{r_k\}$ *is bounded and positive. Then, Eq.* (3.1.57) *is oscillatory for*

$$M\left(\left\{r_k^{-1}\right\}\right) M(\{s_k\}) > \frac{1}{4}$$

and non-oscillatory for

$$M\left(\left\{r_k^{-1}\right\}\right) M(\{s_k\}) < \frac{1}{4}. \tag{3.1.58}$$

Based on results of [75] (see also [77, 76]), the conjecture is given in paper [37] that the border case $M\left(\left\{r_k^{-1}\right\}\right) M(\{s_k\}) = 1/4$ from Corollary 3.1.1 is not solvable for general coefficients, i.e., in the border case, there exist oscillatory equations in the form of Eq. (3.1.57) and, at the same time, there exist non-oscillatory equations in this form. (The situation is similar to the behavior of the continuous equations.)

In addition, using the Sturm type comparison theorem, we get the next new result concerning non-oscillatory half-linear difference equations when the coefficient in the difference term does not need to be bounded.

Theorem 3.1.7. *Let us consider Eq.* (3.1.52) *and* Γ *introduced in* (3.1.53). *Let the coefficients* $\{\tilde{r}_k\}_{k\in\mathbb{N}_a}$, $\{\tilde{s}_k\}_{k\in\mathbb{N}_a}$ *be such that the mean values of sequences* $\left\{\tilde{r}_k^{1-q}\right\}$, $\{\tilde{s}_k\}$ *exist and* $\{\tilde{r}_k\}$ *is positive. Then, Eq.* (3.1.52) *is non-oscillatory if* $M(\{\tilde{s}_k\}) < \Gamma$.

Proof. The inequality $M(\{\tilde{s}_k\}) < \Gamma$ can be trivially rewritten into the form

$$M(\{\tilde{s}_k\}) < q^{-p}\left[M\left(\{\tilde{r}_k^{1-q}\}\right)\right]^{1-p} - \delta$$

for some $\delta > 0$. In particular, there exists $\vartheta > 0$ for which

$$M(\{\tilde{s}_k\}) < q^{-p}\left[M\left(\{\tilde{r}_k^{1-q}\}\right) + \vartheta\right]^{1-p}. \tag{3.1.59}$$

From Definition 3.1.2 and (3.1.56), it is seen that the sequence $\{\tilde{R}_k\}_{k\in\mathbb{N}_a}$ given by

$$\tilde{R}_k := \left(\tilde{r}_k^{1-q} + \vartheta\right)^{1-p}, \quad k \in \mathbb{N}_a,$$

satisfies

$$M\left(\left\{\tilde{R}_k^{1-q}\right\}\right) = M\left(\{\tilde{r}_k^{1-q} + \vartheta\}\right) = M\left(\{\tilde{r}_k^{1-q}\}\right) + \vartheta > 0. \tag{3.1.60}$$

In addition, sequence $\{\tilde{R}_k\}$ is bounded. Thus, we can apply Theorem 3.1.6 which guarantees that the equation

$$\Delta\left[\tilde{R}_k \Phi(\Delta x_k)\right] + \frac{\tilde{s}_k}{(k+1)^{(p)}}\,\Phi(x_{k+1}) = 0 \tag{3.1.61}$$

is non-oscillatory (see (3.1.59) and (3.1.60)). Of course, Eq. (3.1.61) is a majorant of Eq. (3.1.52) because $\tilde{R}_k \le \tilde{r}_k$ for all considered k (see again (3.1.56)). Finally, it suffices to use Theorem 3.1.3. □

Again, from the theorem above, we obtain a new result in the linear case. The linear version of Theorem 3.1.7 reads as follows.

Corollary 3.1.2. *Let us consider Eq.* (3.1.57) *with the coefficients* $\{r_k\}_{k\in\mathbb{N}_a}$ *and* $\{s_k\}_{k\in\mathbb{N}_a}$ *such that* $M(\{r_k^{-1}\})$ *and* $M(\{s_k\})$ *exist and* $\{r_k\}$ *is positive. Then, Eq.* (3.1.57) *is non-oscillatory if* (3.1.58) *is valid.*

§ 3.1.3 Examples

In this paragraph, we give some simple examples of oscillatory and non-oscillatory equations whose oscillatory properties do not follow from any previously known oscillation or non-oscillation criteria. To illustrate Theorems 3.1.5, 3.1.6, 3.1.7 and Corollaries 3.1.1, 3.1.2, we mention Examples 3.1.1, 3.1.2, 3.1.4 and Examples 3.1.3, 3.1.5, respectively.

Example 3.1.1. Let $a, b > 0$ be arbitrary. The equation

$$\Delta\left[\Phi\left(\Delta x_k\right)\right] + \frac{a\,|\sin k| + b\cos k}{(k+1)^{(p)}}\,\Phi(x_{k+1}) = 0 \tag{3.1.62}$$

has evidently the form of Eq. (3.1.13). Since

$$M\left(\{a\,|\sin k| + b\cos k\}\right) = aM\left(\{|\sin k|\}\right) + bM\left(\{\cos k\}\right) = \frac{2a}{\pi},$$

Eq. (3.1.62) is oscillatory for $2a > q^{-p}\pi$ and non-oscillatory for $2a < q^{-p}\pi$.

Example 3.1.2. Let $\lambda, \mu \in \mathbb{R}$ be arbitrarily given, where $|\mu| > |\gamma|$. Let us consider the equation

$$\Delta\left[\left|\sin\frac{(3k+1)\pi}{9}\right|^{-\frac{p}{q}}\Phi\left(\Delta x_k\right)\right] + \frac{\lambda + (-1)^k\mu}{k^{(p)}}\,\Phi(x_{k+1}) = 0 \tag{3.1.63}$$

which has the form of Eq. (3.1.52) for

$$\tilde{r}_k = \left|\sin\frac{(3k+1)\pi}{9}\right|^{-\frac{p}{q}}, \quad \tilde{s}_k = \left[\lambda + (-1)^k\mu\right]\frac{(k+1)^{(p)}}{k^{(p)}}.$$

Since

$$M\left(\left\{\tilde{r}_k^{1-q}\right\}\right) = M\left(\left\{\left|\sin\frac{(3k+1)\pi}{9}\right|\right\}\right) = \frac{1}{6}\sum_{k=1}^{6}\left|\sin\frac{(3k+1)\pi}{9}\right|$$
$$= \frac{1}{3}\left(\sin\frac{\pi}{9} + \sin\frac{2\pi}{9} + \sin\frac{4\pi}{9}\right) = \frac{2}{3}\left(\sin\frac{\pi}{9} + \sin\frac{2\pi}{9}\right)$$

and

$$M(\{\tilde{s}_k\}) = M\left(\left\{\lambda + (-1)^k\mu\right\}\right) = \lambda,$$

considering Theorem 3.1.6, we know that Eq. (3.1.63) is oscillatory for

$$\lambda > \Gamma := q^{-p}\left[\frac{2}{3}\left(\sin\frac{\pi}{9} + \sin\frac{2\pi}{9}\right)\right]^{1-p}$$

and non-oscillatory for $\lambda < \Gamma$.

Example 3.1.3. Let $K_1, L_1, K_2, L_2 > 0$. We define the sequence $\{r_k\}_{k\in\mathbb{N}}$ by the formula

$$r_k := \begin{cases} K_1 + L_1, & k \in \{2^n;\, n \in \mathbb{N}\}; \\ K_1, & k \in \mathbb{N} \setminus \{2^n;\, n \in \mathbb{N}\} \end{cases}$$

and the sequence $\{s_k\}_{k\in\mathbb{N}}$ by

$$s_k := \begin{cases} K_2 + (-1)^n L_2, & k \in \{3^n;\, n \in \mathbb{N}\}; \\ K_2, & k \in \mathbb{N} \setminus \{3^n;\, n \in \mathbb{N}\}. \end{cases}$$

If we consider these functions as the coefficients in Eq. (3.1.57), then this equation is oscillatory for

$$M\left(\left\{r_k^{-1}\right\}\right) M(\{s_k\}) = \frac{K_2}{K_1} > \frac{1}{4}$$

and non-oscillatory for $K_1 > 4K_2$. Indeed, we can apply Corollary 3.1.1.

Example 3.1.4. Let $\gamma > 0$. We use Theorem 3.1.7 for the following equation

$$\Delta\left[\frac{1}{1+\cos k\cdot\sin\left(\sqrt{2}k\right)}\cdot\frac{\Delta x_k}{\sqrt{|\Delta x_k|}}\right] + \frac{1}{\gamma\sqrt{k^3}}\cdot\frac{x_{k+1}}{\sqrt{|x_{k+1}|}} = 0, \tag{3.1.64}$$

where $p = 3/2$ (i.e., $q = 3$) and

$$\tilde{r}_k = \frac{1}{1+\cos k\cdot\sin\left(\sqrt{2}k\right)}, \qquad \tilde{s}_k = \frac{(k+1)^{(3/2)}}{\gamma\sqrt{k^3}}$$

for all large $k \in \mathbb{N}$. One can easily verify that

$$\begin{aligned} M\left(\left\{\tilde{r}_k^{-2}\right\}\right) &= M\left(\left\{\left[1+\cos k \cdot \sin\left(\sqrt{2}k\right)\right]^2\right\}\right) \\ &= 1 + M\left(\left\{\cos^2 k \cdot \sin^2\left(\sqrt{2}k\right)\right\}\right) = \frac{5}{4} \end{aligned}$$

and that

$$M\left(\{\tilde{s}_k\}\right) = \frac{1}{\gamma} \lim_{k\to\infty} \frac{(k+1)^{(3/2)}}{\sqrt{k^3}} = \frac{1}{\gamma}.$$

Thus, Eq. (3.1.64) is non-oscillatory if $2\gamma > 3\sqrt{15}$.

Example 3.1.5. For any $c < 1/4$, the linear equation

$$\Delta\left[\frac{\Delta x_k}{1+\cos k^2}\right] + \frac{c}{(k+1)k}\, x_{k+1} = 0 \tag{3.1.65}$$

satisfies all assumptions of Corollary 3.1.2. It is seen that

$$M\left(\left\{r_k^{-1}\right\}\right) M(\{s_k\}) = M\left(\left\{1+\cos k^2\right\}\right) M(\{c\}) = c < \frac{1}{4},$$

which means that Eq. (3.1.65) is non-oscillatory.

Now, we briefly explain why the oscillatory problems in the above examples are not covered by any previously known results (see also Theorem 3.1.1). In both of Examples 3.1.1 and 3.1.2, the second coefficient changes its sign. In Example 3.1.3, the coefficients are not asymptotically almost periodic. In Example 3.1.4, the coefficient in the difference term is not bounded. In the last example, the first coefficient is not asymptotically almost periodic and, at the same time, it is not bounded.

As a final remark, we focus our attention on the denominators of the potentials considered in Examples 3.1.2 and 3.1.4, where $(k+1)^{(p)}$ and $(k+1)^{(3/2)}$ is replaced by $k^{(p)}$ and $\sqrt{k^3}$, respectively. In fact, all presented results remain true if we replace the coefficients $\{s_k\}$ or $\{\tilde{s}_k\}$ by $\{f_k \cdot s_k\}$ or $\{f_k \cdot \tilde{s}_k\}$ for any sequence of real numbers f_k satisfying $\lim_{k\to\infty} f_k = 1$. Indeed, the existence of $M(\{h_k\})$ implies that $M(\{h_k \cdot g_k\}) = M(\{h_k\})$ whenever $\lim_{k\to\infty} g_k = 1$ (consider Definition 3.1.2 and Lemma 3.1.1). Note that we consider the denominator $(k+1)^{(p)}$ due to the form of previously known results.

Chapter

4 Dynamic equations on time scales

4.1 Half-linear Euler-type dynamic equations on time scales

In this chapter, we analyze oscillatory properties of the second-order half-linear Euler-type dynamic equation

$$\left[r(t)\Phi(y^{\Delta})\right]^{\Delta} + c(t)\Phi(y^{\sigma}) = 0, \quad \Phi(y) = |y|^{p-1} \operatorname{sgn} y, \quad p > 1, \tag{4.1.1}$$

on time scale $\mathbb{T}$ with

$$c(t) = \frac{\gamma s(t)}{t^{(p-1)}\sigma(t)}, \tag{4.1.2}$$

where $t^{(p)}$ is generalized power function (for the definition see below), the functions r, s are rd-continuous, positive, α-periodic with $\inf\{r(t),\, t \in \mathbb{T}\} > 0$ and $\gamma \in \mathbb{R}$ is an arbitrary constant.

Among others, the Sturmian theory extends verbatim for dynamic half-linear equations, therefore we can classify equations as oscillatory and non-oscillatory. For the full theory background and comprehensive literature overview, we refer to [2, 3, 21] (see also [66, 67, 69]).

