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Dear committee members,

This letter reports on John Bourke's very impressive habilitation thesis, which is a compendium of his post-PhD work. Bourke has been extremely productive since his PhD thesis, but what is more impressive than the quantity of his output—which is substantially larger than the articles included here—is the quality and originality of his contributions. Of the five projects he highlights here, all but the most recent preprint have been published in top journals in the field. Prior to being asked to review this habilitation thesis, I was aware of all five of these projects and have cited two of them heavily in my own work.

Dr. Bourke is a beautiful writer and expositor and provides a useful introduction as a guide to his recent work that surveys the motivating questions in each area, the contributions he has made, some applications of his results, and the open problems that remain. What I took away from this was that the influence of Bourke's theorems is even broader than I had realized. I was unaware of, though not surprised by, the use of double-category-generated algebraic weak factorization systems in homotopy type theory, for instance. Given how recent this work is, the breadth of applications is quite impressive.

A final general thing that strikes me about this work is the broad range of technical tools it uses: from Australian-style 2-category theory, to cutting edge applications of enriched category theory (such as $\mathbb{F}$-categories), to Quillen model categories and abstract homotopy theory, to skew structures, to locally presentable and accessible category theory, and so on. While many mathematicians search for occasions to re-use their familiar tools in new settings, for Dr. Bourke it appears the questions come first while the techniques needed to solve them are selected for the purpose at hand.

Two-dimensional monadicity. In the first paper, "Two-dimensional monadicity," Bourke makes one of the most substantial contributions to two-dimensional universal algebra in the past several decades, since perhaps the celebrated paper "Two-dimensional monad theory" of Blackwell, Kelly, and Power and contemporaneous work. The main theorem in this paper characterizes the categories of algebra and pseudo or (co)lax morphisms for a 2-monad via a universal property expressed in the language of $\mathbb{F}$-categories, so that the statement of the universal property can refer to the strict morphisms as well as the weak ones. As is typical of Bourke's work, he has found exactly the right categorical tool to study the relevant behavior of such 2-categories, shedding light on classical phenomena such as doctrinal adjunctions.

Algebraic weak factorization systems. More structured so-called "algebraic" weak factorization systems have attracted increasing interest, motivated for instance by the desire of the homotopy type theory community for *constructive* variants of Quillen model categories appropriate for the categorical semantics of various proposed type theories. As someone who has read every paper in this area and written a few of them myself, I can say definitively that the most important developments are contained in the two papers of Bourke and Garner—"Algebraic weak factorisation systems I: Accessible AWFS" and "Algebraic weak factorisation systems II: Categories of weak maps"—the first of which makes up the second part of this habilitation thesis.

A main theme in this paper is a complete characterization of those algebraic weak factorization systems whose underlying functorial factorization is given by an accessible functor. Bourke and Garner prove that an awfs is accessible if and only if it is "cofibrantly generated" in a sense that had not previously been considered but in retrospect is very natural: by a small *double* category whose vertical morphisms are the generating cofibrations (as classically understood), whose squares enumerate "horizontal coherence conditions" between the generators, and whose vertical composites express further "vertical coherences." This result, together with a monadicity theorem that recognizes awfs by their double category of algebras, form the definitive treatise on the subject, and has already received a large number of citations.

Skew structures. The third paper "Skew structures in 2-category theory and homotopy theory" introduces a lovely idea of a "homotopical" skew closed monoidal structure borne by a Quillen model category that descends to a genuine monoidal closed structure on the homotopy category. The motivating family of examples concerns a skew closed structure on the 2-category of strict algebras and strict morphisms for an accessible 2-monad. The skewness allows the internal homs to include the pseudo morphisms as well as the strict ones. Bourke shows that a suitable model structure on this category is homotopically skew closed monoidal and deduces the monoidal bicategory structure on the pseudo morphisms of Hyland and Power as a slick corollary.

Monads and theories. Of the five papers in this habilitation thesis, "Monads and theories," is the one that I am the least familiar with. As I understand it, a deficiency of the earlier work is that it did not encompass some key examples, such as Grothendieck's model of globular weak ω-groupoids. While these are models of a globular theory in the sense of Berger, they are not "theories with arities" in the sense of Berger, Melliès, and Weber's "Monads with arities and their associated theories." Thus, the existing monad-theory correspondence was not sufficiently general, motiving the present work.

Adjoint functor theorems and accessible ∞-cosmoi. The final paper included in the habilitation thesis, "Adjoint functor theorems for homotopically enriched categories," joint with Steve Lack and Lukáš Vokřínek, is really part of a series starting with Bourke's solo-authored paper "Accessible aspects of 2-category theory" and continuing with the forthcoming "Accessible ∞-cosmoi," joint with Steve Lack. The adjoint functor theorem in this paper, which unifies nearly every strict, weak, and enriched version of Freyd's adjoint functor theorem known to the literature, is really lovely, but the results I'm most excited about concern the new notion of *accessible* ∞-cosmos they introduce.

Roughly, an ∞-*cosmos* is a setting for abstract ∞-category theory, much like a simplicial model category is a setting for abstract homotopy theory. In a forthcoming book with Dominic Verity, we show that a large portion of the theory of $(\infty, 1)$-categories can be developed from the axioms of an ∞-cosmos, which is essentially a category of fibrant objects enriched over the Joyal model structure for quasi-categories. In this paper, Bourke, Lack, and Vokřínek observe that if the ∞-cosmos is accessible, then the ∞-cosmos has all *flexible weighted homotopy colimits*. This result can be understood as a consequence of their adjoint functor theorem, which in this context says that any accessible structure-preserving (and in particular limit-preserving) functor between accessible ∞-cosmoi admits a homotopical left adjoint. These results will greatly expand the amount of $(\infty, 1)$-category theory that can be developed "synthetically" using the axiomatics of an accessible ∞-cosmos. I personally cannot wait to explore such applications further.

Reviewer's question for the habilitation thesis defense

Do you suspect there is an analogue of the Makkai-Paré theorem that the 2-category of accessible categories and accessible functors has PIE limits that would apply to accessible ∞-cosmoi and accessible cosmological functors?

Conclusion

The habilitation thesis entitled "Categorical structures for higher-dimensional universal algebra" by John Bourke fulfills requirements expected of a habilitation thesis in the field of Mathematics — Algebra and Theory of Numbers.

Sincerely,

Emily Riehl
Associate Professor