As well as in the previous chapters, we are interested in the conditional oscillation of equation (4.1.1) with (4.1.2). It means that our aim is to prove that there exists a so-called critical constant, dependent only on coefficients r and s, which establishes a sharp borderline between oscillation and non-oscillation of these equations. More precisely, let us consider the equation

$$\left[\hat{r}(t)\Phi(y^{\Delta})\right]^{\Delta} + \hat{\gamma} d(t)\Phi(y^{\sigma}) = 0, \quad \hat{\gamma} \in \mathbb{R}. \tag{4.1.3}$$

We say, that equation (4.1.3) is conditionally oscillatory, if there exists a positive constant Γ such that equation (4.1.3) is oscillatory if $\hat{\gamma} > \Gamma$ and non-oscillatory if $\hat{\gamma} < \Gamma$. Since the Sturmian theory (especially the comparison theorem) is valid in the theory of half-linear dynamic equations, conditionally oscillatory equations are good testing equations. E.g., let $r, \hat{r} \equiv 1$, and let d be arbitrary positive rd-continuous function. Then equation (4.1.1) is oscillatory if $\liminf_{t\to\infty} c(t)/d(t) > \Gamma$ and non-oscillatory if $\limsup_{t\to\infty} c(t)/d(t) < \Gamma$ (see Corollary 4.1.1).

We note, that the case $\gamma = \Gamma$ is resolved for differential equations (i.e., for $\mathbb{T} = \mathbb{R}$) as non-oscillatory. However, the oscillation behavior of the discrete equation ($\mathbb{T} = \mathbb{Z}$)

for $\gamma = \Gamma$ is generally not known (see Chapter 3). Moreover, it can be shown that even differential equations cannot be generally classified as (non-)oscillatory in the critical case for larger classes of coefficients.

Our aim is to prove that equation (4.1.1) with (4.1.2) is conditionally oscillatory. We will also find its critical constant Γ. Evidently, this result covers the mentioned (i.e., $p = 2$) case and results for equations (1.2.3) and (3.1.3) as well as the result of [78] for the Euler-type dynamic equation with α-periodic positive coefficients

$$[r(t)y^{\Delta}]^{\Delta} + \frac{\gamma s(t)}{t\sigma(t)} y^{\sigma} = 0 \tag{4.1.4}$$

and its critical oscillation constant

$$\Gamma = \frac{\alpha^2}{4} \left(\int_a^{a+\alpha} \frac{\Delta t}{r(t)} \right)^{-1} \left(\int_a^{a+\alpha} s(t)\Delta t \right)^{-1}.$$

We note that, in the literature, one can find Euler type half-linear dynamic equation in forms different from the one treated in this chapter. More precisely, the potential (4.1.2) is sometimes considered with the standard power function in the denominator (i.e., $c(t) = \gamma s(t)/t^p$ or $c(t) = \gamma s(t)/(\sigma(t))^p$) or in differential form (see, e.g., [67]). Nevertheless, we have selected the potential in the form of (4.1.2), because there is a direct correspondence with the difference as well as with differential equations and for $p = 2$ it corresponds to Euler-type dynamic equation (4.1.4).

The chapter is organized as follows. The notion of time scales is recalled in the next paragraph together with the definition of the generalized power function. The (non-)oscillation theory for half-linear dynamic equation with lemmata that we need in the rest of the chapter can the reader find in § 4.1.1 as well. Then, in § 4.1.2, we formulate and prove the main result concerning the conditional oscillation of the mentioned Euler-type half-linear dynamic equation (4.1.1) with (4.1.2) and illustrate it with an example. The chapter is finished by corollaries and concluding remarks given in § 4.1.3.

§ 4.1.1 Preliminaries

At the beginning, let us remind a notation on time scales. The theory of time scales was introduced by Stefan Hilger in his Ph.D. thesis in 1988, see [47], in order to unify the continuous and discrete calculus. Nowadays, it is well-known calculus and it is often studied in applications. Remind that a time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of reals. Note that $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$ (resp. $[a, \infty)_{\mathbb{T}} := [a, \infty) \cap \mathbb{T}$) stands for an arbitrary finite (resp. infinite) time scale interval. Symbols σ, ρ, μ, f^{σ}, f^{Δ}, and $\int_a^b f(t)\Delta t$ stand for the forward jump operator, backward jump operator, graininess, $f \circ \sigma$, Δ-derivative of f, and Δ-integral of f from a to b, respectively. Further, we use the symbols $C_{\mathrm{rd}}(\mathbb{T})$ and $C^1_{\mathrm{rd}}(\mathbb{T})$ for the class of rd-continuous and rd-continuous Δ-differentiable functions defined on the time scale $\mathbb{T}$. Recall that the time scale $\mathbb{T}$ is α-periodic if there exists constant $\alpha > 0$ such that if $t \in \mathbb{T}$ then $t \pm \alpha \in \mathbb{T}$. We note, that any α-periodic time scale $\mathbb{T}$ is infinite and, naturally, unbounded from above. For further information and background on time scale calculus, see [48], which is the initiating paper of the time scale theory, and the books [9, 10], which contain a lot of information on time scale calculus.

For further reading, it is necessary to remind a definition of n-th composition of operator ρ, see also [9]. We define

$$\rho^{-1}(t) := \sigma(t), \quad \rho^0(t) := t, \quad \rho^1(t) := \rho(t), \quad \rho^2(t) := \rho(\rho(t)), \quad \dots, \quad \rho^n(t) = \rho(\rho^{n-1}(t)).$$

If $-\infty < a = \min \mathbb{T}$, then we define $\rho^n(a) = a$ for each $n \in \mathbb{N}$.

Definition 4.1.1. For arbitrary $t \in \mathbb{T}$ and $p \in \mathbb{N}$, we define the *generalized power function* on time scales as

$$t^{(p)} := t\rho(t) \cdots \rho^{p-1}(t).$$

For $p = 0$, we define $t^{(0)} := 1$.

The following definition naturally extends the previous one for arbitrary real $p \geq 0$.

Definition 4.1.2. Let $p \in \mathbb{R}$ and $\lfloor p \rfloor$ denote the greatest integer less then or equal to p (the floor function). For arbitrary $t \in \mathbb{T}$ and $p \geq 0$, we define the *generalized power function* on time scales as

$$t^{(p)} := t^{(\lfloor p \rfloor)} \left\{ \left(\rho^{\lfloor p-1 \rfloor}(t)\right)^{1-p+\lfloor p \rfloor} \cdot \left(\rho^{\lfloor p \rfloor}(t)\right)^{p-\lfloor p \rfloor} \right\}^{p-\lfloor p \rfloor}.$$

Example 4.1.1. Let us illustrate the generalized power function with two simple examples involving the backward and the forward jump operator, respectively.

(i) $t^{(7/3)} = t^{(2)} \left\{ (\rho(t))^{2/3} \cdot (\rho^2(t))^{1/3} \right\}^{1/3} = t \cdot (\rho(t))^{11/9} \cdot (\rho^2(t))^{1/9}$,

(ii) $t^{(3/4)} = \left\{ (\sigma(t))^{1/4} \cdot t^{3/4} \right\}^{3/4} = (\sigma(t))^{3/16} \cdot t^{9/16}$.

Note that for $\mathbb{T} = \mathbb{R}$ we get the classic power function and for $\mathbb{T} = \mathbb{Z}$, $p \in \mathbb{N}$, we get generalized discrete power function, see Section 3.1 or directly [51, Chapter 2]. In the following, we show some properties of the generalized power function, which will be useful later.

Lemma 4.1.1. *Let $\mathbb{T}$ be an α-periodic time scale and $p \geq 0$. Then the function $f(p) = t^{(p)}$ is continuous and increasing in p for large $t \in \mathbb{T}$ and*

$$\lim_{t \to \infty} \frac{t^{(p)}}{t^p} = 1. \tag{4.1.5}$$

Proof. For the sake of clarity, we will use $p \in [1, 2]$ in the first part of the proof and $p \in [1, 2)$ in the second part. Nevertheless, for any other intervals $[k, k+1]$ and $[k, k+1)$, $k \in \mathbb{N} \cup \{0\}$, it can be verified analogously.

Let $p \in [1, 2]$. We show a continuity from the right-side at a point $p = 1$ and a continuity from the left-side at a point $p = 2$ (for any other $p \in (1, 2)$ the continuity is obvious)

$$\lim_{p\to 1+} t^{(p)} = t \lim_{p\to 1+} \left\{t^{2-p} \cdot (\rho(t))^{p-1}\right\}^{p-1} = t = t^{(1)}$$

and

$$\lim_{p\to 2-} t^{(p)} = t \lim_{p\to 2-} \left\{t^{2-p} \cdot (\rho(t))^{p-1}\right\}^{p-1} = t\rho(t) = t^{(2)}.$$

Next, we show that f is increasing for $p \in [1, 2)$. Let $p_1, p_2 \in [1, 2)$, $p_1 < p_2$. On the contrary, let $t^{(p_1)} > t^{(p_2)}$, i.e.,

$$\left\{t^{2-p_1} \cdot (\rho(t))^{p_1-1}\right\}^{p_1-1} > \left\{t^{2-p_2} \cdot (\rho(t))^{p_2-1}\right\}^{p_2-1}.$$

It is easy to see that the last inequality can be written in the form

$$t^{p_1-p_2} \cdot (t/\rho(t))^{(p_1-p_2)(2-p_1-p_2)} > 1. \tag{4.1.6}$$

Hence, for the arbitrary fixed p_1 and p_2, we can see that $t^{p_1-p_2} \to 0$ as $t \to \infty$ and

$$(t/\rho(t))^{(p_1-p_2)(2-p_1-p_2)} \to 1 \quad \text{as} \quad t \to \infty,$$

thus the inequality (4.1.6) is not valid for large $t \in \mathbb{T}$ and we get a contradiction.

Finally, for arbitrary fixed $p \in [1, 2)$, we show that (4.1.5) holds. Let $p \in [1, 2)$, then

$$\frac{t^{(p)}}{t^p} = \frac{t\left\{t^{2-p} \cdot (\rho(t))^{p-1}\right\}^{p-1}}{t^p} = \frac{t\left\{t^{2-p} \cdot t^{p-1}[1 - (\mu(t)/t)]^{p-1}\right\}^{p-1}}{t^p} = [1 - (\mu(t)/t)]^{(p-1)^2}.$$

Hence, in view of $\mu(t)/t \to 0$ as $t \to \infty$ (due to $\mu(t) \le \alpha$ for every t), we get (4.1.5). □

Now, we recall basic elements of the oscillation theory of dynamic equations on time scales. Throughout this chapter, we assume that the time scale $\mathbb{T}$ is α-periodic, which implies $\sup \mathbb{T} = \infty$. Consider the second order half-linear dynamic equation

$$[r(t)\Phi(y^\Delta)]^\Delta + c(t)\Phi(y^\sigma) = 0, \quad \Phi(y) = |y|^{p-1} \operatorname{sgn} y, \quad p > 1, \tag{4.1.7}$$

on a time scale $\mathbb{T}$, where $c, r \in C_{\mathrm{rd}}(\mathbb{T})$ and $\inf\{r(t),\, t \in \mathbb{T}\} > 0$. We note that $\Phi^{-1}(y) = |y|^{q-1} \operatorname{sgn} y$, where $q > 1$ is the conjugate number of p, i.e., $p + q = pq$. It is easy to see that any solution y of (4.1.7) satisfies $r\,\Phi(y^\Delta) \in C^1_{\mathrm{rd}}(\mathbb{T})$.

Further, we note that it is not sufficient to assume only $r(t) > 0$ (instead of $\inf\{r(t),\, t \in \mathbb{T}\} > 0$), because it may happen that $\lim_{t\to t_0-} r(t) = 0$ and $r(t_0) > 0$, which would not be convenient in our case. Indeed, we need $1/r \in C_{\mathrm{rd}}(\mathbb{T})$ due to the integration of $1/r^{q-1}(t)$, which is now fulfilled, see also [68], where this and similar problems are discussed.

Definition 4.1.3. We say that a nontrivial solution y of (4.1.7) has a *generalized zero* at t if

$$r(t)y(t)y(\sigma(t)) \le 0.$$

If $y(t) = 0$, we say that solution y has a *common zero* at t (the common zero is a special case of the generalized zero).

Definition 4.1.4. We say that a solution y of equation (4.1.7) is *non-oscillatory* on $\mathbb{T}$ if there exists $\tau \in \mathbb{T}$ such that there does not exist any generalized zero at t for $t \in [\tau, \infty)_{\mathbb{T}}$. Otherwise, we say that it is *oscillatory.*

Remark 4.1.1. Oscillation may be equivalently defined as follows. A nontrivial solution y of (4.1.7) is called oscillatory on $\mathbb{T}$, if y has a generalized zero on $[\tau, \infty)_{\mathbb{T}}$ for every $\tau \in \mathbb{T}$.

From the Sturm type separation theorem (see, e.g., [65]) it follows that if one solution of (4.1.7) is oscillatory (resp. non-oscillatory), then every solution of (4.1.7) is oscillatory (resp. non-oscillatory). Hence we can speak about *oscillation* or *non-oscillation* of equation (4.1.7).

Next, let us recall the generalization of the Sturm type comparison theorem for time scale calculus, which will be useful later.

Theorem 4.1.1 (Sturm type comparison theorem, [65, p. 388])**.** *Consider the equation*

$$[R(t)\Phi(y^{\Delta})]^{\Delta} + C(t)\Phi(y^{\sigma}) = 0 \tag{4.1.8}$$

and equation (4.1.7)*, where* $R, C \in C_{\mathrm{rd}}(\mathbb{T})$ *with* $\inf\{|R(t)|,\, t \in \mathbb{T}\} > 0$.

(i) *Let* $R(t) \geq r(t)$ *and* $C(t) \leq c(t)$ *for every* $t \in \mathbb{T}$*. If* (4.1.7) *is non-oscillatory then* (4.1.8) *is also non-oscillatory.*

(ii) *Let* $R(t) \leq r(t)$ *and* $C(t) \geq c(t)$ *for every* $t \in \mathbb{T}$*. If* (4.1.7) *is oscillatory then* (4.1.8) *is also oscillatory.*

Our approach to the oscillatory and non-oscillatory problems of (4.1.7) is based mainly on the application of the generalized Riccati dynamic equation

$$w^{\Delta}(t) + c(t) + \mathcal{S}[w, r, \mu](t) = 0, \tag{4.1.9}$$

where

$$\mathcal{S}[w, r, \mu] = \lim_{\lambda \to \mu} \frac{w}{\lambda}\left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \lambda\Phi^{-1}(w))}\right).$$

It is not difficult to observe that

$$\mathcal{S}[w, r, \mu](t) = \begin{cases} \left\{\frac{p-1}{\Phi^{-1}(r)}|w|^q\right\}(t) & \text{at right-dense } t, \\ \left\{\frac{w}{\mu}\left(1 - \frac{r}{\Phi(\Phi^{-1}(r)+\mu\Phi^{-1}(w))}\right)\right\}(t) & \text{at right-scattered } t. \end{cases}$$

Note that using the Lagrange mean value theorem on time scales (see, e.g., [10]), one can show that the operator $\mathcal{S}$ can be written in the form

$$\mathcal{S}[w, r, \mu](t) = \frac{(p-1)|w(t)|^q|\eta(t)|^{p-2}}{\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t))]}, \tag{4.1.10}$$

where $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(w(t))$. The form (4.1.10) will be convenient for our purpose.

The relation between (4.1.7) and (4.1.9) is the following. If $y(t)$ is a solution of (4.1.7) with $y(t)y^\sigma(t) \neq 0$ for $t \in [t_1, t_2]_\mathbb{T}$ and we denote

$$w(t) = \frac{r(t)\Phi(y^\Delta(t))}{\Phi(y(t))},$$

then, for $t \in [t_1, t_2]_\mathbb{T}$, $w = w(t)$ satisfies equation (4.1.9). Now, we are ready to formulate the "time scale version" of the Reid roundabout theorem, which can be understood as a central statement of the oscillation theory for equation (4.1.7).

Theorem 4.1.2 (Roundabout theorem, [65, p. 383])**.** *Let $a \in \mathbb{T}$. The following statements are equivalent.*

(i) *Every nontrivial solution of* (4.1.7) *has at most one generalized zero on $[a, \infty)_\mathbb{T}$.*

(ii) *Equation* (4.1.7) *has a solution having no generalized zeros on $[a, \infty)_\mathbb{T}$.*

(iii) *Equation* (4.1.9) *has a solution w with*

$$\left\{\Phi^{-1}(r) + \mu\, \Phi^{-1}(w)\right\}(t) > 0 \quad \textit{for} \quad t \in [a, \infty)_\mathbb{T}. \tag{4.1.11}$$

The following theorem is a consequence of the Roundabout theorem 4.1.2 and the Sturm type comparison theorem 4.1.1. The method of oscillation theory for (4.1.7), which uses the ideas of this theorem, is usually referred to as the *Riccati technique.*

Theorem 4.1.3 (Riccati technique, [65, p. 390])**.** *The following statements are equivalent.*

(i) *Equation* (4.1.7) *is non-oscillatory.*

(ii) *There is $a \in \mathbb{T}$ and a function $w : [a, \infty)_\mathbb{T} \to \mathbb{R}$ such that* (4.1.11) *holds and $w(t)$ satisfies* (4.1.9) *for $t \in [a, \infty)_\mathbb{T}$.*

(iii) *There is $a \in \mathbb{T}$ and a function $w : [a, \infty)_\mathbb{T} \to \mathbb{R}$ such that* (4.1.11) *holds and $w(t)$ satisfies*

$$w^\Delta(t) + c(t) + \mathcal{S}[w, r, \mu](t) \leq 0 \quad \textit{for} \quad t \in [a, \infty)_\mathbb{T}.$$

For further considerations, the following lemma plays an important role (see also [65], where the similar result can be found).

Lemma 4.1.2. *Let the equation*

$$[r(t)\Phi(y^\Delta)]^\Delta + c(t)\Phi(y^\sigma) = 0, \tag{4.1.12}$$

where coefficients $c, r \in C_{\mathrm{rd}}(\mathbb{T})$ are positive and

$$0 < \inf\{r(t),\, t \in \mathbb{T}\} \leq \sup\{r(t),\, t \in \mathbb{T}\} < \infty, \tag{4.1.13}$$

be non-oscillatory. Then for every solution $w(t)$ of the associated generalized Riccati equation (4.1.9)*, there exists $T \in \mathbb{T}$ such that $w(t) > 0$ for $t \in [T, \infty)_\mathbb{T}$. Moreover, $w(t)$ is decreasing for large t with*

$$\lim_{t\to\infty} w(t) = 0.$$

Proof. At first, let us suppose that y is a positive solution of non-oscillatory equation (4.1.12), i.e., $y(t) > 0$ for $t \in [S,\infty)_{\mathbb{T}}$, where $S \in \mathbb{T}$ is sufficiently large. By contradiction, we prove that there exists $T \in [S,\infty)_{\mathbb{T}}$ such that $y^{\Delta}(t) > 0$ for $t \in [T,\infty)_{\mathbb{T}}$.

(i) Let $y^{\Delta}(t) < 0$ for $t \in [S,\infty)_{\mathbb{T}}$. Because $c(t)\Phi(y^{\sigma}(t)) > 0$ for $t \in [S,\infty)_{\mathbb{T}}$, we have

$$\left[r(t)\Phi(y^{\Delta}(t))\right]^{\Delta} < 0 \quad \text{for} \quad t \in [S,\infty)_{\mathbb{T}}.$$

Integrating the last inequality from S to t, we have

$$r(t)\Phi(y^{\Delta}(t)) - r(S)\Phi(y^{\Delta}(S)) = \int_S^t \left[r(s)\Phi(y^{\Delta}(s))\right]^{\Delta} \Delta s \le 0.$$

Hence

$$y^{\Delta}(t) \le \frac{r^{q-1}(S)y^{\Delta}(S)}{r^{q-1}(t)} \tag{4.1.14}$$

for $t \in [S,\infty)_{\mathbb{T}}$. Integrating (4.1.14) for $t \ge S$, we get

$$\left[\lim_{t\to\infty} y(t)\right] - y(S) = \int_S^{\infty} y^{\Delta}(s)\Delta s \le r^{q-1}(S)y^{\Delta}(S)\int_S^{\infty} \frac{\Delta s}{r^{q-1}(s)} = -\infty.$$

Note that the last integral is equal to infinity in view of (4.1.13). Hence $y(t) \to -\infty$ as $t \to \infty$, which is a contradiction. Therefore $y^{\Delta}(t) < 0$ cannot hold for large t.

(ii) Let $y^{\Delta}(t) \not> 0$ for large t, i.e., there exists $T_0 \in [S,\infty)_{\mathbb{T}}$ such that $y^{\Delta}(T_0) \le 0$. Thanks to $c(t) > 0$ for $t \in \mathbb{T}$, we have

$$\liminf_{t\to\infty} \int_S^t c(s)\Delta s > 0.$$

Since (4.1.12) is non-oscillatory, then due to Theorem 4.1.3, the function

$$w(t) = \frac{r(t)\Phi(y^{\Delta}(t))}{\Phi(y(t))} \tag{4.1.15}$$

satisfies (4.1.9) with $\{\Phi^{-1}(r) + \mu\,\Phi^{-1}(w)\}(t) > 0$ for $t \in [S,\infty)_{\mathbb{T}}$. Integrating (4.1.9) from T_0 to t, $t \ge T_0$, we get

$$w(t) = w(T_0) - \int_{T_0}^t c(s)\Delta s - \int_{T_0}^t \mathcal{S}[w,r,\mu](s)\Delta s. \tag{4.1.16}$$

Since $w(T_0) \le 0$, the first integral in (4.1.16) is positive for large t, and the second integral in (4.1.16) is nonnegative for large t, we obtain $\limsup_{t\to\infty} w(t) < 0$. For the nonnegativity of function $\mathcal{S}$ see [65, Lemma 13]. Hence, there exists $T_1 \in [S,\infty)_{\mathbb{T}}$ such that $w(t) < 0$ for $t \in [T_1, \infty)_{\mathbb{T}}$, thus $y^{\Delta}(t) < 0$ for $t \in [T_1, \infty)_{\mathbb{T}}$, which is a contradiction to the case (i). We proved that for positive y there exists $T \in \mathbb{T}$ such that $y^{\Delta}(t) > 0$ for $t \in [T, \infty)_{\mathbb{T}}$.

Let $y(t)$ be any negative solution of (4.1.12) for large t. Then $-y(t) > 0$ is a positive solution of (4.1.12) with just proven property (the solution space of half linear equations is homogeneous). Hence $y^{\Delta}(t) < 0$ for $t \in [T, \infty)_{\mathbb{T}}$.

In any case, we get (see (4.1.15)) that $w(t) > 0$ and satisfies (4.1.9) together with (4.1.11) for $t \in [T, \infty)_{\mathbb{T}}$. Moreover, since

$$w^{\Delta} = -c(t) - \mathcal{S}[w, r, \mu](t) < 0,$$

$w(t)$ is decreasing for $t \in [T, \infty)_{\mathbb{T}}$.

Finally, we show that $w(t) \to 0$ as $t \to \infty$. Suppose that a solution y is positive and increasing for large t (the case y is negative and decreasing can be proven analogously). Then it either converges to a positive constant L or diverges to ∞. First, we suppose that $y(t) \to \infty$ as $t \to \infty$. Then, since $r(t)\Phi(y^{\Delta}(t))$ is decreasing (see (4.1.12)), we have

$$w(t) = \frac{r(t)\Phi(y^{\Delta}(t))}{\Phi(y(t))} < \frac{r(T)\Phi(y^{\Delta}(T))}{\Phi(y(t))} \to 0 \quad \text{as} \quad t \to \infty.$$

Hence $w(t) \to 0$ as $t \to \infty$. Second, if $y(t) \to L$ as $t \to \infty$, then $y^{\Delta}(t) \to 0$ as $t \to \infty$. Thus $r(t)\Phi(y^{\Delta}(t)) \to 0$ as $t \to \infty$ and consequently, $w(t)$ tends to zero as $t \to \infty$ (see (4.1.15)). $\square$

In the proof of the main result, we use the so-called adapted generalized Riccati equation. Putting

$$z(t) = -t^{p-1} w(t)$$

and using the form of (4.1.9) with (4.1.10), a direct calculation leads to the adapted generalized Riccati equation

$$\begin{aligned} z^{\Delta}(t) = c(t)(\sigma(t))^{p-1} &+ \frac{(p-1)(\sigma(t))^{p-1}|\eta(t)|^{p-2}|z(t)|^{q}}{t^{p}\, \Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})]} \\ &+ \frac{(p-1)(\zeta(t))^{p-2} z(t)}{t^{p-1}}, \end{aligned} \tag{4.1.17}$$

where $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})$ and $\zeta(t)$ is defined as

$$\zeta(t) := \left[\frac{(t^{p-1})^{\Delta}}{p-1} \right]^{\frac{1}{p-2}}. \tag{4.1.18}$$

Note that using the Lagrange mean value theorem on time scales, we can (after rewriting (4.1.18) on $(t^{p-1})^{\Delta} = (p-1)(\zeta(t))^{p-2}$) see that $\zeta(t)$ exists and satisfies $t \le \zeta(t) \le \sigma(t)$.

Now, we state two auxiliary lemmata concerning equation (4.1.17), which can be regarded as consequences of Lemma 4.1.2.

Lemma 4.1.3. *Let* (4.1.12) *be non-oscillatory. Then for every solution $z(t)$ of the associated adapted generalized Riccati equation* (4.1.17)*, there exists sufficiently large $t_0 \in \mathbb{T}$ such that $z(t) < 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.*

Proof. The statement of the lemma follows from Lemma 4.1.2. □

Lemma 4.1.4. *If there exists a solution $z(t)$ of the equation* (4.1.17) *satisfying $z(t) < 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, then its original equation* (4.1.12) *is non-oscillatory. Moreover,*

$$z(t)/t^{p-1} \to 0 \quad \text{as} \quad t \to \infty.$$

Proof. From $z(t) < 0$ it follows that $\{\Phi^{-1}(r) + \mu\, \Phi^{-1}(w)\}(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Hence, thanks to Theorem 4.1.2, we get that every solution of (4.1.12) is non-oscillatory and (4.1.12) is non-oscillatory as well. Further, $z(t)/t^{p-1} = -w(t) \to 0$ as $t \to \infty$ follows from Lemma 4.1.2. □

§ 4.1.2 Conditional oscillation

In this paragraph, we formulate and prove the main result of the chapter. At first, for reader's convenience, let us recall, that we deal with the Euler-type half-linear dynamic equation

$$\left[r(t)\Phi(y^{\Delta})\right]^{\Delta} + \frac{\gamma s(t)}{t^{(p-1)}\sigma(t)}\Phi(y^{\sigma}) = 0, \quad \Phi(y) = |y|^{p-1}\operatorname{sgn} y, \quad p > 1, \tag{4.1.19}$$

on an α-periodic ($\alpha > 0$) time scale interval $[a, \infty)_{\mathbb{T}}$, $a \in \mathbb{T}$ with $a > 0$, where $t^{(p)}$ is generalized power function, the functions r, s are rd-continuous, positive, α-periodic with $\inf\{r(t),\, t \in [a, \infty)_{\mathbb{T}}\} > 0$, and $\gamma \in \mathbb{R}$ is an arbitrary constant. Now, we can formulate the main theorem as follows.

Theorem 4.1.4. *Let $\gamma \in \mathbb{R}$ be a given constant and let $r, s \in C_{\mathrm{rd}}([a, \infty)_{\mathbb{T}})$ be positive α-periodic functions satisfying $\inf\{r(t),\, t \in [a, \infty)_{\mathbb{T}}\} > 0$. Further let*

$$\Gamma := \left(\frac{\alpha}{q}\right)^p \left[\int_a^{a+\alpha} r^{1-q}(t)\Delta t\right]^{1-p} \left[\int_a^{a+\alpha} s(t)\Delta t\right]^{-1}. \tag{4.1.20}$$

Then the Euler-type half-linear dynamic equation (4.1.19) *is oscillatory for $\gamma > \Gamma$ and non-oscillatory for $\gamma < \Gamma$.*

Proof. Since the functions r and s are α-periodic, we have that $\mu(t) \le \alpha$ for every $t \in [a, \infty)_{\mathbb{T}}$ and that a written in limits of integrals in (4.1.20) can be replace by arbitrary $\tau \in [a, \infty)_{\mathbb{T}}$ with same resulting value Γ.

Throughout the proof, we will use the following estimates in which we assume that $\gamma > 0$ and $z(t) < 0$ for large t. Denote

$$r^+ := \sup\{r(t), t \in [a, \infty)_{\mathbb{T}}\}, \quad r^- := \inf\{r(t), t \in [a, \infty)_{\mathbb{T}}\}$$

and

$$s^+ := \sup\{s(t), t \in [a,\infty)_{\mathbb{T}}\}, \quad s^- := \inf\{s(t), t \in [a,\infty)_{\mathbb{T}}\}.$$

Note that due to rd-continuity and α-periodicity of the functions r and s,

$$0 < r^- \le r^+ < \infty \quad \text{and} \quad 0 \le s^- \le s^+ < \infty$$

hold. In view of (4.1.17), the adapted Riccati equation associated to (4.1.19) has the form

$$\begin{aligned} z^\Delta(t) = \frac{\gamma s(t)(\sigma(t))^{p-2}}{t^{(p-1)}} &+ \frac{(p-1)(\sigma(t))^{p-1}|\eta(t)|^{p-2}|z(t)|^q}{t^p\,\Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})]} \\ &+ \frac{(p-1)(\zeta(t))^{p-2}z(t)}{t^{p-1}}, \end{aligned} \tag{4.1.21}$$

where $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t))+\mu(t)\Phi^{-1}(-z(t)/t^{p-1})$, and $t \le \zeta(t) \le \sigma(t)$. Let us define the function

$$h(t) := \mu(t) r^{1-q}(t)\Phi^{-1}(-z(t)/t^{p-1}).$$

It is easy to see (in view of Lemma 4.1.4) that

$$0 \le h(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{4.1.22}$$

Therefore, equation (4.1.21) can be written in the form

$$\begin{aligned} z^\Delta(t) = {} & \frac{\gamma s(t)(\sigma(t))^{p-2}}{t^{(p-1)}} \\ & + (p-1)|z(t)|\frac{((\sigma(t))^{p-1}/t)|\eta(t)|^{p-2}|z(t)|^{q-1} - (\zeta(t))^{p-2}F(t)}{t^{p-1}F(t)}, \end{aligned} \tag{4.1.23}$$

where

$$F(t) := \Phi[\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})] = r(t)[1+h(t)]^{p-1} > 0. \tag{4.1.24}$$

Hence, we get for large t and for $p \ge 2$

$$\begin{aligned} z^\Delta(t) &\ge \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)| \cdot \frac{(\sigma(t))^{p-2}|\eta(t)|^{p-2}|z(t)|^{q-1} - (\sigma(t))^{p-2}r(t)[1+h(t)]^{p-1}}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\ &> \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)|(\sigma(t))^{p-2} \cdot \frac{r^{(q-1)(p-2)}(t)|z(t)|^{q-1} - 2^{p-1}r(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\ &= \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)|(\sigma(t))^{p-2}r^{2-q}(t) \cdot \frac{|z(t)|^{q-1} - 2^{p-1}r^{q-1}(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}}. \end{aligned}$$

Analogously, for large t and for $p < 2$, we have

$$\begin{aligned}
z^{\Delta}(t) &\geq \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)| \cdot \frac{(\sigma(t))^{p-2}|\eta(t)|^{p-2}|z(t)|^{q-1} - (\sigma(t))^{p-2}r(t)[1+h(t)]^{p-1}}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\
&> \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)| \cdot \frac{(\sigma(t))^{p-2}(2r^{q-1}(t))^{p-2}|z(t)|^{q-1} - t^{p-2}2^{p-1}r(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\
&= \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)|(\sigma(t))^{p-2}2^{p-2}r^{2-q}(t) \cdot \frac{|z(t)|^{q-1} - (\sigma(t)/t)^{2-p}2r^{q-1}(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}} \\
&\geq \frac{\gamma s^-}{\sigma(t)} + (p-1)|z(t)|(\sigma(t))^{p-2}2^{p-2}r^{2-q}(t) \cdot \frac{|z(t)|^{q-1} - (1+\alpha)^{2-p}2r^{q-1}(t)}{t^{p-1}r(t)[1+h(t)]^{p-1}}
\end{aligned}$$

and thus

$$z^{\Delta}(t) > \frac{\gamma s^-}{\sigma(t)} \quad \text{if} \quad z(t) < \min\left\{-2^{(p-1)^2}r^+, -2^{\frac{p}{q}}(1+\alpha)^{\frac{2-p}{q-1}}r^+\right\}. \tag{4.1.25}$$

Simultaneously, we estimate $|z^{\Delta}(t)|$ for $z(t) \in (-C, 0)$ and large t. We denote

$$\begin{aligned}
D := \max\Bigg\{ &\sup\left\{\frac{\sigma(t)}{t}, t \in [a,\infty)_{\mathbb{T}}\right\}, \sup\left\{\frac{(\sigma(t))^{p-2}}{t^{p-2}}, t \in [a,\infty)_{\mathbb{T}}\right\}, \\
&\sup\left\{\frac{t(\sigma(t))^{p-2}}{t^{(p-1)}}, t \in [a,\infty)_{\mathbb{T}}\right\}\Bigg\} > 0.
\end{aligned}$$

Then, we get thanks to (4.1.23) for $p \geq 2$ (i.e., $q \leq 2$)

$$\begin{aligned}
|z^{\Delta}(t)| &< \frac{\gamma s^+ D}{t} + (p-1)C\frac{(\sigma(t))^{p-2}D[2r^{q-1}(t)]^{p-2} \cdot C^{q-1} + (\sigma(t))^{p-2}2^{p-1}r(t)}{t^{p-1}r(t)} \\
&\leq \frac{\gamma s^+ D}{t} + \frac{2^{p-2}C(p-1)(\sigma(t))^{p-2}[C^{q-1}Dr^{2-q}(t) + 2r(t)]}{t^{p-1}r^-} \\
&\leq \frac{\gamma s^+ D}{t} + \frac{2^{p-2}C(p-1)D[C^{q-1}D(r^+)^{2-q} + 2r^+]}{tr^-} \\
&= \frac{\gamma s^+ r^- D + 2^{p-2}C(p-1)D[C^{q-1}D(r^+)^{2-q} + 2r^+]}{tr^-},
\end{aligned} \tag{4.1.26}$$

and for $p < 2$ (i.e., $q > 2$)

$$\begin{aligned}
|z^{\Delta}(t)| &< \frac{\gamma s^+ D}{t} + (p-1)C\frac{(\sigma(t))^{p-2}D[r^{q-1}(t)]^{p-2} \cdot C^{q-1} + t^{p-2}2^{p-1}r(t)}{t^{p-1}r(t)} \\
&\leq \frac{\gamma s^+ D}{t} + \frac{(p-1)C^q D^2(r^-)^{2-q}}{tr^-} + \frac{(p-1)2^{p-1}r^+C}{tr^-} \\
&= \frac{\gamma s^+ r^- D + (p-1)C^q D^2(r^-)^{2-q} + (p-1)2^{p-1}r^+C}{tr^-}.
\end{aligned} \tag{4.1.27}$$

Therefore, we have

$$|z^{\Delta}(t)| < \frac{H(C)}{t}, \tag{4.1.28}$$

where

$$\begin{aligned} H(C) := \max\Bigg\{ & \frac{\gamma s^{+} r^{-} D + 2^{p-2} C(p-1) D[C^{q-1} D (r^{+})^{2-q} + 2r^{+}]}{r^{-}}, \\ & \frac{\gamma s^{+} r^{-} D + (p-1) C^{q} D^{2} (r^{-})^{2-q} + (p-1) 2^{p-1} r^{+} C}{r^{-}} \Bigg\} \end{aligned} \tag{4.1.29}$$

is a positive constant which exists due to (4.1.26) and (4.1.27).

Next, from (4.1.25) and (4.1.28) it follows that if $z(t) < 0$ for every $t \in [t_0, \infty)_{\mathbb{T}}$, $t_0 \geq a$, then there exists a constant $K > 0$ such that

$$z(t) \in (-K, 0) \quad \text{for every} \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.1.30}$$

Indeed, according to (4.1.25), $z(t)$ is increasing if $z(t)$ is sufficiently small. Otherwise, thanks to (4.1.28), $z(t)$ cannot drop arbitrarily low.

Next, using the fact that the graininess $\mu(t) \leq \alpha$ for all $t \in [a, \infty)_{\mathbb{T}}$ together with the definition of ζ given in (4.1.18) and taking into account that $\eta(t)$ is between $\Phi^{-1}(r(t))$ and $\Phi^{-1}(r(t)) + \mu(t)\Phi^{-1}(-z(t)/t^{p-1})$, we obtain (see also Lemma 4.1.1), that there exists a constant $\varepsilon \in [0, 1/2)$ such that

$$\begin{gathered} 1 - \varepsilon \leq \frac{(\sigma(t))^{p-2}}{t^{(p-1)}/t} \leq 1 + \varepsilon, \qquad 1 - \varepsilon \leq \frac{(\sigma(t))^{p-1} |\eta(t)|^{p-2}}{t^{p-1} r^{2-q}(t)} \leq 1 + \varepsilon, \\ 1 - \varepsilon \leq \frac{(\zeta(t))^{p-2}}{t^{p-2}} \leq 1 + \varepsilon \end{gathered} \tag{4.1.31}$$

are fulfilled for arbitrary $p > 1$ and large t. More precisely, ε can be chosen arbitrarily near to zero in (4.1.31) if t is sufficiently large.

Using the above estimates, we can turn our attention to the proof of the theorem. We start with the oscillatory part. In this part of the proof, let $\gamma > \Gamma$. By contradiction, we suppose that (4.1.19) is non-oscillatory. According to Lemma 4.1.3, for every solution $z(t)$ of the associated adapted Riccati equation (4.1.21) there exists sufficiently large $t_0 \in \mathbb{T}$ such that $z(t) < 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. Moreover, from previous estimates, there exists $K > 0$ such that (4.1.30) holds. Using (4.1.28) and (4.1.29), we get

$$|z^{\Delta}(t)| < \frac{H(K)}{t}, \qquad t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.1.32}$$

Now, we introduce the average value $\xi(t)$ of the function $z(t)$ on an arbitrary interval $[t, t+\alpha]_{\mathbb{T}}$, where t is sufficiently large. Using $\xi(t)$, we will obtain a contradiction with $z(t) \in (-K, 0)$. Obviously,

$$\xi(t) \in (-K, 0) \quad \text{and} \quad \xi(t) := \frac{1}{\alpha} \int_{t}^{t+\alpha} z(\tau) \Delta\tau, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.1.33}$$

Using (4.1.23), (4.1.24), (4.1.31), and (4.1.33) we get

$$
\begin{aligned}
\xi^{\Delta}(t) &= \frac{1}{\alpha}\int_t^{t+\alpha} z^{\Delta}(\tau)\Delta\tau \\
&= \frac{1}{\alpha}\int_t^{t+\alpha} \frac{1}{\tau}\left[\frac{\gamma s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau} + \frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^{q}}{\tau^{p-1}r(\tau)[1+h(\tau)]^{p-1}}\right]\Delta\tau \\
&\qquad + \frac{1}{\alpha}\int_t^{t+\alpha} \frac{1}{\tau}\cdot\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\Delta\tau \\
&\geq \frac{1}{\alpha}\cdot\frac{1-\varepsilon}{t+\alpha}\int_t^{t+\alpha}\left[\gamma s(\tau) + \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^{q}}{[1+h(\tau)]^{p-1}}\right]\Delta\tau \\
&\qquad + \frac{1}{\alpha}\cdot\frac{1+\varepsilon}{t}\int_t^{t+\alpha}(p-1)z(\tau)\Delta\tau \qquad (4.1.34)\\
&= \frac{1-\varepsilon}{t+\alpha}\left[\frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau + \frac{1}{\alpha}\int_t^{t+\alpha}\frac{(p-1)r^{1-q}(\tau)|z(\tau)|^{q}}{[1+h(\tau)]^{p-1}}\Delta\tau\right] \\
&\qquad + \frac{(1+\varepsilon)(p-1)\xi(t)}{t} \\
&= \frac{1-\varepsilon}{t+\alpha}\Bigg\{\frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau - \frac{A^{p}(t)}{p} \\
&\qquad + \frac{1}{\alpha}\int_t^{t+\alpha}\frac{(p-1)r^{1-q}(\tau)|z(\tau)|^{q}}{[1+h(\tau)]^{p-1}}\Delta\tau - \frac{B^{q}(t)}{q} \\
&\qquad + \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{t+\alpha}{t}(p-1)\xi(t) + \frac{A^{p}(t)}{p} + \frac{B^{q}(t)}{q}\Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
A(t) &= (p-1)\left(\frac{p}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\Delta\tau\right)^{-1/q}, \\
B(t) &= |\xi(t)|\left(\frac{p}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\Delta\tau\right)^{1/q}, \qquad t \geq t_0.
\end{aligned}
\qquad (4.1.35)
$$

We will estimate $\xi^{\Delta}(t)$ using (4.1.34) in three steps.

Step I. We show that there exists $M > 0$ such that

$$\frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau - \frac{A^p(t)}{p} = M \tag{4.1.36}$$

holds for every $t \in [t_0, \infty)_{\mathbb{T}}$. Using $p/q = p-1$, we have for every $t \in [t_0, \infty)_{\mathbb{T}}$

$$\begin{aligned}
&\frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau - \frac{A^p(t)}{p}\\
&= \frac{\gamma}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau - \frac{(p-1)^p}{p}\left(\frac{p}{\alpha}\int_t^{t+\alpha} r^{1-q}(\tau)\Delta\tau\right)^{-p/q}\\
&= \left(\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau\right)\left[\gamma - \frac{(p-1)^p\alpha^{1+p/q}}{p^{1+p/q}}\left(\int_t^{t+\alpha} r^{1-q}(\tau)\Delta\tau\right)^{-p/q}\left(\int_t^{t+\alpha} s(\tau)\Delta\tau\right)^{-1}\right]\\
&= \left(\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau\right)\left[\gamma - \left(\frac{p-1}{p}\right)^p\alpha^p\left(\int_t^{t+\alpha} r^{1-q}(\tau)\Delta\tau\right)^{1-p}\left(\int_t^{t+\alpha} s(\tau)\Delta\tau\right)^{-1}\right]\\
&= \left(\frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau\right)\left[\gamma - \left(\frac{\alpha}{q}\right)^p\left(\int_a^{a+\alpha} r^{1-q}(t)\Delta t\right)^{1-p}\left(\int_a^{a+\alpha} s(t)\Delta t\right)^{-1}\right] = S(\gamma-\Gamma),
\end{aligned}$$

where

$$S := \frac{1}{\alpha}\int_t^{t+\alpha} s(\tau)\Delta\tau > 0. \tag{4.1.37}$$

Hence there exists $M = S(\gamma - \Gamma) > 0$ such that (4.1.36) holds for $t \in [t_0, \infty)_{\mathbb{T}}$.

Step II. We prove the existence of $t_1 \in \mathbb{T}$, $t_1 \geq t_0$, satisfying

$$\frac{1}{\alpha}\int_t^{t+\alpha} \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^q}{[1+h(\tau)]^{p-1}}\Delta\tau - \frac{B^q(t)}{q} \geq -\frac{M}{4}, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{4.1.38}$$

where M is taken from *Step I.* To do it, we need three further auxiliary estimates. First, in view of (4.1.22), we can write

$$\frac{1}{[1+h(t)]^{p-1}} = \frac{1}{1+\tilde{h}(t)} = 1 - \frac{\tilde{h}(t)}{1+\tilde{h}(t)} = 1 - \hat{h}(t), \tag{4.1.39}$$

where $\tilde{h}(t)$ and $\hat{h}(t)$ are convenient functions. It is obvious that $0 \leq \tilde{h}(t) \to 0$ as $t \to \infty$ and

$$0 \leq \hat{h}(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{4.1.40}$$

Second, since the function $y = |x|^q$ is continuously differentiable on $(-K, 0)$, there exists $\lambda > 0$ for which

$$|z|^q - |\xi|^q \geq -\lambda |z - \xi|, \quad \text{where} \quad z, \xi \in (-K, 0). \tag{4.1.41}$$

Third, from (4.1.32) we have

$$\begin{aligned} |z(t_m) - z(t_n)| &= \left| \int_{t_n}^{t_m} z^\Delta(\tau)\Delta\tau \right| \leq \int_{t_n}^{t_m} |z^\Delta(\tau)|\Delta\tau \leq \int_{t}^{t+\alpha} |z^\Delta(\tau)|\Delta\tau \\ &< \int_{t}^{t+\alpha} \frac{H(K)}{\tau}\Delta\tau \leq \frac{1}{t}\int_{t}^{t+\alpha} H(K)\Delta\tau = \frac{H(K)\alpha}{t} \end{aligned} \tag{4.1.42}$$

for every $t_m, t_n \in [t, t+\alpha]_{\mathbb{T}}$, where $t \geq t_0$ and (see (4.1.29)) $H(K) > 0$. Because (see (4.1.33))

$$\xi(t) \in [z_{\min}(t), z_{\max}(t)],$$

where

$$z_{\min}(t) := \min\{z(\tau), \tau \in [t, t+\alpha]_{\mathbb{T}}\}, \qquad z_{\max}(t) := \max\{z(\tau), \tau \in [t, t+\alpha]_{\mathbb{T}}\},$$

there exist $t_m, t_n \in [t, t+\alpha]_{\mathbb{T}}$ (see (4.1.42)) such that for every $\tau \in [t, t+\alpha]_{\mathbb{T}}$

$$|z(\tau) - \xi(t)| \leq |z(t_m) - z(t_n)| < \frac{H(K)\alpha}{t}. \tag{4.1.43}$$

Now, we are ready to finish *Step II.* Using (4.1.35), (4.1.39), (4.1.41), (4.1.43), and again

$p/q = p - 1$, we can estimate

$$
\begin{aligned}
&\frac{1}{\alpha}\int_t^{t+\alpha} \frac{(p-1)r^{1-q}(\tau)|z(\tau)|^q}{[1+h(\tau)]^{p-1}}\Delta\tau - \frac{B^q(t)}{q} \\
&\quad = \frac{1}{\alpha}\int_t^{t+\alpha} (p-1)r^{1-q}(\tau)|z(\tau)|^q(1-\hat{h}(\tau))\Delta\tau - \frac{|\xi(t)|^q p}{\alpha q}\int_t^{t+\alpha} r^{1-q}(\tau)\Delta\tau \\
&\quad = \frac{p-1}{\alpha}\int_t^{t+\alpha} \left[r^{1-q}(\tau)|z(\tau)|^q(1-\hat{h}(\tau)) - r^{1-q}(\tau)|\xi(t)|^q\right]\Delta\tau \\
&\quad = \frac{p-1}{\alpha}\int_t^{t+\alpha} \frac{|z(\tau)|^q - |\xi(t)|^q - \hat{h}(\tau)|z(\tau)|^q}{r^{q-1}(\tau)}\Delta\tau \qquad (4.1.44) \\
&\quad \geq -\frac{\lambda(p-1)}{\alpha}\int_t^{t+\alpha} \frac{|z(\tau)-\xi(t)|}{r^{q-1}(\tau)}\Delta\tau - \frac{(p-1)}{\alpha}\int_t^{t+\alpha} \frac{\hat{h}(\tau)|z(\tau)|^q}{r^{q-1}(\tau)}\Delta\tau \\
&\quad > -\frac{\lambda(p-1)H(K)}{t}\int_t^{t+\alpha} \frac{1}{r^{q-1}(\tau)}\Delta\tau - \frac{(p-1)}{\alpha}\int_t^{t+\alpha} \frac{\hat{h}(\tau)|z(\tau)|^q}{r^{q-1}(\tau)}\Delta\tau \\
&\quad \geq -\frac{\lambda(p-1)H(K)\alpha}{t(r^-)^{q-1}} - \frac{(p-1)}{\alpha(r^-)^{q-1}}\int_t^{t+\alpha} \hat{h}(\tau)|z(\tau)|^q\Delta\tau, \quad t \in [t_0,\infty)_{\mathbb{T}}.
\end{aligned}
$$

Finally, (4.1.44) (see also (4.1.40), which ensures that the value of the last integral in (4.1.44) tends to zero for large t) implies that there exists $t_1 \geq t_0$ such that (4.1.38) is fulfilled.

Step III. From Young's inequality ($A^p/p + B^q/q \geq AB$), from the fact that

$$(p-1)|\xi(t)| = A(t)B(t)$$

(see (4.1.35)), and from (4.1.33), we obtain that

$$
\begin{aligned}
\frac{t+\alpha}{t}(p-1)\xi(t) + \frac{A^p(t)}{p} + \frac{B^q(t)}{q} &= \frac{A^p(t)}{p} + \frac{B^q(t)}{q} - (p-1)|\xi(t)| - \frac{\alpha(p-1)}{t}|\xi(t)| \\
&= \frac{A^p(t)}{p} + \frac{B^q(t)}{q} - A(t)B(t) + \frac{\alpha(p-1)\xi(t)}{t} > -\frac{\alpha(p-1)K}{t}
\end{aligned}
$$

is fulfilled for $t \in [t_0,\infty)_{\mathbb{T}}$. Hence there exists $t_2 \in \mathbb{T}$, $t_2 \geq t_1$, such that

$$\frac{t+\alpha}{t}(p-1)\xi(t) + \frac{A^p(t)}{p} + \frac{B^q(t)}{q} \geq -\frac{M}{8}, \qquad t \in [t_2,\infty)_{\mathbb{T}}, \qquad (4.1.45)$$

where M is taken from *Step I*. Finally, we know that the constant ε in (4.1.34) can be taken arbitrarily near to zero for sufficiently large t. Hence, and in view of (4.1.45), there exists $t_3 \in \mathbb{T}$, $t_3 \geq t_2$, such that

$$\frac{1+\varepsilon}{1-\varepsilon} \cdot \frac{t+\alpha}{t}(p-1)\xi(t) + \frac{A^p(t)}{p} + \frac{B^q(t)}{q} \geq -\frac{M}{4}, \qquad t \in [t_3, \infty)_{\mathbb{T}}. \tag{4.1.46}$$

Altogether, from the previous three steps, we show that $\xi(t) \to \infty$ if $t \to \infty$. Indeed, in view of (4.1.34) and estimates (4.1.36), (4.1.38), and (4.1.46), we can easily see that

$$\xi^{\Delta}(t) \geq \frac{1-\varepsilon}{t+\alpha}\left(M - \frac{M}{4} - \frac{M}{4}\right) = \frac{M(1-\varepsilon)}{2(t+\alpha)} > \frac{M}{4(t+\alpha)}, \qquad t \in [t_3, \infty)_{\mathbb{T}}. \tag{4.1.47}$$

Integrating (4.1.47) from t_3 to ∞, we get (thanks to $\mu(t) \leq \alpha$)

$$\left[\lim_{t\to\infty} \xi(t)\right] - \xi(t_3) \geq \frac{M}{4}\int_{t_3}^{\infty} \frac{\Delta t}{t+\alpha} \geq \frac{M}{4}\sum_{n=1}^{\infty} \frac{\alpha}{n\alpha + t_3 + \alpha} = \infty,$$

thus $\xi(t) \to \infty$ if $t \to \infty$. Therefore, $\xi(t) > 0$ for every sufficiently large $t \in \mathbb{T}$, which means that $z(t) > 0$ for every sufficiently large $t \in \mathbb{T}$. This contradiction gives that equation (4.1.19) is oscillatory for $\gamma > \Gamma$.

To prove the non-oscillatory part of the theorem, we start with $\gamma \leq 0$. In this case, Eq. (4.1.19) is non-oscillatory in view of Theorem 4.1.1, part (i). It suffices to consider the non-oscillatory equation $\left[r(t)\Phi(y^{\Delta})\right]^{\Delta} = 0$. Then

$$c(t) = 0 \geq \frac{\gamma s(t)}{t^{(p-1)}\sigma(t)} = C(t), \qquad t \in [a, \infty)_{\mathbb{T}}.$$

Therefore, using this comparison, (4.1.19) is non-oscillatory as well.

To prove the last part of the theorem, we show that Eq. (4.1.19) is non-oscillatory for $0 < \gamma < \Gamma$. To do it, we show that there exists $t^* \in \mathbb{T}$ such that a solution $z(t)$ of (4.1.21) with

$$z(t^*) = -\left(\frac{q}{\alpha}\int_{t^*}^{t^*+\alpha} \frac{\Delta\tau}{r^{q-1}(\tau)}\right)^{1-p} := -Z \tag{4.1.48}$$

is negative for every $t \in [t^*, \infty)_{\mathbb{T}}$. Since

$$-r^{+} < -\frac{r^{+}}{q^{p-1}} \leq -\left(\frac{q}{\alpha}\int_{t^*}^{t^*+\alpha} \frac{\Delta\tau}{r^{q-1}(\tau)}\right)^{1-p},$$

and using (4.1.25) and (4.1.28), there exists $T_1 \in \mathbb{T}$ sufficiently large such that

$$z(t) \in (-2r^{+}, 0) \qquad \text{for } t \in [t^*, t^* + \alpha]_{\mathbb{T}},\ t^* \geq T_1. \tag{4.1.49}$$

More precisely, according to (4.1.25), $z(t)$ is increasing if $z(t) \in (-2r^+, -r^+)$. Otherwise, from (4.1.28), we have that $z(t) \in (-2r^+, 0)$ is varying arbitrarily small for large t. Hence, for $t \in [t^*, t^* + \alpha]_{\mathbb{T}}$, (4.1.49) holds.

Next, using (4.1.49) (see also (4.1.28) and (4.1.42)), there exists constant $c > 0$ such that

$$|z(t_m) - z(t_n)| < \frac{c}{t^*} \qquad \text{for } t_m, t_n \in [t^*, t^* + \alpha]_{\mathbb{T}},\ t^* \geq T_1. \tag{4.1.50}$$

Analogously as in the first part of the proof, we use the average value $\xi(t^*)$, i.e.,

$$\xi(t^*) \in (-2r^+, 0) \quad \text{and} \quad \xi(t^*) := \frac{1}{\alpha} \int_{t^*}^{t^*+\alpha} z(\tau)\Delta\tau, \quad t^* \geq T_1. \tag{4.1.51}$$

From (4.1.50) it follows (compare (4.1.43))

$$|\xi(t^*) - z(\tau)| < \frac{c}{t^*}, \qquad \tau \in [t^*, t^* + \alpha]_{\mathbb{T}},\ t^* \geq T_1. \tag{4.1.52}$$

Now (similarly as before, see (4.1.34)), we estimate $\xi^\Delta(t^*)$. Using (4.1.21), (4.1.31), and (4.1.51), we get

$$\begin{aligned}
\xi^\Delta(t^*) &= \frac{1}{\alpha} \int_{t^*}^{t^*+\alpha} z^\Delta(\tau)\Delta\tau \\
&\leq \frac{1}{\alpha} \cdot \frac{1}{t^*} \int_{t^*}^{t^*+\alpha} \left[\frac{\gamma s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau} + \frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^q}{\tau^{p-1} r(\tau)[1+h(\tau)]^{p-1}} \right] \Delta\tau \\
&\qquad + \frac{1}{\alpha} \cdot \frac{1}{t^*+\alpha} \int_{t^*}^{t^*+\alpha} \frac{(p-1)(\zeta(\tau))^{p-2} z(\tau)}{\tau^{p-2}} \Delta\tau \\
&= \frac{1}{t^*} \Bigg\{ \frac{\gamma}{\alpha} \int_{t^*}^{t^*+\alpha} \frac{s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau} \Delta\tau - \frac{A^p(t^*)}{p} \\
&\qquad - \frac{1}{t^*+\alpha} \int_{t^*}^{t^*+\alpha} \frac{(p-1)(\zeta(\tau))^{p-2} z(\tau)}{\tau^{p-2}} \Delta\tau \\
&\qquad + \frac{1}{\alpha} \int_{t^*}^{t^*+\alpha} \frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^q}{\tau^{p-1} r(\tau)[1+h(\tau)]^{p-1}} \Delta\tau - \frac{B^q(t^*)}{q} \\
&\qquad + \frac{1}{\alpha} \int_{t^*}^{t^*+\alpha} \frac{(p-1)(\zeta(\tau))^{p-2} z(\tau)}{\tau^{p-2}} \Delta\tau + \frac{A^p(t^*)}{p} + \frac{B^q(t^*)}{q} \Bigg\},
\end{aligned} \tag{4.1.53}$$

where $A(t)$ and $B(t)$ are given in (4.1.31). Again, we will estimate $\xi^{\Delta}(t^*)$ using (4.1.53) in three steps.

Step I. Let $S > 0$ be defined by (4.1.37) for t^*. Then, using (4.1.31), (4.1.35), and (4.1.51), we get

$$\frac{\gamma}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau}\Delta\tau - \frac{A^p(t^*)}{p} - \frac{1}{t^*+\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\Delta\tau$$
$$\leq \gamma S(1+\varepsilon) - \Gamma S + \frac{2\alpha r^+(p-1)(1+\varepsilon)}{t^*+\alpha} = S\big[(1+\varepsilon)\gamma - \Gamma\big] + \frac{2\alpha r^+(p-1)(1+\varepsilon)}{t^*+\alpha}.$$

Therefore, there exist $T_2 \in \mathbb{T}$, $T_2 \geq T_1$, and $N > 0$ such that for $t^* \geq T_2$ ($t^* \in \mathbb{T}$) we have

$$\frac{\gamma}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{s(\tau)(\sigma(\tau))^{p-2}}{\tau^{(p-1)}/\tau}\Delta\tau - \frac{A^p(t^*)}{p} - \frac{1}{t^*+\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\Delta\tau \leq -N. \quad (4.1.54)$$

Note that we use the fact that ε tends to zero for large t.

Step II. Using (4.1.31), (4.1.39), (4.1.49), and (4.1.52), we have

$$\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^q}{\tau^{p-1}r(\tau)[1+h(\tau)]^{p-1}}\Delta\tau - \frac{B^q(t^*)}{q}$$
$$= \frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\sigma(\tau))^{p-1}|\eta(\tau)|^{p-2}|z(\tau)|^q}{\tau^{p-1}r(\tau)[1+h(\tau)]^{p-1}}\Delta\tau - |\xi(t^*)|^q\frac{p}{q\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\Delta\tau$$
$$\leq \frac{(1+\varepsilon)(p-1)}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{r^{1-q}(\tau)|z(\tau)|^q}{[1+h(\tau)]^{p-1}}\Delta\tau - (p-1)\frac{|\xi(t^*)|^q}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\Delta\tau$$
$$= \frac{(1+\varepsilon)(p-1)}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)|z(\tau)|^q(1-\hat{h}(\tau))\Delta\tau \quad (4.1.55)$$
$$- (p-1)\frac{|\xi(t^*)|^q}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\Delta\tau$$
$$= \frac{p-1}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)\left[|z(\tau)|^q(1-\hat{h}(\tau)) - |\xi(t^*)|^q\right]\Delta\tau$$
$$+ \frac{\varepsilon(p-1)}{\alpha}\int_{t^*}^{t^*+\alpha} r^{1-q}(\tau)|z(\tau)|^q(1-\hat{h}(\tau))\Delta\tau \leq \frac{N}{4}$$

for $t^* \in [T_3, \infty)_T$, where $T_3 \geq T_2$ is sufficiently large. Indeed, T_3 exists due to the facts, that r, z, ξ are bounded, $\hat{h}, \varepsilon$ tend to zero, and due to the continuity of the function $|x|^q$ (compare (4.1.41)). Of course, the constant N is taken from *Step I.*

Step III. Using (4.1.31), (4.1.35), and (4.1.51) in this part of the proof, we have

$$\begin{aligned}\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\Delta\tau+\frac{A^p(t^*)}{p}+\frac{B^q(t^*)}{q}\\ \le(1+\varepsilon)(p-1)\xi(t^*)+\frac{(p-1)^p}{p}\left(\frac{p}{\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\Delta\tau\right)^{-p/q}\\ +(p-1)\left|\xi(t^*)\right|^q\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\Delta\tau,\end{aligned}$$

which is, according to (4.1.52), asymptotically the same as

$$\begin{aligned}&(1+\varepsilon)(p-1)z(t^*)+\frac{(p-1)^p}{p}\left(\frac{p}{\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\Delta\tau\right)^{-p/q}\\ &\qquad+(p-1)\left|z(t^*)\right|^q\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\Delta\tau\\ &=-(1+\varepsilon)(p-1)q^{1-p}\left(\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\Delta\tau\right)^{1-p}\\ &\qquad+\left(\frac{p-1}{p}\right)^p\left(\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\Delta\tau\right)^{1-p}\\ &\qquad+(p-1)q^{-p}\left(\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\Delta\tau\right)^{1-p}\\ &=\left(\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}r^{1-q}(\tau)\Delta\tau\right)^{1-p}\left[-(1+\varepsilon)(p-1)q^{1-p}+\left(\frac{p-1}{p}\right)^p+(p-1)q^{-p}\right].\end{aligned}$$

By a direct calculation one can verify, that $p+q=pq$ implies

$$(p-1)q^{-p}-(p-1)q^{1-p}+\left(\frac{p-1}{p}\right)^p=0.$$

Therefore, there exists $T_4\in\mathbb{T}$, $T_4\ge T_3$, such that

$$\frac{1}{\alpha}\int_{t^*}^{t^*+\alpha}\frac{(p-1)(\zeta(\tau))^{p-2}z(\tau)}{\tau^{p-2}}\Delta\tau+\frac{A^p(t^*)}{p}+\frac{B^q(t^*)}{q}\le\frac{N}{4},\quad t^*\in[T_4,\infty)_{\mathbb{T}},\tag{4.1.56}$$

where N is, again, taken from *Step I.*

Finally, using (4.1.54), (4.1.55), and (4.1.56) in (4.1.53), we have

$$\xi^{\Delta}(t^*) \leq \frac{1}{t^*}\left(-N + \frac{N}{4} + \frac{N}{4}\right) = -\frac{N}{2t^*}, \qquad t^* \in [T_4, \infty)_{\mathbb{T}}, \tag{4.1.57}$$

and taking into account (4.1.57), we obtain

$$\xi^{\Delta}(t^*) = \frac{1}{\alpha}\int\limits_{t^*}^{t^*+\alpha} z^{\Delta}(\tau)\Delta\tau = \frac{z(t^*+\alpha) - z(t^*)}{\alpha} < 0, \qquad t^* \in [T_4, \infty)_{\mathbb{T}},$$

i.e.,

$$z(t^* + \alpha) < z(t^*), \qquad t^* \in [T_4, \infty)_{\mathbb{T}}. \tag{4.1.58}$$

In particular, if (4.1.48) holds for some $t^* \in [T_4, \infty)_{\mathbb{T}}$, then (4.1.49) and (4.1.58) assure the negativity of $z(t)$ for the whole period, more precisely,

$$z(t) < 0 \quad \text{for} \quad t \in [t^*, t^* + \alpha]_{\mathbb{T}} \qquad \text{with} \quad z(t^* + \alpha) < z(t^*).$$

To finish the proof, it suffices to show the existence of $\vartheta > 0$ (depending only on r and α) such that if $z(t) \in (-\vartheta - Z, -Z)$ for some $t \in (t^*, \infty)_{\mathbb{T}}, t^* > T_4$, then $z(t+\alpha) < z(t)$. Immediately, we have that if $z(t) \in (-\vartheta - Z, -Z)$ then $z(t + \alpha) < -Z$. Next, using (4.1.50), if $z(t) \leq -\vartheta - Z$ then $z(t+\alpha) \leq -\vartheta - Z$ as well. Further, the initial value $-Z$ was not used in (4.1.53), (4.1.54), and (4.1.55). Moreover, (4.1.56) is valid for (4.1.48) with a sufficiently small negative perturbation depending only on the coefficient r and the period α. Therefore, the number ϑ exists, which guarantees the existence of negative solution $z(t)$ of (4.1.21) for large t.

Altogether, we have shown, that the initial value problem (4.1.21), (4.1.48) has a solution $z(t)$ satisfying $z(t) < 0$ for every $t \in [t^*, \infty)_{\mathbb{T}}$ (where t^* is sufficiently large), which, combined with Lemma 4.1.4, means that equation (4.1.19) is non-oscillatory. □

The following example demonstrates the previous theorem.

Example 4.1.2. Consider an arbitrary finite time scale interval $[3, 3 + \alpha]_{\mathbb{T}}$ with $\alpha > 0$, where $3 \in \mathbb{T}$ and $3 + \alpha \in \mathbb{T}$. Let us define infinite time scale interval $[3, \infty)_{\mathbb{T}}$ such that

$$\text{if} \quad t \in [3, 3+\alpha]_{\mathbb{T}}, \qquad \text{then} \quad \{t + \alpha n\}_{n=1}^{\infty} \subseteq [3, \infty)_{\mathbb{T}}$$

and moreover, $[3, \infty)_{\mathbb{T}}$ does not contain any other points. Consider the dynamic equation

$$\left[\left(3 - 2\cos\left(\frac{2\pi t}{\alpha}\right)\right)\Phi(y^{\Delta})\right]^{\Delta} + \frac{\gamma\left(1 + \frac{2}{3}\sin\left(\frac{2\pi t}{\alpha}\right)\right)}{t^{(p-1)}\sigma(t)}\Phi(y^{\sigma}) = 0 \tag{4.1.59}$$

on $[3, \infty)_{\mathbb{T}}$. Then (4.1.59) is oscillatory if $\gamma > \tilde{\Gamma}$ and non-oscillatory if $\gamma < \tilde{\Gamma}$, where

$$\tilde{\Gamma} = \left(\frac{\alpha}{q}\right)^{p}\left[\int\limits_{3}^{3+\alpha}\left(3 - 2\cos\left(\frac{2\pi t}{\alpha}\right)\right)^{1-q}\Delta t\right]^{1-p}\left[\int\limits_{3}^{3+\alpha}\left(1 + \frac{2}{3}\sin\left(\frac{2\pi t}{\alpha}\right)\right)\Delta t\right]^{-1}.$$

For the concrete time scale interval $[3,\infty)_{\mathbb{T}}$ and numbers α and p, we can compute the exact value of constant $\tilde{\Gamma}$. We illustrate this fact, e.g., for

$$\mathbb{T} = \left\{ \bigcup_{k=0}^{\infty} [3+3k, 4+3k] \right\} \cup \{5+3k\}_{k=0}^{\infty},$$

$\alpha = 3$, and $p = 3/2$ (which implies $q = 3$). For this choice we get

$$\begin{aligned}
\tilde{\Gamma} &= \left[\int_3^6 \left(3 - 2\cos\left(\frac{2\pi t}{3}\right)\right)^{-2} \Delta t \right]^{-1/2} \left[\int_3^6 \left(1 + \frac{2}{3}\sin\left(\frac{2\pi t}{3}\right)\right) \Delta t \right]^{-1} \\
&= \left[\int_3^4 \left(3 - 2\cos\left(\frac{2\pi t}{3}\right)\right)^{-2} \mathrm{d}t \right]^{-1/2} \left[\int_3^4 \left(1 + \frac{2}{3}\sin\left(\frac{2\pi t}{3}\right)\right) \mathrm{d}t \right]^{-1} \\
&\quad + \left[\sum_{k=4}^{5} \left(3 - 2\cos\left(\frac{2k\pi}{3}\right)\right)^{-2} \right]^{-1/2} \left[\sum_{k=4}^{5} \left(1 + \frac{2}{3}\sin\left(\frac{2k\pi}{3}\right)\right) \right]^{-1} \\
&= \sqrt{2} + \frac{20\sqrt{6\pi^3}}{3(2\pi+3)\sqrt{5\sqrt{3} + 24\sqrt{5}\arctan\sqrt{15}}} \doteq 2.513\,492\,637.
\end{aligned}$$

Note that we used a software to obtain this value (namely, we used Maple 16).

§ 4.1.3 Applications and concluding remarks

The result of Theorem 4.1.4 can be used as an oscillation test also to equations that are not Euler-type. For example, we can combine Theorem 4.1.4 and Sturm type comparison theorem 4.1.1 to obtain the following Kneser-type oscillation criteria.

Corollary 4.1.1. *Let us consider the equation*

$$\left[\Phi(y^\Delta)\right]^\Delta + d(t)\Phi(y^\sigma) = 0, \tag{4.1.60}$$

where $d \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}}), a \in \mathbb{T}, a > 0$.

(i) If there exists a positive α*-periodic function* $s \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}})$ *such that*

$$\limsup_{t\to\infty} \frac{t^{(p-1)}\sigma(t)d(t)}{s(t)} < \frac{\alpha}{q^p} \left[\int_a^{a+\alpha} s(t)\Delta t \right]^{-1},$$

then Eq. (4.1.60) *is non-oscillatory.*

(ii) If there exists a positive α-periodic function $s \in C_{\mathrm{rd}}([a,\infty)_{\mathbb{T}})$ such that

$$\liminf_{t\to\infty} \frac{t^{(p-1)}\sigma(t)d(t)}{s(t)} > \frac{\alpha}{q^p}\left[\int_a^{a+\alpha} s(t)\Delta t\right]^{-1},$$

then Eq. (4.1.60) *is oscillatory.*

Proof. Let the assumptions of the first part hold. We consider the Euler-type equation

$$\left[\Phi(y^\Delta)\right]^\Delta + \frac{\gamma s(t)}{t^{(p-1)}\sigma(t)}\Phi(y^\sigma) = 0 \tag{4.1.61}$$

together with its oscillation constant

$$\Gamma = \frac{\alpha}{q^p}\left[\int_a^{a+\alpha} s(t)\Delta t\right]^{-1}.$$

Then, for some positive number $\varepsilon \in \mathbb{R}$, we have

$$d(t) < \frac{(\Gamma-\varepsilon)s(t)}{t^{(p-1)}\sigma(t)}.$$

From Theorem 4.1.4 we have that Eq. (4.1.61) is non-oscillatory for $\gamma = \Gamma - \varepsilon$. Using Sturm type comparison theorem 4.1.1, part (i), we obtain that Eq. (4.1.60) is non-oscillatory.

The second part follows from an analogical idea and Sturm type comparison theorem 4.1.1, part (ii). □

Next, let us mention a corollary that (partially) covers the cases of negative coefficients.

Corollary 4.1.2. *Let us consider Eq.* (4.1.19) *with rd-continuous, α-periodic functions r, s satisfying*

$$\inf\{|r(t)|,\ t \in [a,\infty)_{\mathbb{T}}\} > 0, \qquad s(t) \not\equiv 0,\ t \in [a,\infty)_{\mathbb{T}}.$$

Further denote

$$\bar{\Gamma} := \left(\frac{\alpha}{q}\right)^p \left[\int_a^{a+\alpha} |r(t)|^{1-q}\Delta t\right]^{1-p} \left[\int_a^{a+\alpha} |s(t)|\Delta t\right]^{-1}.$$

Then the following statements hold.

(i) If $r(t)$ is positive for $t \in [a,\infty)_{\mathbb{T}}$ and $\gamma < \bar{\Gamma}$, then Eq. (4.1.19) *is non-oscillatory.*

(ii) If $s(t)$ is positive for $t \in [a,\infty)_{\mathbb{T}}$ and $\gamma > \bar{\Gamma}$, then Eq. (4.1.19) *is oscillatory.*

Proof. The corollary comes directly from Theorem 4.1.4, Sturm type comparison theorem 4.1.1, and the fact that the absolute value preserves periodicity. □

Finally, as a possible direction of future research, we conjecture that Eq. (4.1.19) with more general coefficients remains conditionally oscillatory. This conjecture is based on continuous and discrete cases. More precisely, in [37], there is found the oscillation constant for Euler-type half-linear difference equations with asymptotically almost periodic coefficients. Concerning the continuous case, in [35] is shown that Euler-type half-linear differential equations with coefficients having mean values (which covers periodic and almost periodic cases) are conditionally oscillatory. However, extension of these types for dynamic equations on time scales appear to be much more technically difficult.

For another natural possible direction, we should mention papers [17, 19, 39], where perturbed half-linear differential equations are studied. Typically, the perturbations are placed in the potential of the given equation, which leads to the equations of the form

$$[r(t)\Phi(y')]' + \left[\frac{c(t)}{t^2} + \frac{d(t)}{t^2 \log^2 t}\right] \Phi(y) = 0, \qquad \mathbb{T} = \mathbb{R},$$

which is referred to as the Riemann–Weber half-linear equation (see also Section 2.6). Eventually, the perturbation in the potential can be replaced by a more complex one involving the iterated logarithms (i.e., $\log(\log(\dots(\log t))))$. In the above mentioned papers, it is proved that such equations are conditionally oscillatory and from the behavior of the "more perturbed" equation, there is shown, that the "less perturbed" equation with the critical constant is non-oscillatory, e.g., the results concerning the Riemann–Weber equation give that the border case of the Euler equation is non-oscillatory.

Appendix

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Appendix

B Curriculum vitae

Personal Details

Name	Mgr. Petr Hasil, Ph.D.
Date of Birth	23. 3. 1982
Nationality	Czech
E-mail	hasil@mail.muni.cz
Web	`http://www.math.muni.cz/~hasil`

Education

2010	Ph.D.	Mathematical Analysis (Masaryk University) Supervisor: prof. RNDr. Ondřej Došlý, DrSc.
2006	Master's degree	Mathematical Analysis (Masaryk University)
2004	Bachelor's degree	(General) Mathematics (Masaryk University)

Professional experience

since 2013	Assistant Professor	Masaryk University Department of Mathematics and Statistics
since 2010	Research Assistant	Mendel University in Brno Department of Mathematics
2008 – 2010	Assistant	Mendel University in Brno Department of Mathematics

Professional stays

March – July 2009	Ulm University (Ulm, Germany) Teaching Assistant (Lineare Algebra II, summer semester 2009)

Membership in scientific societies

since 2010	International Society of Difference Equations (ISDE)
since 2013	Reviewer for MathSciNet (American Mathematical Society)

Main pedagogical activities

MU	Lecturer, Examiner, Instructor, Bachelor and master students supervisor Member of bachelor and master exam committees
MENDELU	Course supervisor, Lecturer, Examiner, Instructor

Teaching activities

Ulm University: Faculty of Mathematics and Economics	Linear algebra
Masaryk University: Faculty of Science	Mathematical Analysis
Masaryk University: Faculty of Informatics	Differential and Integral Calculus
Mendel University in Brno: Faculty of Forestry and Wood Technology	Engineering mathematics Advanced Mathematics Applied Mathematics
Mendel University in Brno: Faculty of Agronomy	Matematics Rudiments of Higher Mathematics

Awards Related to Science and Research

2010 Prize of the Dean of the Faculty of Science (Masaryk University) for excellent study results and success in the research

2005 Prize of the Department of Mathematics (Faculty of Science, Masaryk University) for excellent study results and creative approach to solving problems in mathematics

Conferences and invited talks

2008–2015 14 international conferences
(13 talks, 1 poster — CA, CZ, DE, ES, FR, LV, OM, PT, SK)

2008–2015 6 lectures at "Seminar on Differential Equations"
Dep. Math. Stat., Fac. Sci., MU

Publications *(including accepted publications)*

Total number of research papers	24
Papers in journals with IF	21
Citations without self-citations	38
Citations without self-citations on WoS	29
Citations without self-citations of all authors	26
h-index (SCOPUS)	6
h-index (WoS)	5

Pedagogical publications	4

Appendix

C Publications

Research Papers

1. Petr Hasil, Michal Veselý:
   Values of limit periodic sequences and functions.
   Accepted in Mathematica Slovaca, ISSN 0139-9918,
   impact factor (2014): 0.409.

2. Petr Hasil, Michal Veselý:
   Oscillation constant for modified Euler type half-linear equations.
   Electronic Journal of Differential Equations (2015), ISSN 1072-6691, vol. 2015, no. 220, pp. 1–14
   impact factor (2014): 0.524.

3. Petr Hasil, Michal Veselý:
   Oscillation constants for half-linear difference equations with coefficients having mean values.
   Advances in Difference Equations (2015), ISSN 1687-1847, vol. 2015, no. 210, pp. 1–18,
   impact factor (2014): 0.640, doi: 10.1186/s13662-015-0544-1.

4. Petr Hasil, Michal Veselý:
   Non-oscillation of perturbed half-linear differential equations with sums of periodic coefficients.
   Advances in Difference Equations (2015), ISSN 1687-1847, vol. 2015, no. 190, pp. 1–17,
   impact factor (2014): 0.640, doi: 10.1186/s13662-015-0533-4.

5. Petr Hasil, Michal Veselý:
   Limit periodic homogeneous linear difference systems.
   Applied Mathematics and Computation (2015), ISSN 0096-3003, vol. 265, pp. 958–972,
   impact factor (2014): 1.551, doi: 10.1016/j.amc.2015.06.008.

   Cited in

   - Martin Chvátal:
     Asymptotically almost periodic solutions of limit and almost periodic linear difference systems.

Electron. J. Qual. Theory Differ. Equ. (2015), ISSN 1417-3875, vol. 2015, no. 75, pp. 1–17.

- Martin Chvátal:
  Non-almost periodic solutions of limit periodic and almost periodic homogeneous linear difference systems.
  Electron. J. Qual. Theory Differ. Equ. (2014), ISSN 1417-3875, vol. 2014, no. 76, pp. 1–20.

6. Petr Hasil, Jiří Vítovec:
   Conditional oscillation of half-linear Euler-type dynamic equations on time scales.
   Electronic Journal of Qualitative Theory of Differential Equations (2015), ISSN 1417-3875, vol. 2015, no. 6, pp. 1–21,
   impact factor (2014): 0.817.

7. Petr Hasil, Michal Veselý:
   Non-oscillation of half-linear differential equations with periodic coefficients.
   Electronic Journal of Qualitative Theory of Differential Equations (2015), ISSN 1417-3875, vol. 2015, no. 1, pp. 1–24,
   impact factor (2014): 0.817.

8. Petr Hasil, Michal Veselý:
   Conditional oscillation of Riemann–Weber half-linear differential equations with asymptotically almost periodic coefficients.
   Studia Scientiarum Mathematicarum Hungarica (2014), ISSN 0081-6906, vol. 51, no. 3, pp. 303–321,
   impact factor (2014): 0.205, doi: 10.1556/SScMath.51.2014.3.1283.

   Cited in

   - Ondřej Došlý, Michal Veselý:
     Oscillation and non-oscillation of Euler type half-linear differential equations.
     J. Math. Anal. Appl. (2015), ISSN 0022-247X, vol. 429, no. 1, pp. 602–621, doi: 10.1016/j.jmaa.2015.04.030.

9. Petr Hasil, Robert Mařík, Michal Veselý:
   Conditional oscillation of half-linear differential equations with coefficients having mean values.
   Abstract and Applied Analysis (2014), ISSN 1085-3375, vol. 2014, no. 258159, pp. 1–14,
   impact factor (2013): 1.274, doi: 10.1155/2014/258159.

   Cited in

   - Ondřej Došlý, Michal Veselý:
     Oscillation and non-oscillation of Euler type half-linear differential equations.
     J. Math. Anal. Appl. (2015), ISSN 0022-247X, vol. 429, no. 1, pp. 602–621, doi: 10.1016/j.jmaa.2015.04.030.

- Masakazu Onitsuka, Tomomi Soeda:
Uniform asymptotic stability implies exponential stability for nonautonomous half-linear differential systems.
Adv. Differ. Equ. (2015), ISSN 1687-1847, vol. 2015, no. 158, pp. 1–24.
doi: 10.1186/s13662-015-0494-7.

10. Petr Hasil, Michal Veselý:
Limit periodic linear difference systems with coefficient matrices from commutative groups.
Electronic Journal of Qualitative Theory of Differential Equations (2014), ISSN 1417-3875, vol. 2014, no. 23, pp. 1–25,
impact factor (2014): 0.817.

    Cited in

    - Martin Chvátal:
    Asymptotically almost periodic solutions of limit and almost periodic linear difference systems.
    Electron. J. Qual. Theory Differ. Equ. (2015), ISSN 1417-3875, vol. 2015, no. 75, pp. 1–17.
    - Martin Chvátal:
    Non-almost periodic solutions of limit periodic and almost periodic homogeneous linear difference systems.
    Electron. J. Qual. Theory Differ. Equ. (2014), ISSN 1417-3875, vol. 2014, no. 76, pp. 1–20.

11. Petr Hasil, Michal Veselý:
Oscillation and non-oscillation of asymptotically almost periodic half-linear difference equations.
Abstract and Applied Analysis (2013), ISSN 1085-3375, vol. 2013, no. 432936, pp. 1–12, impact factor (2013): 1.274, doi: 10.1155/2013/432936.

    Cited in

    - Ondřej Došlý, Michal Veselý:
    Oscillation and non-oscillation of Euler type half-linear differential equations.
    J. Math. Anal. Appl. (2015), ISSN 0022-247X, vol. 429, no. 1, pp. 602–621,
    doi: 10.1016/j.jmaa.2015.04.030.
    - Masakazu Onitsuka, Tomomi Soeda:
    Uniform asymptotic stability implies exponential stability for nonautonomous half-linear differential systems.
    Adv. Differ. Equ. (2015), ISSN 1687-1847, vol. 2015, no. 158, pp. 1–24.
    doi: 10.1186/s13662-015-0494-7.
    - Jiří Vítovec:
    Critical oscillation constant for Euler-type dynamic equations on time scales.

Appl. Math. Comput. (2014), ISSN 0096-3003, vol. 243, no. 7, pp. 838–848, doi: 10.1016/j.amc.2014.06.066.

12. Petr Hasil, Michal Veselý:
Oscillation of half-linear differential equations with asymptotically almost periodic coefficients.
Advances in Difference Equations, ISSN 1687-1847, vol. 2013, no. 122, pp. 1–15, impact factor (2013): 0.634, doi: 10.1186/1687-1847-2013-122.

    Cited in

    - Ondřej Došlý, Michal Veselý:
    Oscillation and non-oscillation of Euler type half-linear differential equations.
    J. Math. Anal. Appl. (2015), ISSN 0022-247X, vol. 429, no. 1, pp. 602–621, doi: 10.1016/j.jmaa.2015.04.030.
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    Critical oscillation constant for Euler-type dynamic equations on time scales.
    Appl. Math. Comput. (2014), ISSN 0096-3003, vol. 243, no. 7, pp. 838–848, doi: 10.1016/j.amc.2014.06.066.
    - Ondřej Došlý, Hana Funková:
    Euler type half-linear differential equation with periodic coefficients.
    Abstract Appl. Anal. (2013), ISSN 1085-3375, vol. 2013, no. 714263, pp. 1–6, doi: 10.1155/2013/714263.
    - Simona Fišnarová, Robert Mařík:
    Oscillation of half-linear differential equations with delay.
    Abstract Appl. Anal. (2013), ISSN 1085-3375, vol. 2013, no. 583147, pp. 1–6, doi: 10.1155/2013/583147.

13. Petr Hasil, Michal Veselý:
Critical oscillation constant for difference equations with almost periodic coefficients.
Abstract and Applied Analysis (2012), ISSN 1085-3375, vol. 2012, no. 471435, pp. 1–19, impact factor (2012): 1.102, doi: 10.1155/2012/471435.

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    Oscillation and non-oscillation of Euler type half-linear differential equations.
    J. Math. Anal. Appl. (2015), ISSN 0022-247X, vol. 429, no. 1, pp. 602–621, doi: 10.1016/j.jmaa.2015.04.030.
    - Jiří Vítovec:
    Critical oscillation constant for Euler-type dynamic equations on time scales.
    Appl. Math. Comput. (2014), ISSN 0096-3003, vol. 243, no. 7, pp. 838–848, doi: 10.1016/j.amc.2014.06.066.

14. Petr Hasil, Michal Veselý:
    Criticality of one term 2n-order self-adjoined differential equations.
    Electronic Journal of Qualitative Theory of Differential Equations (2012), Proc. 9'th Coll. QTDE, ISSN 1417-3875, vol. 2012, no. 18, pp. 1–12,
    impact factor (2012): 0.740.

15. Petr Hasil, Michal Veselý:
    Almost periodic transformable difference systems.
    Applied Mathematics and Computation (2012), ISSN 0096-3003, 2012, vol. 218, no. 9, pp. 5562–5579,
    impact factor: (2012): 1.349, doi: 10.1016/j.amc.2011.11.050.

    Cited in

    - Martin Chvátal:
      Asymptotically almost periodic solutions of limit and almost periodic linear difference systems.
      Electron. J. Qual. Theory Differ. Equ. (2015), ISSN 1417-3875, vol. 2015, no. 75, pp. 1–17.
    - Martin Chvátal:
      Non-almost periodic solutions of limit periodic and almost periodic homogeneous linear difference systems.
      Electron. J. Qual. Theory Differ. Equ. (2014), ISSN 1417-3875, vol. 2014, no. 76, pp. 1–20.
    - Michal Veselý:
      Almost periodic skew-symmetric differential systems.
      Electron. J. Qual. Theory Differ. (2012), ISSN 1417-3875, vol. 2012, no. 72, pp. 1–16,
      doi: 10.1504/IJDSDE.2011.038498.

16. Petr Hasil, Petr Zemánek:
    Friedrichs extension of operators defined by even order Sturm–Liouville equations on time scales.
    Applied Mathematics and Computation (2012), ISSN 0096-3003, vol. 218, no. 22, pp. 10829–10842,
    impact factor (2012): 1.349, doi: 10.1016/j.amc.2012.04.027.

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    - Petr Zemánek:
      A note on the equivalence between even order Sturm–Liouville equations and symplectic systems on time scales.
      Appl. Math. Lett. (2013), ISSN 0893-9659, vol. 26, no. 1, pp. 134–139,
      doi: 10.1016/j.aml.2012.04.009.

17. Petr Hasil:
Conjugacy of self-adjoint difference equations of even order.
Abstract and Applied Analysis (2011), ISSN 1085-3375, vol. 2011, no. 814962, pp. 1–16, impact factor (2011): 1.318, doi: 10.1155/2011/814962.

Cited in

- Peter Šepitka, Roman Šimon Hilscher:
Recessive solutions for nonoscillatory discrete symplectic systems.
Linear Alg. Appl. (2015), ISSN 0024-3795, vol. 469, pp. 243–275,
doi: 10.1016/j.laa.2014.11.029.

18. Petr Hasil:
Criterion of p-criticality for one term 2n-order difference operators.
Archivum Mathematicum (2011), ISSN 0044-8753, vol. 47, no. 2, pp. 99–109,
Scopus, SciNet: MR2813536.

19. Ondřej Došlý, Petr Hasil:
Critical higher order Sturm–Liouville difference operators.
Journal of Difference Equations and Applications (2011), ISSN 1023-6198, vol. 17, no. 9, pp. 1351–1363,
impact factor (2011): 0.800, doi: 10.1080/10236190903527251.

20. Ondřej Došlý, Petr Hasil:
Critical oscillation constant for half-linear differential equations with periodic coefficients.
Annali di Matematica Pura ed Applicata (2011), ISSN 0373-3114, vol. 190, no. 3, pp. 395–408,
impact factor (2011): 0.838, doi: 10.1007/s10231-010-0155-0.

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- Ondřej Došlý, Michal Veselý:
Oscillation and non-oscillation of Euler type half-linear differential equations.
J. Math. Anal. Appl. (2015), ISSN 0022-247X, vol. 429, no. 1, pp. 602–621,
doi: 10.1016/j.jmaa.2015.04.030.
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